McCarthy's amb has no least fixpoint semantics

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Abstract

This note presents a formal argument why it is impossible to give a least fixpoint semantics for McCarthy's *amb* [3] that discriminates programs with different divergence behaviour and satisifies some equational axioms.

1 Introduction

McCarthy's ambiguous choice operator *amb* [3] differs from erratic choice in its divergence behaviour: *amb* evaluates all branches in fair parallel and may only diverge if every branch may diverge, whereas erratic choice selects one branch up-front and thus may diverge if any branch may diverge. Therefore the characteristics of *amb* is only brought out by semantics that take divergence behaviour into account. Let us say that a semantics is *adequate* if it discriminates programs with different divergence behaviours.

With this definition of adequacy, *amb* is known to be non-monotone with respect to known adequate least fixpoint semantics partial orders [4, 5]. Specifically, *amb* is monotone with respect to the "Hoare" partial-correctness partial order used in the lower powerdomain construction, which is not adequate in our sense, and *amb* is non-monotone with respect to the "Smyth" total-correctness partial order and the "Egli-Milner" partial order used in the upper and convex powerdomains, which are adequate.

This note goes further and proves that *amb* is not monotone with respect to any partial order semantics with divergence as bottom element, provided the semantics is adequate and satisfies a set of equational axioms, namely β-conversion and basic algebraic properties of *amb* and erratic choice.

Outline Section 2 defines the syntax and operational semantics of the λ-calculus extended with *amb*. Section 3 specifies the equational axioms for the language. In Section 4 we prove that there is no adequate least fixpoint semantics for *amb* which satisfies the equational axioms.

2 Syntax and operational semantics

Consider the pure untyped λ-calculus extended with *amb*:

$$\begin{array}{llcl} \text{Variables} & x, y, z & & \\ \text{Terms} & t & ::= & x \quad | \quad \lambda x.\, t \quad | \quad t_1\, t_2 \quad | \quad amb(t_1, t_2) \end{array}$$

Terms are identified up to renaming of bound variables and $t_1[t_2/x]$ denotes the capture-free substitution of t_2 for the free occurrences of x in t_1.

The term $amb(t_1, t_2)$ non-deterministically chooses between t_1 and t_2. Informally, it does so by evaluating t_1 and t_2 in fair parallel and choosing the outcome of the branch that first evaluates to a result. It may only diverge if both threads of execution in the fair parallel evaluation may diverge.

Recall the following standard combinators: the identity function $I = \lambda x.\, x$, true $T = \lambda x.\, \lambda y.\, x$, false $F = \lambda x.\, \lambda y.\, y$, and Curry's fixed point combinator $Fix = \lambda x.\, (\lambda y.\, x\, (y\, y))\, (\lambda y.\, x\, (y\, y))$. Let $\Omega = Fix\, I$.

Erratic choice can be defined in terms of *amb* as follows:

$$err(t_1, t_2) = amb(T, F)\, t_1\, t_2.$$

Following [4], we specify the operational semantics as an inductively defined *may evaluation* relation $t \rightsquigarrow_{\mathrm{wh}} w$ between closed terms t and λ-abstractions w, and a co-inductively defined *may divergence* predicate $t{\Uparrow_{\mathrm{wh}}}$ on closed terms t.

$$\frac{}{\lambda x.\, t \rightsquigarrow_{\mathrm{wh}} \lambda x.\, t} \qquad \frac{t_1 \rightsquigarrow_{\mathrm{wh}} \lambda x.\, t \quad t[t_2/x] \rightsquigarrow_{\mathrm{wh}} w}{t_1\, t_2 \rightsquigarrow_{\mathrm{wh}} w}$$

$$\frac{t_1 \rightsquigarrow_{\mathrm{wh}} w}{amb(t_1, t_2) \rightsquigarrow_{\mathrm{wh}} w} \qquad \frac{t_2 \rightsquigarrow_{\mathrm{wh}} w}{amb(t_1, t_2) \rightsquigarrow_{\mathrm{wh}} w}$$

$$\frac{t_1{\Uparrow_{\mathrm{wh}}}}{t_1\, t_2{\Uparrow_{\mathrm{wh}}}}\, - \qquad \frac{t_1 \rightsquigarrow_{\mathrm{wh}} \lambda x.\, t \quad t[t_2/x]{\Uparrow_{\mathrm{wh}}}}{t_1\, t_2{\Uparrow_{\mathrm{wh}}}}\, - \qquad \frac{t_1{\Uparrow_{\mathrm{wh}}} \quad t_2{\Uparrow_{\mathrm{wh}}}}{amb(t_1, t_2){\Uparrow_{\mathrm{wh}}}}\, -$$

(The minuses mean that the rules are co-inductive. May divergence is defined as the largest predicate on closed terms that is dense with respect to the rules.)

Given the operational definition of divergence, we define adequacy as follows.

Definition 2.1. A relation on terms is *adequate* if and only if related terms have the same divergence behaviour, that is, if t is related to t' then $t{\Uparrow_{\mathrm{wh}}} \Leftrightarrow t'{\Uparrow_{\mathrm{wh}}}$.

3 Equations

Let $\asymp$ be the equational theory generated by the following equations:

$$(\lambda x.\, t_1)\, t_2 \asymp t_1[{}^{t_2}\!/x] \tag{1}$$
$$amb(t_1, amb(t_2, t_3)) \asymp amb(amb(t_1, t_2), t_3) \tag{2}$$
$$amb(t_1, t_2) \asymp amb(t_2, t_1) \tag{3}$$
$$amb(t, t) \asymp t \tag{4}$$
$$amb(\Omega, t) \asymp amb(t, \Omega) \asymp t \tag{5}$$
$$amb(err(t_1, t_2), t_3) \asymp err(amb(t_1, t_3), amb(t_2, t_3)) \tag{6}$$
$$err(amb(t_1, t_2), t_3) \asymp amb(err(t_1, t_3), err(t_2, t_3)) \tag{7}$$
$$amb(\lambda x_1.\, t_1, \lambda x_2.\, t_2) \asymp err(\lambda x_1.\, t_1, \lambda x_2.\, t_2) \tag{8}$$
$$err(t_1, err(t_2, t_3)) \asymp err(err(t_1, t_2), t_3) \tag{9}$$
$$err(t_1, t_2) \asymp err(t_2, t_1) \tag{10}$$
$$err(t, t) \asymp t \tag{11}$$
$$err(t_1, t_2)\, t \asymp err(t_1\, t, t_2\, t) \tag{12}$$

These equations are all included in the equational theories for *amb* in [2, 1], called *Kleene equivalence*, *whnf bisimilarity*, *whnf simulation equivalence*, and *contextual equivalence*. The theories are all adequate.

4 Least fixpoints

Every term t is a fixpoint of I, up to β-equivalence (1), that is, $I\, t \asymp t$. Therefore, if $\Omega = \mathit{Fix}\, I$ is the least fixpoint of I with respect to some ordering on terms, then Ω is the least element in the ordering.

Theorem 1. *Suppose $\precsim$ is a pre-congruence relation on terms and $\cong$ is the induced congruence relation, $t \cong t' \Leftrightarrow t \precsim t'$ & $t' \precsim t$. Furthermore, suppose $\cong\, \supseteq\, \asymp$ and $\cong$ is adequate. Then Ω cannot be least with respect to $\precsim$.*

Proof. Let $t = amb(\Omega, F)\, \Omega$ and $t' = amb(I, F)\, \Omega$. Then t and t' have different divergence behaviour, $t' {\Uparrow}_{\mathrm{wh}}$ and $\neg t {\Uparrow}_{\mathrm{wh}}$. Therefore, by the assumption that $\cong$ is adequate, $t \not\cong t'$. We will use this fact to prove that Ω is not least, by contradiction, namely we will show that Ω is least implies $t \cong t'$.

Suppose Ω is least. Then $\Omega \precsim I$ and $\Omega \precsim t$. Since $\precsim$ is a pre-congruence, $\Omega \precsim I$ implies

$$t \precsim t' \tag{13}$$

and $\Omega \precsim t$ implies

$$err(\Omega, t) \precsim err(t, t). \tag{14}$$

From (14) and the calculation

$$\begin{aligned} t' = amb(I, F)\,\Omega &\asymp err(I, F)\,\Omega && \text{by (8)} \\ &\asymp err(I\,\Omega, F\,\Omega) && \text{by (12)} \\ &\asymp err(\Omega, F\,\Omega) && \text{by (1)} \\ &\asymp err(\Omega, amb(\Omega, F)\,\Omega) && \text{by (5)} \\ &= err(\Omega, t) \end{aligned}$$

and (11) and the assumption $\cong \supseteq \asymp$, derive

$$t' \cong err(\Omega, t) \precsim err(t, t) \cong t. \tag{15}$$

Finally, by (13), (15), and the assumption $t \cong t' \Leftrightarrow t \precsim t' \;\&\; t' \precsim t$, deduce $t \cong t'$. □

Not all the equations from section 3 are used in the proof of the theorem. The condition $\cong \supseteq \asymp$ in the statement of theorem serves only to ensure that $t \cong err(t, t)$ and $err(\Omega, t) \cong amb(I, F)\,\Omega$, where $t = amb(\Omega, F)\,\Omega$.

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