The Formal Statement of the Pumping Lemma and its Negation

The pumping lemma states that for any regular language L, there is a positive integer p such that for any string s in the language L with the length of s greater than or equal to p there are three strings x, y, z such that $s = xyz$ with the length of xy less than or equal to p and the length of y strictly positive such that for any natural number i the string xy^iz is in L.

In symbols this is written, for L a regular language

$$\exists p.(p > 0) \wedge \forall s.(s \in L \wedge |s| \geq p) \Rightarrow \exists x, y, z.(s = xyz \wedge |xy| \leq p \wedge |y| > 0 \wedge \forall i.xy^iz \in L).$$

We negate this in stages as follows:

$$\neg[\exists p.(p > 0) \wedge \forall s.(s \in L \wedge |s| \geq p) \Rightarrow \exists x, y, z.(s = xyz \wedge |xy| \leq p \wedge |y| > 0 \wedge \forall i.xy^iz \in L)].$$

which is

$$\forall p.\neg(p > 0) \vee \neg[\forall s. \ldots]$$

Before we delve any deeper, recall that $P \Rightarrow Q$ is the same as $\neg P \vee Q$ so in the above we can write

$$\forall p.(p > 0) \Rightarrow \neg[\forall s. \ldots].$$

Using this conversion to implication gives something more readable, though it may confuse those who expect to see the *and*s become *or*s. Similarly $\neg(P \Rightarrow Q) = \neg(\neg P \vee Q) = P \wedge \neg Q$. Now we can look deeper into the expression and we get

$$\forall p.(p > 0) \Rightarrow \exists s.(s \in L) \wedge (|s| \geq p) \wedge \neg[\exists x, y, z.(s = xyz \wedge |xy| \leq p \wedge |y| > 0) \wedge \forall i.xy^iz \in L].$$

Then pushing the negation further inside we get

$$\forall p.(p > 0) \Rightarrow \exists s.(s \in L) \wedge (|s| \geq p) \wedge \forall x, y, z.(s = xyz \wedge |xy| \leq p \wedge |y| > 0) \Rightarrow \neg[\forall i.xy^iz \in L].$$

Finally,

$$\forall p.(p > 0) \Rightarrow \exists s.(s \in L) \wedge (|s| \geq p) \wedge \forall x, y, z.(s = xyz \wedge |xy| \leq p \wedge |y| > 0) \Rightarrow \exists i.xy^iz \notin L.$$