Def An NFA is a 4-tuple (fix Σ as the alphabet)
$$N = (Q, Q_0, \Delta, F)$$
Q: set of states, $Q_0 \subseteq Q$ (start states; note plural)
$\Delta: Q \times \Sigma \to 2^Q$ (2^Q is the powerset of Q)
[$\Delta \subseteq Q \times \Sigma \times Q$ or $\forall a \in \Sigma$ Δ_a is a binary relation on Q]
Given Δ we can define $\Delta^*: 2^Q \times \Sigma^* \to 2^Q$

Def $\Delta^*(A, \varepsilon) = A$ ‖ $A \subseteq Q,\ a \in \Sigma,\ \omega \in \Sigma^*$
$\Delta^*(A, \omega a) \stackrel{?}{=} \Delta(\cancel{A}, \Delta^*(A,\omega), a)$ ‖ ⇐ NOT QUITE RIGHT
$$= \bigcup_{q \in \Delta^*(A,\omega)} \Delta(q, a)$$

FACT (1) $\Delta^*(A \cup B, \omega) = \Delta^*(A,\omega) \cup \Delta^*(B,\omega)$
(2) $\Delta^*(A, xy) = \Delta^*(\Delta^*(A,x), y)$

DEF $L(N) \stackrel{def}{=} \{\omega \in \Sigma^* \mid \Delta^*(Q_0, \omega) \cap F \neq \phi\}$

Thm Given an NFA N there exists a DFA M such that $L(M) = L(N)$.

Proof Let $M = (S, s_0, \delta, \hat{F})$; we will describe it explicitly: $S = 2^Q$; $s_0 = Q_0$ [Do the types make sense?]
$\hat{F} = \{A \subseteq Q \mid A \cap F \neq \phi\}$
$\delta(A, a) = \bigcup_{q \in A} \Delta(q, a) = \Delta^*(A, a)$.
Now we must prove $L(M) = L(N)$.

Lemma $\Delta^*(A,\omega) = \delta^*(A,\omega)$ $\forall \omega \in \Sigma^*$

Proof By induction on $|\omega|$

Base $\omega = \varepsilon$. $\Delta^*(A,\varepsilon) = A = \delta^*(A,\varepsilon)$

Ind. Case Let $\omega = xa$ & assume $\forall A \subseteq Q$
$$\Delta^*(A,x) = \delta^*(A,x)$$

$$
\begin{aligned}
\delta^*(A, xa) &= \delta(\delta^*(A,x), a) && [\text{Def. of } \delta^*] \\
&= \delta(\Delta^*(A,x), a) && [\cancel{\text{Def of } \delta}\ \text{Ind. Hyp}] \\
&= \Delta^*(\Delta^*(A,x), a) && [\text{Def of } \delta] \\
&= \Delta^*(A, xa) && [\text{Fact (2)}]
\end{aligned}
$$

Lemma is proved.

Completion of the proof of the theorem:

$$
\begin{aligned}
L(N) &= \{\omega \mid \Delta^*(Q_0, \omega) \cap F \neq \phi\} \\
&= \{\omega \mid \Delta^*(Q_0, \omega) \in \hat{F}\} && [\text{Def of } \hat{F}] \\
&= \{\omega \mid \delta^*(Q_0, \omega) \in \hat{F}\} && \text{by Lemma} \\
&= \{\omega \mid \delta^*(s_0, \omega) \in \hat{F}\} && \text{by def. of } s_0 \\
&= L(M). \quad \blacksquare
\end{aligned}
$$

NFA with ε- moves

$$N = (Q, Q_0, \Delta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q, F)$$

<u>Def</u> ϵ-closure of $q \in Q \stackrel{def}{=}$

$$\{q' \mid \text{there is an } \epsilon\text{-path from } q \text{ to } q'\}.$$

We modify Δ^* to $\hat{\Delta} : 2^Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$

$$\hat{\Delta}(A, \varepsilon) = \epsilon\text{-closure}(A) = \bigcup_{q \in A} \epsilon\text{-closure}(q).$$

$$\hat{\Delta}(A, xa) = \epsilon\text{-cl}(\Delta(\hat{\Delta}(A,x), a))$$

Define $N' = (Q, Q_0, \Delta', F')$

$$\Delta'(q,a) = \hat{\Delta}(\{q\}, a)$$

$$F' = \begin{cases} F \cup \{q_0\} \text{ if } \epsilon\text{-closure}(q_0) \cap F \neq \phi \\ F \quad \text{otherwise} \end{cases}$$

Not too hard to see $L(N) = L(N')$

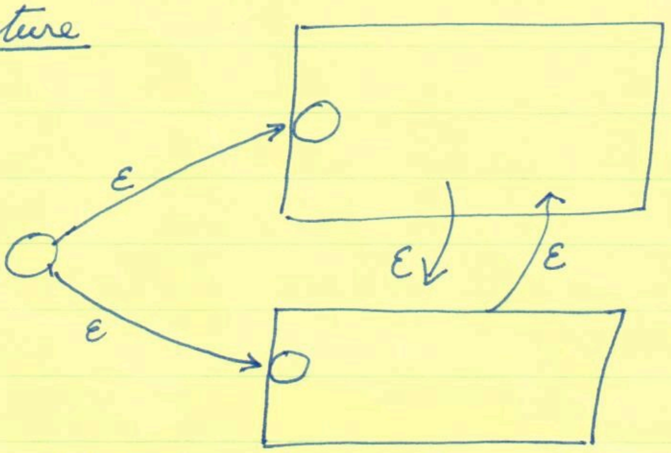
Therefore DFA, NFA & NFA with ϵ-moves all have the same power.

<u>Example</u> Suppose L_1, L_2 are regular languages

$$L_1 \| L_2 = \{ x_1 y_1 x_2 y_2 \cdots x_k y_k \mid x_1 x_2 \cdots x_k \in L_1 \ \& \ y_1 y_2 \cdots y_k \in L_2 \}$$

The shuffle of two languages is also regular
How do we prove this?

<u>Picture</u>



M_1 to recognize L_1

M_2 to recognize L_2

Use ε - transitions to go back and forth.
We need to remember where we were.

$M_1 = (S_1, s_1, \delta_1, F_1)$ $\quad M_2 = (S_2, s_2, \delta_2, F_2)$

New NFA + ε m/c (Q, Q_0, Δ, F)

$Q = (S_1 \times S_2 \times \{1\}) \cup (S_1 \times S_2 \times \{2\}) \cup \{q_0\}$

$Q_0 = \{q_0\}$

$\Delta(q_0, \varepsilon) = \{(s_1, s_2, 1), (s_1, s_2, 2)\}$

$\Delta((s, s', 1), a) = \{(\delta_1(s, a), s', 1)\}$

$\Delta((s, s', 2), a) = \{(s, \delta_2(s', a), 2)\}$

$\Delta((s, s', 1), \varepsilon) = \{(s, s', 2)\}$

$\Delta((s, s', 2), \varepsilon) = \{(s, s', 1)\}$

$F = \{(s, s', 1) \mid s \in F_1 \ \& \ s' \in F_2\} \cup \{(s, s', 2) \mid s \in F_1 \ \& \ s' \in F_2\}$