The Swapping Argument

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Suppose that we have an input space X where the elements are distributed according to D. We draw a sample of m items independently and without replacement to obtain $S_1 = \{x_1, \ldots, x_m\}$. Such samples are distributed according to D^m; we assume that drawing an item makes a negligible difference to D. We draw a second sample of size m, $S_2 = \{y_1, \ldots, y_m\}$ also distributed the same way. Now for every pair (x_i, y_i) we toss a coin $(p, 1-p)$ and if it lands "heads" we swap x_i and y_i so that x_i will be placed in S_2 and y_1 will be placed in S_1. If the coin lands tails we leave the items where they are. Thus we obtain two new samples S_1' and S_2'.

I claim that the distribution of the S_1, S_2 samples is exactly the same as the S_1', S_2' samples. The probability of drawing the samples S_1, S_2 described above is

$$\prod_{i=1}^{m} D(x_i) \times \prod_{j=1}^{m} D(y_j).$$

The probability of getting the same sample using the swapping procedure is calculated as follows. The ith item in each sample that we got before the swapping process is u_i and v_i. With probability p they get swapped so we get the same items as in the first process if $u_i = y_i$ and $v_i = x_i$. The other possibility is that with probability $(1-p)$ they did not get swapped and we have $u_i = x_i$ and $v_i = y_i$. Thus, the probability that the ith elements in the samples S_1' and S_2' are respectively x_i and y_i is

$$D(x_i)D(y_i)p + D(y_i)D(x_i)(1-p) = D(x_i)D(y_i)$$

This holds for all pairs so the probability of obtaining the same samples by the swapping procedure is the same as without the swapping procedure. Notice, there is no particular assumption about the nature of D or the fairness of the coin.