Assignment 6

COMP 599 Fall 2020 McGill University
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Due $2nd$ December 2020

Question 1[15 points] The Hilbert space $\ell_2(\mathbf{N})$ or just ℓ_2 is the set of sequences of real numbers that are square-summable:

$$\ell_2 := \{(a_0, a_1, a_2, \ldots, a_n, \ldots) | \forall i, a_i \in \mathbf{R} \text{ and } \sum_{i=0}^{\infty} a_i^2 < \infty\}.$$

The inner product on this Hilbert space is

$$\langle (a_0, a_1, \ldots), (b_0, b_1, \ldots) \rangle = \sum_{i=0}^{\infty} a_i b_i.$$

Show that this is a reproducing kernel Hilbert space.

Question 2[20 points] Suppose that H_1 and H_2 are two different reproducing kernel Hilbert spaces defined on an underlying space X. Suppose that the canonical embedding maps are

$$\Phi_i : X \longrightarrow H_i, i = 1, 2.$$

Show that if we define the embedding map

$$\Phi_1 \otimes \Phi_2 : X \times X \longrightarrow H_1 \otimes H_2 \text{ by } \Phi_1 \otimes \Phi_2(x_1, x_2)\Phi_1(x_1) \otimes \Phi_2(x_2)$$

then the associated kernel is given by $K_1(x_1, y_1)K_2(x_2, y_2)$, where K_1 and K_2 are the kernels associated with Φ_1 and Φ_2 respectively.

Question 3[25 points] Suppose that $H \subset \mathcal{F}(X)$ is an reproducing kernel Hilbert space defined on X. Let P be a probability measure on X (don't worry about X having to be a measurable space). Show how to embed P into H in such a way that if P is the Dirac distribution (point mass) at x_0 then $\langle f, \mu_P \rangle = f(x_0)$. [Hint: Use the Riesz representation theorem.] Describe μ_P explicitly when X is $\mathbf{N}$ and H is ℓ_2.

Question 4[20 points] We consider symmetric kernels over $\mathbf{R}^N$. Fix a positive integer n. Show that

$$K(\vec{x}, \vec{y}) = \sum_{i=1}^{N} \cos^n(x_i^2 - y_i^2)$$

defined for all $(\vec{x}, \vec{y}) \in \mathbf{R}^N \times \mathbf{R}^N$ is a positive semi-definite kernel.

Question 5[20 points] We consider kernels over the unit square $[0,1] \times [0,1]$. Show that $K(x, y) = \min(x, y)$ is a PSD kernel.

[Hint for 4 and 5: working directly from the definition will lead you into hopelessly messy calculations. Use properties of PSD to simplify the problem, then show that the kernel you are now dealing with arises from an embedding into a reproducing kernel Hilbert space.]