Assignment 1 Solutions

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Question 1.[30 points] Let u, v be two fixed vectors in d-dimensions. Let $r \in \mathbb{R}^d$ be sampled by picking its coordinates independently according to the standard Gaussian. The notation $\langle x,\ y\rangle$ stands for the usual inner product between vectors in $\mathbb{R}^d$.

1. What is the expected value of $\langle u,\ r\rangle$?
2. What is the expected value of $|\langle u,\ r\rangle|$?
3. What is the expected value of $\langle u,\ r\rangle \cdot \langle v,\ r\rangle$

Solutions

1. $$\mathbb{E}[\langle u,\ r\rangle] = \mathbb{E}[\textstyle\sum_i u_i r_i] = \displaystyle\sum_i \mathbb{E}[u_i r_i].$$

   Each term in this sum is zero so the sum is zero.

2. By rotational symmetry the answer cannot change if we align the axis with the vector u. So we set $u = (1, 0, 0, \ldots, 0)$. Now we have

   $$\mathbb{E}[|\langle u,\ r\rangle|] = \mathbb{E}[|r_1|].$$

   Now each component is chosen randomly according to the standard gaussian, which is $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. So we need to calculate $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{1}{2}x^2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} 2x e^{-\frac{1}{2}x^2} \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u} \mathrm{d}u = \sqrt{\frac{2}{\pi}}$.

3. Here we have

   $$\mathbb{E}[\langle u,\ r\rangle \cdot \langle v,\ r\rangle] = \mathbb{E}\left[\left(\sum_i u_i r_i\right)\left(\sum_j v_j r_j\right)\right].$$

   Now using linearity of expectations we get that this is equal to

   $$\sum_{i,j} u_i v_j \mathbb{E}[r_i r_j].$$

   However, we have $\mathbb{E}[r_i] = 0$ so all the cross terms vanish and $\mathbb{E}[r_i^2] = 1$. Thus the answer is $\langle u,\ v\rangle$.

Here are some standard tricks for evaluating integrals of the form

$$\int_{-\infty}^{+\infty} x^{2n} \exp(-\frac{1}{2}x^2)\mathrm{d}x.$$

Note that the integral $\int_{-\infty}^{+\infty} \exp(-\alpha x^2)\mathrm{d}x$ is $\sqrt{\frac{\pi}{\alpha}}$. Differentiating with respect to α we get

$$\int_{-\infty}^{+\infty} x^2 \exp(-\alpha x^2)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}.$$

Now we get the value we want by putting $\alpha = \frac{1}{2}$. To get higher n we can differentiate repeatedly with respect to α.

Question 2.[20 points] Prove that

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right).$$

Solution Using the usual Bunyakovsky-Cauchy-Schwartz inequality we get

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} (b_k c_k)^2\right).$$

Now we introduce $\hat{c_k} := c_k^2/\sum_i c_i^2$ in the sum in the second factor. This gives us

$$\left(\sum_{k=1}^{n} (b_k c_k)^2\right) = \left(\sum_{k=1}^{n} (b_k^2 \hat{c_k})\right)\left(\sum_{k=1}^{n} c_k^2\right) \leq \left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right).$$

The last inequality is because all the $\hat{c_k}$ are less than 1. This completes the proof.

Question 3.[50 points] Consider a probabilistic process that picks a real number uniformly at random from the interval $[0, 1]$. What is the expected number of times that a number is chosen before the sum of the numbers chosen is greater than or equal to 1? If you prefer to think of programs[1]

```
X: real;
I: int;
X = 0.0;
I = 0;
while ( X ≤ 1.0) do
     {X = X + choose-uniformly(0.0,1.0);
     I++;}
```

[1] This is, of course, not a realistic program. It uses an idealized notion of the real numbers which does not correspond to floating point numbers.

What is the expected value of I at termination?

Solution We will define $T(a)$ to be the expected time (number of steps) taken to exit if the value of x starts at a instead of 0. This is in fact a *conditional expectation*. We will determine this as a function of a and then easily compute $T(0)$, which is what we want.

How can we calculate $T(a)$? In order to exit in one step, we need to draw a value bigger than $1-a$ and the length of the interval $[1-a, 1]$ is clearly a. Now suppose that we draw a value b less than $1-a$ so that x is now $a+b$. In this case we need $T(a+b)$ further steps in addition to the one we have just taken. Now the probability of getting *exactly* b is of course 0 but the probability of being in a small interval of length db around b is just db so we can integrate over b. Thus adding these two cases we get

$$T(a) = 1 \cdot a + \int_0^{1-a} [1 + T(a+b)]\mathrm{d}b.$$

This is an integral equation which is not so easy to solve directly but we can differentiate with respect to a. Differentiating the integral appearing in the right hand side can be done from first principles or by Googling (it is more fun to do it from first principles) to give the ODE[2]

$$T'(a) = -T(a),$$

yes, miraculous cancellations do happen! The initial condition is $T(1) = 1$. This ode is easy to solve: $T(a) = \exp(c-a)$ for some constant c. Now using the initial condition we get $\exp(c-1) = 1$ or $c = 1$. Thus the desired solution is $T(a) = \exp(1-a)$ and hence $T(0) = e$.

[2] A poetic name for ordinary differential equation.