Assignment 1

COMP 599/MATH 597 Autumn 2020 McGill University
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Due 18^{th} September 2020 via myCourses

This assignment has 6 questions; the first three are meant to be turned in for grading. The remaining 3 are meant to be done to sharpen your skills. But please do not submit them.

Please submit answers through myCourses using the COMP 599 site. The solutions should be pdfs in LaTeXformat. This assignment is an exercise in basic probability and some concentration inequalities.

Question 1.[30 points] Let u, v be two fixed vectors in d-dimensions. Let $r \in \mathbb{R}^d$ be sampled by picking its coordinates independently according to the standard Gaussian. The notation $\langle x,\ y\rangle$ stands for the usual inner product between vectors in $\mathbb{R}^d$.

1. What is the expected value of $\langle u,\ r\rangle$?
2. What is the expected value of $|\langle u,\ r\rangle|$?
3. What is the expected value of $\langle u,\ r\rangle \cdot \langle v,\ r\rangle$

Please justify your answers.

Question 2.[20 points] Prove that

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right).$$

Question 3.[50 points] Consider a probabilistic process that picks a real number uniformly at random from the interval $[0, 1]$. What is the expected number of times that a number is chosen before the sum of the numbers chosen is greater than or equal to 1? If you prefer to think of programs[1]

```
X: real;
I: int;
X = 0.0;
```

[1]This is, of course, not a realistic program. It uses an idealized notion of the real numbers which does not correspond to floating point numbers.

```
I = 0;
while ( X ≤ 1.0) do
    {X = X + choose-uniformly(0.0,1.0);
    I++;}
```

What is the expected value of I at termination?
[Hint: the answer is e. In order to get marks you will need a *proof* of this fact. I used differential equations to solve this. Define $T(a)$ as the conditional expectation of the number of iterations needed to cross 1 given than X is initialized at $a \in [0, 1)$. Then try to set up a differential equation for T and solve it. Others I know have given combinatorial arguments but I found that the combinatorial arguments required a lot more cleverness than my solution.]

Question 4.[0 points] **Practice question, do not turn in: Chebyshev inequality** Show that Chebyshev's inequality is sharp. Use a random variable $X = (-1, 0, 1)$ with probability $(\epsilon, 1 - 2\epsilon, \epsilon)$ respectively. Find the value of ϵ that makes the inequality sharp for a given value of t.

Solution: Look up the Wikipedia page on Chebyshev's inequality under "sharpness of bounds."

Question 5.[0 points] **Practice question, do not turn in: Generating Functions**

1. Show that for a standard normal distribution, X, the moment generating function is given by $M_X(t) = e^{t^2/2}$.
2. Use the Chernoff bound to show that $P(X > a) \leq e^{-a^2/2}$.

Question 6.[0 points] **Practice question, do not turn in: Generating Functions** Read the proof of Theorem 5.5 from Calder's notes.