McGill University COMP531 Winter 2010 **Instructor: Phuong Nguyen**

Assignment 2

Due February 17 in lecture

The work you submit must be your own. You may discuss problems with each others; however, you should prepare written solutions alone. Copying assignments is a serious academic offense, and will be dealt with accordingly.

(It is helpful to give a high level description of a proof or an algorithm before giving the details.)

Question 1 (5pt) Recall from lecture that QBF-TAUT is the language of all valid quantified Boolean formula. A proof system for quantified Boolean logic (or QBF proof system for short) is a polytime onto function

$$F : \{0,1\}^* \longrightarrow \text{QBF-TAUT}$$

A string π is called an F-proof of a valid formula A if $F(\pi) = A$. F is called p-bounded if there is a polynomial p so that for all valid QBF formula A there is an F-proof of A of size at most $p(|A|)$ ($|A|$ denotes the size of A).

Show that there is a p-bounded QBF proof system if and only if $\mathbf{NP} = \mathbf{PSPACE}$.

Question 2 (5pt) Show that for any oracle A, $\mathbf{NP}^A = \mathbf{NP}$ if and only if A is in $\mathbf{NP} \cap co\text{-}\mathbf{NP}$.

Question 3 (Exercise 5.13(a) in the text) (5pt) Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be a family of subsets of a set U. The VC-dimension of $\mathcal{S}$ is defined to be the largest size of a subset $X \subseteq U$ so that every subset $X' \subseteq X$ such that for some S_i, $S_i \cap X = X'$. (We say that $\mathcal{S}$ shatters X.)

Take U to be $\{0,1\}^n$ and $m = 2^\ell$, then we say that a Boolean circuit C that takes input of the form (i, x) where i has length ℓ and x has length n represents the family $\mathcal{S} = \{S_1, S_2, \ldots, S_{2^\ell}\}$ if for each i, $1 \leq i \leq 2^\ell$,

$$S_i = \{x \ : \ C(i, x) = 1\}$$

Here we view i as a binary string of length ℓ (take the binary representation of i and pad with preceding 0's if necessary) and x is of length n.

The language VC-DIMENSION is defined as:

$$\text{VC-DIMENSION} = \{(C, k) \ : \ \text{the family represented by } C \text{ has VC-dimension } \geq k\}$$

Show that VC-DIMENSION belongs to Σ_3^p.

Question 4 (10pt) For binary strings x and y let $MULT(x, y)$ be the product, written in binary, of x and y when they are viewed as natural numbers. Show that the function $MULT$ is computable in log space. (In other words, $MULT \in \mathbf{FL}$.)

Question 5 (10pt) Recall that HornSAT is the language of all satisfiable Horn formulas. (A Horn formula is a conjunction of Horn clauses, i.e., clauses that contain at most one positive literal.) Show that HornSAT is $\mathbf{P}$-complete (with respect to logspace reduction).

Question 6 (10pt) Let UDIST be the language that consists of all tuples (G, s, t, d) such that G is an undirected graph, s, t are vertices of G, and the distance between s and t is exactly d. Show that UDIST is $\mathbf{NL}$-complete (with respect to logspace reduction).