McGill University COMP360 Winter 2011 **Instructor: Phuong Nguyen**

Assignment 2 Solution

Question 1 Consider the reduction from SAT to 3CNF-SAT. Give the 3CNF formula that results from transforming the following formula:

$$(\neg x_1 \vee x_2) \wedge \big((x_1 \wedge x_2) \vee \neg(\neg x_1 \wedge x_2)\big)$$

Clearly list the new variables and the clauses. Give a satisfying truth assignment to the resulted 3CNF formula.

Solution The new variables are

- q_B for the formula $B = \neg x_1$;
- q_C for the formula $C = B \vee x_2$;
- q_D for the formula $D = x_1 \wedge x_2$;
- q_E for the formula $E = \neg x_1$;
- q_F for the formula $F = E \wedge x_2$;
- q_G for the formula $G = \neg F$;
- q_H for the formula $H = E \vee G$;
- q_A for the original formula $A = C \wedge H$;
- u, v to make small clauses size exactly 3.

In the first stage, we get the following clauses:

- $q_B \vee x_1$, $\neg q_B \vee \neg x_1$,
- $\neg q_B \vee q_C$, $\neg x_2 \vee q_C$, $\neg q_C \vee B \vee x_2$,
- $\neg q_D \vee x_1$, $\neg q_D \vee x_2$, $\neg x_1 \vee \neg x_2 \vee q_D$,
- $q_E \vee x_1$, $\neg q_E \vee \neg x_1$,
- $\neg q_F \vee q_E$, $\neg q_F \vee x_2$, $\neg q_E \vee \neg x_2 \vee q_F$,
- $q_G \vee q_F$, $\neg q_G \vee \neg q_F$,
- $\neg q_E \vee q_H$, $\neg q_G \vee q_H$, $\neg q_H \vee q_E \vee q_G$,
- $\neg q_A \vee q_C$, $\neg q_A \vee q_H$, $\neg q_C \vee \neg q_H \vee q_A$,
- q_A

Now we use u, v to replace clauses of size 1 and 2 by conjunctions of size-3 clauses. Thus the result of the transformation is the following 3CNF formula:

$$(q_B \vee x_1 \vee u) \wedge q_B \vee x_1 \vee \neg u) \wedge (\neg q_B \vee \neg x_1 \vee u) \wedge (\neg q_B \vee \neg x_1 \vee \neg u) \wedge$$
$$(\neg q_B \vee q_C \vee u) \wedge (\neg q_B \vee q_C \vee \neg u) \wedge (\neg x_2 \vee q_C \vee u) \wedge (\neg x_2 \vee q_C \vee \neg u) \wedge (\neg q_C \vee B \vee x_2) \wedge$$
$$(\neg q_D \vee x_1 \vee u) \wedge (\neg q_D \vee x_1 \vee \neg u) \wedge (\neg q_D \vee x_2 \vee u) \wedge (\neg q_D \vee x_2 \vee \neg u) \wedge (\neg x_1 \vee \neg x_2 \vee q_D) \wedge$$
$$(q_E \vee x_1 \vee u) \wedge (q_E \vee x_1 \vee \neg u) \wedge (\neg q_E \vee \neg x_1 \vee u) \wedge (\neg q_E \vee \neg x_1 \vee \neg u) \wedge$$
$$(\neg q_F \vee q_E \vee u) \wedge (\neg q_F \vee q_E \vee \neg u) \wedge (\neg q_F \vee x_2 \vee u) \wedge (\neg q_F \vee x_2 \vee \neg u) \wedge (\neg q_E \vee \neg x_2 \vee q_F) \wedge$$
$$(q_G \vee q_F \vee u) \wedge (q_G \vee q_F \vee \neg u) \wedge (\neg q_G \vee \neg q_F \vee u) \wedge (\neg q_G \vee \neg q_F \vee \neg u) \wedge$$
$$(\neg q_E \vee q_H \vee u) \wedge (\neg q_E \vee q_H \vee \neg u) \wedge (\neg q_G \vee q_H \vee u) \wedge (\neg q_G \vee q_H \vee \neg u) \wedge (\neg q_H \vee q_E \vee q_G) \wedge$$
$$(\neg q_A \vee q_C \vee u) \wedge (\neg q_A \vee q_C \vee \neg u) \wedge (\neg q_A \vee q_H \vee u) \wedge (\neg q_A \vee q_H \vee \neg u) \wedge (\neg q_C \vee \neg q_H \vee q_A) \wedge$$
$$(q_A \vee u \vee v) \wedge (q_A \vee u \vee \neg v) \wedge (q_A \vee \neg u \vee v) \wedge (q_A \vee \neg u \vee \neg v)$$

A satisfying truth assignment for the original formula A is $x_1 = False, x_2 = False$. Extend this to a satisfying truth assignment for the above 3CNF by letting q_A, q_b, etc. have the values of the corresponding subformulas A, B, etc. Thus $q_B = True$, $q_C = True$, $q_D = False$, $q_E = True$, $q_F = False$, $q_G = True$, $q_H = True$, $q_A = True$. Also u, v take arbitrary values, e.g., $u = v = True$.

Question 2 Consider the following problem. The input consists of

- an $m \times n$ integer matrix A,

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$$

  where all A_{ij} $(1 \le i \le m, 1 \le j \le n)$ are integers, and

- a column vector $\vec{b}$ of m coordinates, $\vec{b} = (b_1, b_2, \dots, b_m)$, where all $b_1, b_2, \dots, b_m$ are integers.

The problem is to decide whether there is a column vector $\vec{x}$ of n coordinates, $\vec{x} = (x_1, x_2, \dots, x_n)$ where each x_i $(1 \le i \le n)$ can take value either 0 or 1, such that $A\vec{x} \le \vec{b}$, that is, whether there exists $\vec{x} = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$ such that

$$\begin{gathered} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n \le b_1 \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n \le b_2 \\ \dots \\ A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n \le b_m \end{gathered}$$

Show that the problem is NP-complete by giving a nondeterministic polytime algorithm for it, and show that 3CNF-SAT is polytime reducible to it.

Solution

Nondeterministic polytime algorithm: The certificate is an assignment of 0-1 value to the variables $x_1, x_2, \ldots, x_n$. The verifier works by evaluating the inequalities

$$A_{i1}x_1 + A_{i2}x_2 + \ldots A_{in}x_n \leq b_i$$

for $i = 1, 2, \ldots, m$.

For each i the above inequality can be evaluated in time $\mathcal{O}(n)$. Therefore the running time of the verifier is $\mathcal{O}(nm)$, i.e., polynomial in the size of the matrix.

Reduction from 3CNFSAT: Let φ be a 3CNF formula of the form

$$\varphi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$$

where each clause C_i contains exactly three literals. Let $v_1, v_2, \ldots, v_n$ be the variables of A. For each clause C_i we introduce an expression E_i as follows. For each literal v_j in C_i we have a term x_j, and for each literal $\neg v_j$ we have a term $(1 - x_j)$. Thus the term is 1 or 0 depending on whether the literal is True or False, and an 0-1 assignment to the variables $x_1, x_2, \ldots, x_n$ determines a truth assignment to the boolean variables $v_1, v_2, \ldots, v_n$.

Now let E_i be the sum of three terms corresponding to three literals of C_i. Since each term takes value 0 or 1, E_i is a nonnegative integer. Moreover, for any 0-1 assignment to variables x_j, $E_i > 0$ precisely when at least one term in E_i is 1, i.e., $E_i \geq 1$ if and only if the truth assignment (to v_j) determined by the 0-1 assignment to the x_j satisfies the clause C_i. To write the inequalities in the required form, i.e., LHS is less than RHS, we write

$$-E_i \leq -1$$

So our system of inequalities are

$$\begin{aligned} -E_1 &\leq -1 \\ -E_2 &\leq -1 \\ &\cdots \\ -E_m &\leq -1 \end{aligned}$$

To explicitly describe the matrix A:

$$A_{i,j} = \begin{cases} -1 & \text{if } v_j \text{ is a literal in } C_i \\ 1 & \text{if } \neg v_j \text{ is a literal in } C_i \\ 0 & \text{if neither } v_j \text{ nor } \neg v_j \text{ appears in } C_i \end{cases}$$

The b_i are:

$$b_i = -1 + \text{number of negative literals in } C_i$$

The coefficients of the i-th row of A and the value of b_i can be computed by a linear pass through the i-th clause of the given formula φ, so the matrix A and vector b can be computed in polynomial time.

Proof of Correctness: We prove two directions.

First, suppose that the given formula φ is satisfiable. Let τ be a satisfying truth assignment to φ. We define a 0-1 assignment to the variables x_j that satisfies the inequalities as follows. Let $x_j = 1$ if and only if $\tau(v_j)$ is TRUE (otherwise $x_j = 0$). The i-th inequality is equivalent to $E_i \geq 1$, and this is satisfied by the 0-1 assignment because τ makes at least one literal in C_i true, i.e., at least one term in E_i is 1.

Second, suppose that the system of inequalities is satisfied by a 0-1 assignment to the variables x_j. Define a truth assignment τ to the boolean variables v_j as follows:

$$\tau(v_j) = TRUE \quad \text{iff} \quad x_j = 1$$

Consider a clause C_i. Since the corresponding expression E_i has value ≥ 1, at least one term in E_i is 1, so at least one literal in C_i is TRUE, hence C_i is true under τ. Thus τ satisfies all clauses in φ, hence τ satisfies φ.

A shorter proof For each 0-1 assignment to the variable x_j associate a truth a truth assignment to the boolean variables v_j by letting v_j be TRUE if and only if $x_j = 1$. Then the i-th inequality is satisfied by an 0-1 assignment if and only if the associated truth assignment satisfies the i-th clause of φ. Therefore there exists a 0-1 assignment to x_j that satisfies all inequalities if and only if there exists a satisfying truth assignment to the variables in the formula φ.