Some properties of DFS

After initialization, each vertex v is colored exactly twice (Gray, at time $s[v]$; then Black, at time $f[v]$). So $s[v]$ and $f[v]$ define a time interval $[s[v], f[v]]$ associated with v. This is precisely the period during which v is Gray, and is on the stack (v may be present on the stack before $s[v]$ and after $f[v]$).

Parenthesis Theorem For every two vertices u and v, exactly one of the following conditions holds:

- the intervals $[s[u], f[u]]$ and $[s[v], f[v]]$ are disjoint;
- one interval contains the other:
  - either $s[u] < s[v] < f[v] < f[u]$
  - or $s[v] < s[u] < f[u] < f[v]$

Proof To prove the theorem it suffices to prove that if $s[u] < s[v] < f[u]$ then $s[u] < s[v] < f[v] < f[u]$ (and similarly if $s[v] < s[u] < f[v]$ then $s[v] < s[u] < f[u] < f[v]$).

So suppose that $s[u] < s[v] < f[u]$. In this case, at time $s[v]$ when v is colored Gray (and pushed back to the stack) u is on the stack and has color Gray. Thus u is never added again to the stack, and therefore it can only become Black after this occurrence of v is taken out and v is colored Black. This means $f[v] < f[u]$. QED.

The DFS algorithm defines a forest (i.e., a collection of tree). Here each tree is created by a call of DFS-Visit. The DFS-Visit(G, u) define a tree rooted at u, so that (v, w) is an edge in the tree if and only if $p[w] = v$.

Note that for directed graph G, even if w is reachable from v, it may happen that w is not in the same tree as v. For example, this happens if DFS-Visit(G, w) is called before DFS-Visit(G, v).

The next corollary provides a useful test for checking whether v is a descendant of u in a DFS tree.

Corollary For $v \neq u$, v is a descendant of u in a DFS tree if and only if $s[u] < s[v] < f[v] < f[u]$.

Proof This is an "if and only if" statement, so we must prove two directions.

($\Longleftarrow$): The condition $s[u] < s[v] < f[v] < f[u]$ implies that v is added to the stack during the time u is Gray. When u is Gray, only descendants of u are added to the stack. Therefore v is a descendant of u.

($\Longrightarrow$): For this direction, it suffices to show that if u is the parent of v in the DFS tree, then $s[u] < s[v] < f[v] < f[u]$. Suppose that u is the parent if v, then u is the last vertex that causes v to be added on the stack. So at time $s[u]$ v is White, i.e., $s[u] < s[v]$. Furthermore, u is not added to the stack after $s[u]$. So when v is colored Black, u is still on stack and has not yet colored Black. Hence $f[v] < f[u]$. QED.

White-path Theorem In a DFS forest of a (directed or undirected) graph G, vertex v is a descendant of vertex u if and only if at time $s[u]$ (just before u is colored Gray), there is a path from u to v that consists of only White vertices.

Proof There are two directions to prove.

($\Longrightarrow$) Suppose that v is a descendant of u. So there is a path in the tree from u to v. (Of course this is also a path in G.) All vertices w on this path are also descendants of u. So by the corollary above, they are colored Gray during the interval $[s[u], f[u]]$. In other words, at time $s[u]$ they are all White.

($\Longleftarrow$) Suppose that there is a White path from u to v at time $s[u]$. Let this path be

$$v_0 = u, v_1, v_2, \ldots, v_{k-1}, v_k = v$$

To show that v is a descendant of u, we will indeed show that all v_i (for $0 \leq i \leq k$) are descendants of u. (Note that this path may not be in the DFS tree.) We prove this claim by induction on i.

Base case: $i = 0$, $v_i = u$, so the claim is obviously true.

Induction step: Suppose that v_i is a descendant of u. We show that v_{i+1} is also a descendant of u. By the corollary above, this is equivalent to showing that

$$s[u] < s[v_{i+1}] < f[v_{i+1}] < f[u]$$

i.e., v_{i+1} is colored Gray during the interval $[s[u], f[u]]$.

Since v_{i+1} is White at time $s[u]$, we have $s[u] < s[v_{i+1}]$. Now, since v_{i+1} is a neighbor of v_i, v_{i+1} cannot stay White after v_i is colored Black. In other words, $s[v_{i+1}] < f[v_i]$. Apply the induction hypothesis: v_i is a descendant of u so $s[u] \leq s[v_i] < f[v_i] \leq f[u]$, we obtain $s[v_{i+1}] < f[u]$. Thus $s[u] < s[v_{i+1}] < f[v_{i+1}] < f[u]$ by the Parenthesis Theorem. QED.