# Correctness of BFS

**Lemma 1** If $(u, v)$ is an edge, then

$$dist(s, v) \leq dist(s, u) + 1$$

**Lemma 2** Upon termination of the BFS,

$$dist(s, u) \leq d[u]$$

**Lemma 3** Suppose that during the execution of BFS, at some point we have

$$Q = \langle v_1, v_2, \ldots, v_r \rangle$$

(where $v_1$ is the head, $v_r$ is the tail of the queue). Then

$$d[v_1] \leq d[v_2] \leq \ldots \leq d[v_r] \leq d[v_1] + 1$$

**Corollary 1** If $v$ is enqueued before $u$, then $d[v] \leq d[u]$.
**Proof** Consider the sequence of vertices that are enqueued in between $v$ and $u$:

$$v_1 = v, v_2, v_3, \ldots, v_{k-1}, v_k = u$$

Then $d[v_1] \leq d[v_2] \leq d[v_3] \leq \ldots \leq d[v_{k-1}] \leq d[v_k]$

**Corollary 2** If $v$ is a neighbor of $u$, then $d[v] \leq d[u] + 1$
**Proof** When $u$ is dequeued, either

- $v$ is not colored, so $v$ will be colored in the for-loop, and therefore $d[v] = d[u] + 1$;
- or $v$ is colored, then either
  - $v$ is black: then $v$ is enqueued before $u$, so $d[v] \leq d[u]$ by Corollary 1, or
  - $v$ is gray: $v$ is in the queue, so by Lemma 3 we have $d[v] \leq d[u] + 1$.

**Theorem** Upon termination of BFS, $d[v] = dist(s, v)$ for all vertices $v$.
**Proof** By contradiction. Suppose for a contradiction that there are vertices $v$ such that $dist(s, v) \neq d[v]$. Let $v$ be such a vertex with smallest $dist(s, v)$. First of all, $v \neq s$ (because $d[s] = dist(s, s) = 0$). So $dist(s, v) \geq 1$.

By Lemma 2, $d[v] \geq dist(s, v)$, so $d[v] \geq dist(s, v) + 1$.

There is a neighbor $u$ of $v$ so that $dist(s, u) + 1 = dist(s, v)$. So $dist(s, u) < dist(s, v)$, and so by the choice of $v$ we have $dist(s, u) = d[u]$. Now by Corollary 2: $d[v] \leq d[u]+1$. So $d[v] \leq dist(s, u)+1$, so $d[v] \leq dist(s, v)$. So $dist(s, v) + 1 \leq dist(s, v)$: contradiction.