COMP 330 - Fall 2010 - Assignment 4

Due 8:00 pm Nov 3, 2010

General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 3rd floor left of the elevators.

1. (25 Points) Let M_1 and M_2 be two Turing machines. Consider the following Turing machine:

   On input w:

   - Step 1: Run M_1 on w. If M_1 accepts w, then accept.
   - Step 2: Run M_2 on w. If M_2 accepts w, then accept.

   What is the language of this Turing Machine? Explain.

   Solution:
   $$\{w : (M_1 \text{ accepts } w) \text{ or } (M_1 \text{ rejects } w \text{ and } M_2 \text{ accepts } w)\}.$$

   If M_1 accepts w then w will be accepts in Step 1. If M_1 loops on w, then the TM will loop on w, and never will reach Step 2. However if M_1 rejects w, then in Step 2, if M_2 accepts w, then the TM will accept w.

2. (25 Points) Is the following language decidable[1]?

   $$L = \{\langle M \rangle \mid M = (\{1, 2, \dots, 100\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, 1, 2, 3) \text{ is a decider}\}.$$

   Solution: This is a finite set (the different possibilities for δ is finite), so it is decidable.

3. A polynomial is called *non-negative*, if it is non-negative for every assignment of real numbers to its variables. For example $x_1^2 + x_2^2 - 2x_1x_2$ is non-negative because we can express it as $(x_1 - x_2)^2$.

   (a) (15 Points) Show that if a multi-variate polynomial p is <u>not</u> *non-negative*, then there is an assignment of *rational numbers* to its variables that makes it negative.

   Solution: Suppose that p is negative for assigning the real values $a_1, ..., a_k$ to its variables. We pick a sequence of rational numbers $r_1(n), .., r_k(n)$ such that $\lim_{n\to\infty} r_i(n) = a_i$. Then $\lim_{n\to\infty} p(r_1(n), .., r_k(n)) = p(a_1, ..., a_k)$ and the statement follows.

   (b) (15 Points) Use the previous part to show that the following language is recursively enumerable (Turing recognizable):

   $L = \{\langle p \rangle \mid p$ is a multi-variate polynomial with integer coefficients, and p is <u>not</u> non-negative.$\}$

[1] Recall that a TM is formally defined as $(Q, \Sigma, \Gamma, \delta, q_{\text{start}}, q_{\text{accept}}, q_{\text{reject}})$. A Turing Machine is called a decider if it halts on every input.

Solution: Let $\mathbb{Q}$ denote the set of rational numbers. First note, for every positive integer k, one can enumerate $\mathbb{Q}^k$: Indeed there is an algorithm that for $m = 1, 2, 3, \ldots$, lists the *finite* set of elements $\left(\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}\right) \in \mathbb{Q}^k$, with $|a_i|, |b_i| \leq m$ for every $1 \leq i \leq k$. Let $s_1, s_2, \ldots$ be this enumeration of the elements of $\mathbb{Q}^k$. Now it follows from Part (a) that the following TM accepts L.

- $N =$ "on input p which is a polynomial on k variables:
- For $i = 1, 2, \ldots$
- Compute the value of p on $s_i = (r_1, \ldots, r_k)$. If it is negative accept."

(c) (30) Consider a multi-variate polynomial p with integer coefficients (e.g. $x_1^5 x_2 + x_3 x_4 - x_1^2$). It is known that if p is *non-negative*, then there exists an integer k and polynomials $q_1, \ldots, q_k$ and r with integer coefficients such that $p = \left(\frac{q_1}{r}\right)^2 + \ldots + \left(\frac{q_k}{r}\right)^2$. Use this fact (and the previous part) to show that the following language is decidable:

$$\{\langle p \rangle \mid p \text{ is a } \underline{\text{non-negative}} \text{ multi-variate polynomial with integer coefficients}\}.$$

Solution: We show that it is possible to enumerate the elements of the following set

$$A = \{(q_1, \ldots, q_k, r) \mid q_1, \ldots, q_k \text{ and } r \text{ are polynomials with integer coefficients}\}$$

Indeed there is an algorithm that for $m = 1, 2, 3, \ldots$, lists the *finite* set of elements $(q_1, \ldots, q_k, r) \in A$, with $k \leq m$, $\deg(q_i) \leq m$ for every $1 \leq i \leq k$, and all the coefficients of $q_1, \ldots, q_k$ and r are bounded in magnitude by m. The we can use this algorithm to enumerate the elements of A as $s_1, s_2, \ldots$. Now the following TM, accepts the complement of L:

- $M =$ "on input p which is a polynomial on k variables:
- For $i = 1, 2, \ldots$
- Check whether $p = \left(\frac{q_1}{r}\right)^2 + \ldots + \left(\frac{q_k}{r}\right)^2$, where $s_i = (q_1, \ldots, q_k, r)$. If it is, accept."

This and Part (b) show that both L and its complement are recursively enumerable. It follows that L and L^c are decidable.