Improving Integrality Gaps via Chvátal-Gomory Rounding

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Abstract

In this work, we study the strength of the Chvátal-Gomory cut generating procedure for several hard optimization problems. For hypergraph matching on k-uniform hypergraphs, we show that using Chvátal-Gomory cuts of low rank can reduce the integrality gap significantly even though Sherali-Adams relaxation has a large gap even after linear number of rounds. On the other hand, we show that for other problems such as k-CSP, unique label cover, maximum cut, and vertex cover, the integrality gap remains large even after adding all Chvátal-Gomory cuts of large rank.

1 Introduction

Linear Programming is an enormously useful tool in the study of combinatorial optimization problems, giving exact algorithms for several problems in P, and approximation algorithms for several NP-hard problems. Typically, one writes an integer linear program for the problem at hand, and solves its linear programming relaxation. For a large number of problems of interest, such a relaxation has an optimum value that is within a small multiplicative factor of the optimal. A more powerful tool that sometimes gives better polynomial time approximations is semidefinite programming. In both cases, the approximation factor one gets depends on the integer linear program (or the vector program) that one starts with. For many problems, a natural linear program suggests itself and can be shown to have the best possible gap (e.g. bipartite matching, set cover). In many other cases (e.g. graph matching, sparsest cut), the "natural" linear program for the problem does not suffice and one needs to add carefully designed constraints that force the linear program to reveal information about optimal solutions.

Cut generating procedures are algorithms for adding constraints to the linear relaxation with the property that every integer solution in the polytope satisfies the new constraints. Starting with a polytope P, such a procedure gives a new polytope that is closer to P_I, the convex hull of integer points in P. Thus they provide a generic way to strengthen the linear relaxation of the integer program, without changing the set of integer feasible solutions. They can thus be thought of as an alternative to the addition of the carefully designed constraints that have been used. Indeed for several problems, the ingeniously added constraints can in hindsight be shown to be also generated by these cut generating procedures. A number of such procedures have been proposed including Chvátal-Gomory (CG) [11, 22, 23], Lovász-Schrijver (LS, LS+) [31], Sherali-Adams (SA) [37] and Lassere [29].

For a large class of combinatorial optimization problems, the best known approximation algorithms are matched by hardness of approximation results, ruling out the possibility of better

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approximations based on smarter LP relaxations (or on other techniques) unless P=NP. Certain interesting problems such as Vertex Cover, Max Cut, Sparsest Cut and Unique Label cover have so far resisted attempts to prove matching upper and lower bounds. For these problems, it is therefore natural and interesting to ask if one can design stronger LP (or SDP) relaxations. A negative answer would rule out a large class of algorithmic approaches, suggesting that computing better approximations may in fact be NP-hard. Arora, Lovász and Bollobas [2] initiated this direction of research, and showed that starting with a natural linear program for vertex cover, and iteratively applying the LS cut generating procedure does not reduce the integrality gap below $(2-\epsilon)$, even after a linear number of rounds. Similar results have been shown for other problems, and for LS+, SA and Lassere, which strengthen LS.

Somewhat disconcertingly however, such gap results can also be shown for some polynomial-time solvable problems. This is not surprising since despite its generality, linear programming does not capture all algorithmic tools at our disposal, and other tools such as dynamic programming and local search are often useful in cases where natural convex relaxations fail. However, such gap results exist even for problems where good LP relaxations exist. Indeed if one starts with the natural LP for maximum matching, it can be shown that the gap is at least $(1+\epsilon)$ even after $\frac{1}{\epsilon}$ rounds of SA [32], even though the problem is polynomial time solvable using an (exponentially sized) LP relaxation. Starker gaps exist for hypergraph matching on k-uniform hypergraphs, where the gap stays above $(k-2)$ even after a linear number of rounds of SA starting with the natural LP. On the other hand, when k is a constant, there is a polynomial sized linear program that has gap at most $\frac{k+1}{2}$ [9]. Thus even for simple combinatorial problems, SA can fail to capture the power of LP based algorithms.

The gap results from these lift-and-project schemes can be interpreted in several different ways. The *guide-the-algorithmicist* viewpoint looks at such result as a strong integrality gap for a family of linear programs. Thus an algorithm designer considering a new strengthened linear program could check whether or not the constraints in her LP are quickly generated by this procedure, and if so, she would conclude that the new LP will not help in the worst case, and thus may be guided towards other constraints to add. With this viewpoint, it is interesting to try to strengthen the integrality gaps to other cut generating procedures that may capture large families of efficient linear programs (even though the cut generating procedure considered in its full generality may not be efficient). A somewhat more controversial viewpoint is the *limits-of-techniques* viewpoint, where one interprets a gap result as suggesting that "LP based approaches" will not be able to give good approximation algorithms. However the above examples of matching and hypergraph matching make such a viewpoint less appealing. Finally, one can view these results as *structural results* which prove the limits of a certain proof system (e.g. SA).

In this work we study Chvátal-Gomory rounding, a popular cut generating procedure that is often used in practice. Buresh-Oppenheim *et al.* [7] previously showed that optimal integrality gaps survive a linear number of rounds of CG for MAX kSAT and MAX kXORSAT, for $k \geq 5$ (see also [3]). For problems such as unique label cover, where known hardness results do not match the best known upper bounds, most of the attention has been diverted to LS and other procedures, and little is known about CG cuts. It is particularly interesting to look at this procedure since it does in fact handle the (graph) matching example above: one round of Chvátal-Gomory suffices to make the matching polytope integral! Further as we show, the polynomial-sized linear program for hypergraph matching from [9] is also captured by a few rounds of C-G. Thus C-G does in fact capture useful and efficient linear programs that SA fails to capture, making it interesting to study C-G gaps from the guide-the-algorithmicist viewpoint. Moreover, C-G is an interesting proof system in its own right.

Chvátal-Gomory rounding is defined as follows. Let P be a polyhedron in $\mathcal{R}^n$, define

$$P' = \{x \in P : a^T x \geq b \text{ whenever } a \in Z^n, b \in Z, \text{ and } \min\{a^T x : x \in P\} > b - 1\}$$

to be the polyhedron obtained after doing a single round of Chvátal-Gomory rounding. Trivially $P \cap Z^n \subseteq P'$, define $P^{(0)} = P$ and recursively,

$$P^{(j)} = (P^{(j-1)})'$$

for all positive integers j. Also let P_I denote the convex hull of $P \cap Z^n$. We clearly have $P_I \subseteq P^{(j)} \subseteq P^{(j-1)}$ for each $j \geq 1$. We call $P^{(j)}$ to be the polyhedron obtained after j rounds of CG rounding.

1.1 Our Contributions and Results

In this work, we study the power of Chvátal-Gomory rounding to reduce integrality gaps for various combinatorial optimization problems as compared to lift and project procedures like Sherali-Adams.

Our first result shows an integrality gap separation between C-G and SA which show that C-G cuts can be much stronger than SA hierarchy.

Theorem 1.1 *For the maximum matching problem in k-uniform hypergraphs, $O(k^2)$ rounds of CG suffice to reduce the integrality gap to $\frac{k+1}{2}$.*

We contrast the above theorem with result from Chan and Lau [9] that the integrality gap remains at least $k-2$ after $\Omega(n)$ rounds of the SA hierarchy. Thus C-G can generate significantly stronger linear programs than SA can.

Can C-G rounding then lead to better LP relaxations for other problems? Our next set of results show that CG rounding performs as poorly as the Sherali-Adams hierarchy on a number of problems. We show integrality gaps for the max-cut problem, Unique Label Cover problem, k-CSP$_q$ and the vertex cover problem. We prove the following theorems.

Theorem 1.2 *For any $\epsilon > 0$, there exists a $\gamma > 0$ such that integrality gap of linear programming relaxation for the max-cut problem obtained using all cuts of CG rank at most r is at least $2 - \epsilon$ where $r = n^\gamma$.*

Theorem 1.3 *For any $\epsilon > 0$, and integer q, there exists a $\gamma > 0$ such that integrality gap of linear programming relaxation for the unique label cover problem on q labels, using all cuts of CG rank at most r is at least $q - \epsilon$ where $r = n^\gamma$.*

Theorem 1.4 *For any $\epsilon > 0$, integer k and prime q, there exists a $\gamma > 0$ such that integrality gap of linear programming relaxation for the k-CSP$_q$ problem using all cuts of CG rank at most r is at least $\frac{q^k}{kq(q-1)} - \epsilon$ where $r = \gamma n$.*

We note that the integrality gaps above resemble closely the bounds obtained for the Sherali-Adams hierarchy for the corresponding problems [10, 38]. Interestingly, the proofs of all the above results follow a similar outline and use the integrality gap instances for the Sherali-Adams hierarchy as a starting point. Using our general technique we also show the following integrality gap for the vertex cover problem.

Theorem 1.5 *For any $\epsilon > 0$, there exists a $\gamma > 0$ such that integrality gap of relaxation for the vertex cover problem obtained after r rounds of C-G rounding is at least $2 - \epsilon$ where $r = n^\gamma$.*

We believe that our positive result gives strong motivation for studying C-G cuts as an algorithmic technique[1]. The resulting hopes are somewhat dashed by our negative results. In the process we enlarge the class of linear programs that are provably ineffectual for the problems studied. Moreover, our results enhance our understanding of C-G as a proof system.

1.2 Related Work

Gomory [22, 23] introduced the Chvátal-Gomory rounding and proved that for every bounded polyhedron P, there exists a non-negative integer j such that $P^{(j)} = P_I$. Chvátal [11] gave an alternate proof of the result. The smallest such integer j is called the *Chvátal* rank of P. There has been a significant work on both lower and upper bounding Chvátal rank of a polyhedron. Although the Chvátal rank can, in general, be very large, Bockmayr *et al.* [6] proved that it is bounded by $O(n^3 \log n)$ when the polytope is contained in the hypercube $[0, 1]^n$. This bound was improved to $O(n^2 \log n)$ by Eisenbrand and Schulz [18]. Chvátal, Cook and Hartmann [12] proved lower bounds on the Chvátal rank of many combinatorial optimization problems including maximum cut problem, stable set problem and traveling salesman problem. We also note that their results can also be used to show $(1 + \epsilon)$ integrality gaps after $\Omega(\frac{1}{\epsilon})$ rounds for the vertex cover problem and maximum cut problem while our results show much stronger integrality gaps. However, the Chvátal-Gomory closure from a theoretical point of view does not behave very well algorithmically; Eisenbrand [17] proved that optimizing over the polytope resulting from one round of C-G cuts is a NP-hard problem in general. Nevertheless, Bienstock and Zuckerberg [5] show that for a large class of polytopes (e.g. covering problems), one can optimize over (a subset of) the rth iterate of the polytope, up to an arbitrarily small error, for any constant r in polynomial time.

Arora, Lovász and Bollobas [2] initiated the study of integrality gaps of linear programming relaxations obtained via lift and project hierarchies. Since then there has been a series of works [1, 24, 10, 16, 36, 38, 35] showing integrality gaps for linear and semi-definite relaxations for various combinatorial optimization problems. Closely related to our work is the work of Charikar, Makarychev and Makarychev [10] who show integrality gaps for linear programming relaxations obtained via Sherali-Adams hierarchy for the maximum cut, vertex cover and the unique games problem. We also note that the integrality gap for the vertex cover problem obtained in Theorem 1.5 can also be obtained using the results of Arora et al [2]. Lift and project hierarchies and CG rounding can also be used as proof systems for satisfiability and other problems. There has been a series of works [7, 34, 33, 14, 15] which lower bound the size or depth of the proofs obtained using these hierarchies. Buresh-Oppenheim *et al.* [7] show that for MAX k-SAT, and MAX k-XOR SAT, a linear number of rounds of CG are needed to reduce the integrality gap.

2 Maximum matching in k-uniform hypergraphs

The maximum matching problem on a hypergraph $G = (V, E)$ is to find the maximum cardinality subset $F \subseteq E$ of hyperedges such that for any vertex $v \in V$, there is at most one hyperedge in F incident on v. A hypergraph $G = (V, E)$ is said to be k-uniform if $|e| = k$ for every $e \in E$. We study the (unweighted) maximum matching problem in k-uniform hypergraphs. We note that the problem is NP-hard and APX-hard even for $k = 3$ [4]. Hazan, Safra and Schwartz [26] show an $\Omega(k/\log k)$-inapproximability result, while Hurkens and Schrijver [27] give a $(\frac{k}{2} + \epsilon)$-approximation algorithm.

[1] One important difference between Chvátal-Gomory rounding and other hierarchies such as SA, is that unlike the latter, C-G does not come with a general efficient algorithmic procedure. Indeed optimizing over the Chvátal-Gomory closure is actually NP-hard in general [17]. Nevertheless, these cuts are commonly used by practitioners [13].

Figure 1 gives the natural linear programming relaxation for the hypergraph matching problem. Here $\delta(v)$ denotes the set of edges incident at vertex $v \in V$. Let P denote the polytope defined by feasible solutions to this linear program. Chan and Lau [9] show that the integrality gap of this

$$\begin{array}{llll} & & \max\sum_{e\in E} & x_e \\ \text{s.t.} & & & \\ \sum_{e\in\delta(v)} x_e & \leq & 1 & \forall v \in V \\ x_e & \geq & 0 & \forall\ e \in E \end{array}$$

Figure 1: Linear program for the Hypergraph Matching Problem

linear program remains at least $k-2$ even after $O(n/k^3)$ rounds of the Sherali-Adams hierarchy. On the other hand, they show a polynomial sized linear program with integrality gap at most $\frac{k+1}{2}$, for any constant k.

This latter result is derived in two steps. First, Chan and Lau [9] define a rather large linear program whose gap is shown to be bounded by $\frac{k+1}{2}$. Next they use a result in extremal combinatorics to construct an equivalent linear program with a polynomial number of constraints. We use similar techniques to show that the polytope $P^{(2k^2)}$ satisfies all the constraints defining the polytope considered by Chan and Lau [9].

A set of hyperedges K is said to be an *intersecting family* if every pair of hyperedges in K has a non-empty intersection. Clearly, for any intersecting family in E, a matching can contain at most one hyperedge. Thus one can add to the linear program the constraint $\sum_{e\in K} x_e \leq 1$ for any intersecting family K. Chan and Lau [9] show that

Theorem 2.1 ([9]) *Consider the linear program in Figure 1 above, augmented with the constraints $\sum_{e\in K} x_e \leq 1$ for all intersecting families $K \subseteq E$. For a k-uniform hypergraph, the integrality gap of this program is bounded by $\frac{k+1}{2}$.*

Next we define a *Kernel*. Given a subset $S \subseteq V$ and a hyperedge e, we let e_S denote $e \cap S$. For a subset K of hyperedges, we can then define $K_S = \{e_S : e \in K\}$. A subset $S \subseteq V$ is a kernel for an intersecting family K, if the family K_S is intersecting. In other words, S is a kernel for K if every pair of hyperedges in K has an non-empty intersection in S. It can be shown [8] that every intersecting family has a Kernel of size $s(k)$ for some function $s(k)$ independent of $|V|$.

Theorem 2.2 ([8]) *There exists a function $s(k)$ such that for any k-uniform hypergraph H, and any intersecting family K of hyperedges in H, there is a kernel S containing at most $s(k)$ vertices.*

The best bounds on $s(k)$ are $\Theta(\binom{2k}{k})$ [19, 39, 40].

Note that if S is a kernel of K, then the constraint $\sum_{e\in K} x_e \leq 1$ is equivalent to the constraint $\sum_{f\in K_S}\sum_{e\in K: e_S=f} x_e \leq 1$. We next argue that all constraints of the latter form are derived in a small number of rounds of C-G. In the lemma below,

Lemma 2.3 *Let $P^{(0)} = P$ be the polytope in figure 1, let $P^{(j)} = (P^{(j-1)})'$ and let $l_0 = 2$ and $l_{t+1} = 2l_t - 1$. Then for any S and any intersecting family K_S on S, $P^{(j)}$ satisfies all constraints of the form*

$$\sum_{f\in L}\sum_{e: e_S=f} x_e \leq 1,$$

where $L \subseteq K_S$ is arbitrary with $|L| = l_j$.

$$\begin{array}{llll}
 & & \max\sum_{\{u,v\}\in E} & w_{uv}x_{uv} \\
\text{s.t.} & & & \\
x_{uv} & \leq & y_u + y_v & \forall \{u,v\} \in E \\
x_{uv} & \leq & 2-(y_u+y_v) & \forall\ \{u,v\} \in E \\
x_{uv} & \geq & 0 & \forall\ \{u,v\} \in E \\
0 \leq y_u & \leq & 1 & \forall\ u \in V
\end{array}$$

Figure 2: Linear program for the Max-Cut Problem

Proof: The proof is by induction on j. For $j = 0$, the claim follows from the definition of an intersecting family. Indeed, in this case, L contains two hyperedges which intersect in a vertex, and the relevant inequality is implied by the packing constraint for that vertex. Now suppose that the claim holds for $j \leq t$. We prove the claim for $j = t+1$. Let $L \subseteq K_S$ be arbitrary with $|L| = l_{t+1}$. By the induction hypothesis, the constraint is satisfied for each of the $\binom{l_{t+1}}{l_t}$ subsets of L of size l_t. Adding up these constraints and dividing by $\binom{l_{t+1}-1}{l_t-1}$, we conclude that $P^{(t)}$ satisfies the constraint

$$\sum_{f\in L}\sum_{e:e_S=f} x_e \leq \frac{\binom{l_{t+1}}{l_t}}{\binom{l_{t+1}-1}{l_t-1}} = \frac{l_{t+1}}{l_t}.$$

Thus $P^{(t+1)}$ satisfies the above constraints with the right hand side replaced by its floor. Since the ratio on the right hand side is strictly smaller than two, this completes the induction. □

It is easy to see that $l_{2t} \geq 2^t$. Moreover, for $|S| < s(k)$, any intersecting family K_S is of size at most $s(k)^k$. It follows that

Theorem 2.4 *Let $P^{(0)} = P$ be the polytope in figure 1, and let $P^{(j)} = (P^{(j-1)})'$. Then the integrality gap of $P^{(2k\log s(k))}$ is bounded by $\frac{k+1}{2}$.*

Using the bound of $s(k)$ above, we conclude that $O(k^2)$ rounds of C-G suffice to bring down the integrality gap to $\frac{k+1}{2}$.

3 Integrality Gaps for Max-Cut

Let P denote the linear programming relaxation for the max-cut problem given in Figure 2. The variables x_{uv} for an edge $\{u,v\} \in E$ denote whether the edge is in the cut. The variable y_u for each vertex $u \in V$ denotes whether the vertex is on the *left* side of the cut.

The following lemma characterizes the constraints for $P^{(k)}$ and is crucial in showing integrality gaps.

Lemma 3.1 *Let $\mathbf{a}^T\mathbf{x} \leq b + \mathbf{c}^T\mathbf{y}$ be a non-trivial facet of $P^{(k)}$ for any k. We can assume without loss of generality that $\mathbf{a}, b$ and $\mathbf{c}$ are integral, $\mathbf{a} \geq 0$.*

Proof: The integrality follows simply from the fact that P is a rational polyhedron and hence $P^{(k)}$ is rational for each integer k. The non-negativity of $\boldsymbol{a}$ follows since using the constraint $x_{uv} \geq 0$, one can obtain a stronger constraint. □

Proof of Theorem 1.2: The proof uses the integrality gap example for the Sherali-Adams hierarchy to argue nearly the same integrality gap. We show that the fractional solution which survives the Sherali-Adams hierarchy, with a small scaling, also survives the Chvátal-Gomory hierarchy. Let the norm of a constraint $\boldsymbol{a}^T\boldsymbol{x} \le b + \boldsymbol{c}^T\boldsymbol{y}$ be defined as the size of the support of $\boldsymbol{a}$. To show that the fractional solution satisfies all the constraints generated by the Chvátal-Gomory rounding, we argue separately for the constraints which have small norm and large norm. Using the properties of the Sherali-Adams hierarchy, one can show that the constraints with small norm are implied by the Sherali-Adams hierarchy and thus the fractional solution to the integrality gap example satisfies these constraints. For the constraints with large norm, we show that in each round of C-G rounding, the constraint is strengthened by at most 1 in the constant term. Since the constraint had large norm, this implies that slight degradation of the original fractional solution satisfies the new tighter constraint. We now expand on the above outline.

We use the following theorem which follows from the integrality gap example given by Charikar, Makarychev and Makarychev [10] for the Sherali-Adams Hierarchy.

Theorem 3.2 ([10]) *For any $\epsilon > 0$, there exists a $\gamma > 0$ and a graph $G = (V, E)$ such that any integral cut has at most $(\frac{1}{2} + \frac{\epsilon}{8})$ fraction of the edges but the fractional solution $x^0_{uv} = 1 - \frac{\epsilon}{16}$ for each $\{u, v\} \in E$ and $y^0_u = \frac{1}{2}$ for each $u \in V$ is in P^t_{SA} for $t = \frac{32n^\gamma}{\epsilon}$. Therefore, for every subset $S \subset V$ of size at most t, there exists a distribution $\mathcal{D}$ of solutions such that (i) expected value of the solutions equals $(\mathbf{x}^0, \mathbf{y}^0)$ and (ii) each of the solution with non-zero probability in $\mathcal{D}$ is integral over S.*

Let G be the graph given by Theorem 3.2. We prove the following lemma.

Lemma 3.3 *Let $x^k_{uv} = (1 - \frac{\epsilon}{16} - \frac{2k}{t})$ for each $(u, v) \in E$ and $y^k_v = \frac{1}{2}$ for each $v \in V$ for any nonnegative integer k. Then the fractional solution $(\mathbf{x}^k, \mathbf{y}^k) \in P^{(k)}$ for each $0 \le k \le n^\gamma$.*

Before we prove Lemma 3.3, we complete the proof of Theorem 1.2. Consider $k = n^\gamma$. Lemma 3.3 implies that

$$x^k_{uv} = (1 - \frac{\epsilon}{16} - \frac{2n^\gamma}{t}) = 1 - \frac{\epsilon}{8}$$

for each $(u, v) \in E$. Consider the weight vector which is uniformly 1. Then

$$\max\{\boldsymbol{w}^T\boldsymbol{x} : (\boldsymbol{x}, \boldsymbol{y}) \in P_I\} \le (\frac{1}{2} + \frac{\epsilon}{8})|E|$$

but

$$\max\{\boldsymbol{w}^T\boldsymbol{x} : (\boldsymbol{x}, \boldsymbol{y}) \in P^{(k)}\} \ge \boldsymbol{1}^T\boldsymbol{x}^k \ge \big(1 - \frac{\epsilon}{8})|E|$$

proving Theorem 1.2.

Now we prove Lemma 3.3. We show $(\boldsymbol{x}^k, \boldsymbol{y}^k) \in P^{(k)}$ by induction on k. For $k = 0$, the claim is trivially true. Suppose that the claim is true for $k - 1 \ge 0$; we prove that the claim holds for k if $k \le r = n^\gamma$.

Let $\boldsymbol{a}^T\boldsymbol{x} \le b + \boldsymbol{c}^T\boldsymbol{y}$ be a non-trivial facet of $P^{(k)}$. First suppose that the size of the support of $\boldsymbol{a}$, $\|\boldsymbol{a}\|_0 \le \frac{t}{2}$. Let S denote the set of vertices at which some edge in support of $\boldsymbol{a}$ is incident. We have $|S| \le t$. From Theorem 3.2, there exists a distribution $\mathcal{D}$ over a set of feasible solutions to P which are integral on S and whose expectation is $(\boldsymbol{x}^0, \boldsymbol{y}^0)$. Modify these integral solutions in the following manner. For each edge not incident at a vertex in S, set $x_e = 0$ and for each vertex v not in S, set $y_v = y_u$ where u is the smallest index vertex in S (or any fixed vertex in S). Thus, we obtain a distribution $\mathcal{D}$ over *integral* feasible solutions. Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ denote the expectation of these solutions under distribution $\mathcal{D}$. We have the following properties for $(\boldsymbol{x}^*, \boldsymbol{y}^*)$.

$$
\begin{array}{llll}
 & \multicolumn{3}{l}{\max \sum_{(u,v)\in E}(1-\sum_{i\in\mathcal{L}} x(uv,i))} \\
\text{s.t.} & & & \\
x(uv,i) & \geq & y(u,i) - y(v,\pi_{uv}(i)) & \forall (u,v)\in E, i\in\mathcal{L} \\
\sum_{i\in\mathcal{L}} y(u,i) & = & 1 & \forall\ u\in V \\
\sum_{i=1}^{t} x(u_{i-1}u_i, l_{i-1}) & \geq & y(u,l_0) & \forall C, \forall u\in C, \forall l_0\in B(u,C) \\
x(uv,i) & \geq & 0 & \forall\ (u,v)\in E \\
y(u,i) & \geq & 0 & \forall\ u\in V
\end{array}
$$

Figure 3: Linear program for the Unique Label Cover Problem

1. $x^*_{uv} = x^0_{uv}$ if both $u, v \in S$.
2. $y^*_v = \frac{1}{2}$ for each $v \in V$.

The second property holds for each vertex $v \in S$ from Theorem 3.2 and for each vertex $v \notin S$ by construction. Observe that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies $\boldsymbol{a}^T\boldsymbol{x}^* \leq b + \boldsymbol{c}^T\boldsymbol{y}^*$ since $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in P_I$. But $\boldsymbol{y}^k = \boldsymbol{y}^*$ and $\boldsymbol{x}^k_e \leq \boldsymbol{x}^0_e = \boldsymbol{x}^*_e$ for each e with $a_e > 0$. Thus $\boldsymbol{a}^T\boldsymbol{x}^k - \boldsymbol{c}^T\boldsymbol{y}^k \leq \boldsymbol{a}^T\boldsymbol{x}^* - \boldsymbol{c}^T\boldsymbol{y}^*$ thus showing that $(\boldsymbol{x}^k, \boldsymbol{y}^k)$ satisfies the constraint.

Now, suppose that $\|\boldsymbol{a}\|_0 > \frac{t}{2}$. Since $\boldsymbol{a}^T\boldsymbol{x} \leq b + \boldsymbol{c}^T\boldsymbol{y}$ is valid for $P^{(k)}$, we must have $\max\{\boldsymbol{a}^T\boldsymbol{x} - \boldsymbol{c}^T\boldsymbol{y} : (\boldsymbol{x}, \boldsymbol{y}) \in P^{(k-1)}\} < b + 1$. But we have $(\boldsymbol{x}^{k-1}, \boldsymbol{y}^{k-1}) \in P^{(k-1)}$. Thus we have

$$
\begin{aligned}
\boldsymbol{a}^T\boldsymbol{x}^k - \boldsymbol{c}^T\boldsymbol{y}^k &= (\boldsymbol{a}^T\boldsymbol{x}^{k-1} - \boldsymbol{a}^T\cdot(\frac{2}{t}\boldsymbol{1})) - \boldsymbol{c}^T\boldsymbol{y}^{k-1} \qquad \text{(By definition of } \boldsymbol{x}^k, \boldsymbol{y}^k\text{)} \\
&= \boldsymbol{a}^T\boldsymbol{x}^{k-1} - \boldsymbol{c}^T\boldsymbol{y}^{k-1} - \frac{2}{t}\|a\|_1 \qquad \text{(Rearranging)} \\
&\leq \boldsymbol{a}^T\boldsymbol{x}^{k-1} - \boldsymbol{c}^T\boldsymbol{y}^{k-1} - \frac{2\|\boldsymbol{a}\|_0}{t} \qquad \text{(For integer vectors, } \|\cdot\|_1 \geq \|\cdot\|_0\text{)} \\
&< b + 1 - 1 \qquad \text{(By definition of CG)} \\
&= b
\end{aligned}
$$

□

4 Integrality Gaps for Unique Games

We now prove Theorem 1.3 and present integrality gap result for the unique games problem. The problem is defined as follows. Given a graph $G = (V, E)$, a set of q labels $\mathcal{L} = \{1, \ldots, q\}$ and permutation $\pi_{uv} : \mathcal{L} \to \mathcal{L}$ for each edge $\{u, v\} \in E$, the task is to assign a label $\Lambda(v)$ to each vertex v of G to maximize the number of *satisfied* edges $\pi_{uv}(\Lambda(u)) = \Lambda(v)$.

Figure 3 is a linear program for the unique label cover problem. Here variable $y(u, i)$ denotes whether the vertex u gets label i. The variable $x(uv, i)$ denotes whether edge $(u, v) \in E$ is violated (value 1) with u getting label i and v not getting label $\pi_{uv}(i)$. Note that the LP here is for maximizing the number of satisfied constraints; the LP for minimizing the number of satisfied constraints can be obtained by changing the objective function to $\sum_{(u,v)\in E}\sum_{i\in\mathcal{L}} x(uv, i)$.

We will in fact look at a richer LP from [25]. Let C be a simple cycle $u = v_0, v_1, \ldots, v_t = u$ in G containing u. Let l_0 be a label for v_0: for each value of $i \in [1, t]$, inductively define l_i as $l_i = \pi_{v_{i-1}v_i}(l_{i-1})$. I.e., the l_i's are defined so that $l_0, l_1, \ldots, l_i$ are labels that satisfy each of the edges $(v_0, v_1), \ldots, (v_{i-1}, v_i)$. Note that this process also defines another label l_t for $u = v_t$ which may or may not agree with the initial label l_0: indeed, we say that the label l_0 is *bad* for u with respect to C if $l_t \neq l_0$. Let $B_{u,C}$ be the set of labels that are bad for u with respect to C. Note that for any labeling f, if the label $f(u) = l_0$ lies in $B_{u,C}$, there must be at least one position i such that the label $f(v_i) = l_i$ and the next label $f(v_{i+1}) \neq l_{i+1}$; i.e., there must be at least one edge (v_i, v_{i+1}) that is violated. Hence for every such cycle C and every label $l_0 \in B_{u,C}$, we can write a constraint $\sum_{i=1}^{t} x(u_{i-1}u_i, l_{i-1}) \geq y(u, l_0)$.

We use the following gap results for the unique label cover problem shown by Charikar, Makarychev and Makarychev [10].

Theorem 4.1 ([10]) *For any $\epsilon > 0$, integer q there exists a $\gamma > 0$ and a unique label cover instance on a graph $G = (V, E)$ on n vertices such that a) Any labeling satisfies at most $(1+\epsilon)/q$ fraction of the constraints, but b) for any set S of $t = n^\gamma$ vertices, there is a distribution $\mathcal{D}$ over assignments Λ_S of labels to these vertices such that (i) the marginal on any vertex is uniform over the labels, i.e. $\Pr_{\Lambda_S \sim \mathcal{D}}[\Lambda_S(v) = l] = \frac{1}{q}$ for any $l \in [q]$, and $v \in S$, and (ii) for any $e = (u, v) \in E$ with $u, v \in S$, $\Pr_{\Lambda_S \sim \mathcal{D}}[\Lambda_S(v) = \pi_{uv}(\Lambda_S(u)) = l] \geq \frac{1-\epsilon}{q}$.*

The result then follows along lines similar to the previous section. We inductively construct feasible solutions for the polytope $P^{(k)}$. Valid constraints involving few x variables are handled by the fact that local distributions Λ_S exist with the right marginals. Valid constraints involving many x variables are satisfied by induction due to the right scaling.

We set $(\boldsymbol{x}^k, \boldsymbol{y}^k)$ as follows: $y^k(u, i)$ is set to $\frac{1}{q}$ for each $u \in V, i \in [q]$. $x^k(uv, i)$ is set to $\frac{\epsilon}{q} + \frac{2(k+1)}{t}$. We will show by induction that $(\boldsymbol{x}^k, \boldsymbol{y}^k)$ lies in $P^{(k)}$.

We first show that any constraint in $P^{(k)}$ has a specific structure.

Lemma 4.2 *Let $\mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} \geq c$ be a valid non-trivial constraint for $P^{(k)}$. Then the following hold without loss of generality.*

- $\mathbf{a}, \mathbf{b}$ *and c are integral and $a_i \geq 0$ for each i.*
- *Every vector $(\mathbf{x}, \mathbf{y})$ in P_I satisfies $\mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} \geq c$.*
- $\mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} > c - 1$ *for any $(\mathbf{x}, \mathbf{y}) \in P^{(k-1)}$.*

Proof: The first property follows by observing that they hold for the inequalities in P, and are preserved under summation. The last two properties are a consequence of the definition of $P^{(k)}$. □

Lemma 4.3 *Under the definitions above, $(\mathbf{x}^k, \mathbf{y}^k) \in P^{(k)}$.*

Proof: For the base case, note that $\boldsymbol{x}^0, \boldsymbol{y}^0$ satisfies all equation of the type $x(uv, i) \geq y(u, i) - y(v, \pi_{uv}(i))$ since the right hand side is zero. Also $\sum_i y(u, i)$ is indeed 1. For the cycle constraints, note that any cycle of length greater than $\lceil \frac{t}{2} \rceil$ is satisfied since each $x^0(uv, i)$ is at least $2/t$. For a constraint $\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{y} \geq c$ corresponding to a shorter cycle C, let $F = \{e \in E : \exists i : a_e^i > 0\}$ denote the set of edges with a positive a, and let $S = \{u \in V : \exists e \in F \cap \delta(u)\}$. Thus $|S| \leq |C| \leq \frac{t}{2}$. Let $\mathcal{D}$ denote the distribution of labelings of S guaranteed by Theorem 4.1. For a partial labeling Λ_S, let $Comp(\Lambda_S)$ denote a *completion* of Λ_S to all of V giving each vertex the same label as the lexicographically smallest vertex in S, and let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ denote the expected value of the integer solution defined by $Comp(\Lambda_S)$, when Λ_S is drawn from $\mathcal{D}$. Clearly $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in P_I$ so that $\boldsymbol{a}^T\boldsymbol{x}^* + \boldsymbol{b}^T\boldsymbol{y}^* \geq c$. By Theorem 4.1, for any $u, v \in S$, $x^*(uv, i) \leq \frac{\epsilon}{q} \leq x^0(uv, i)$. Moreover, for any $u \in S$, $y^*(u, i) = \frac{1}{q} = y^0(u, i)$. Thus $\boldsymbol{a}^T\boldsymbol{x}^0 + \boldsymbol{b}^T\boldsymbol{y}^0 \geq \boldsymbol{a}^T\boldsymbol{x}^* + \boldsymbol{b}^T\boldsymbol{y}^* \geq c$.

$$
\begin{array}{lcll}
 & & \max\sum_{e\in E}\sum_{\alpha\in\mathbb{F}_q^k} & C_e(\alpha)x(e,\alpha) \\
\text{s.t.} & & & \\
\sum_{i\in\mathbb{F}_q} y(u,i) & = & 1 & \forall\ u\in V \\
x(e,\alpha) & \leq & y(u_j,\alpha_j) & \forall e=(u_1,\ldots,u_k)\in E, \alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_q^k, j\in[k] \\
x(e,\alpha) & \geq & 0 & \forall\ e\in E,\alpha\in\mathbb{F}_q^k \\
y(u,i) & \geq & 0 & \forall\ u\in V, i\in\mathbb{F}_q
\end{array}
$$

Figure 4: Linear program for k-CSP over $\mathbb{F}_q$

Suppose that the claim holds for $k-1$, i.e. $(\boldsymbol{x}^{k-1},\boldsymbol{y}^{k-1})\in P^{(k-1)}$. We argue that the claim holds for k. Now let $\boldsymbol{a}^T\boldsymbol{x}+\boldsymbol{b}^T\boldsymbol{y}\geq c$ be a constraint in $P^{(k)}$. We wish to argue that the solution $(\boldsymbol{x}^k,\boldsymbol{y}^k)$ above satisfies this constraint. Let $F=\{e\in E:\exists i: a_e^i>0\}$ denote the set of edges with in support of $\boldsymbol{a}$, and let $S=\{u\in V:\exists e\in F\cap\delta(u)\}$. It is easy to see that $|S|\leq 2|F|\leq 2\|\boldsymbol{a}\|_0$.

First suppose that $|S|\leq t$. Let $\mathcal{D}$ denote the distribution of labelings of S guaranteed by theorem 4.1. For $Comp(\Lambda_S)$ as above, let $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ denote the expected value of the integer solution defined by $Comp(\Lambda_S)$, when Λ_S is drawn from $\mathcal{D}$. Clearly $(\boldsymbol{x}^*,\boldsymbol{y}^*)\in P_I$ so that $\boldsymbol{a}^T\boldsymbol{x}^*+\boldsymbol{b}^T\boldsymbol{y}^*\geq c$. By theorem 4.1, for any $u,v\in S$, $x^*(uv,i)\leq\frac{\epsilon}{q}\leq x^k(uv,i)$. Moreover, for any $u\in V$, $y^*(u,i)=\frac{1}{q}=y^k(u,i)$. Thus $\boldsymbol{a}^T\boldsymbol{x}^k+\boldsymbol{b}^T\boldsymbol{y}^k\geq\boldsymbol{a}^T\boldsymbol{x}^*+\boldsymbol{b}^T\boldsymbol{y}^*\geq c$.

Now suppose that $|S|>t$. Then $\sum_i a_i\geq\|\boldsymbol{a}\|_0>\frac{t}{2}$. By the last property is Lemma 4.2, $\boldsymbol{a}^T\boldsymbol{x}^{k-1}+\boldsymbol{b}^T\boldsymbol{y}^k>c-1$. Thus

$$
\boldsymbol{a}^T\boldsymbol{x}^k+\boldsymbol{b}^T\boldsymbol{y}^k=\boldsymbol{a}^T\boldsymbol{x}^{k-1}+\frac{2}{t}\sum_i a_i+\boldsymbol{b}^T\boldsymbol{y}^{k-1}\geq\boldsymbol{a}^T\boldsymbol{x}^{k-1}+1+\boldsymbol{b}^T\boldsymbol{y}^{k-1}\geq c-1+1=c
$$

This completes the induction and the claim follows. □

Proof of Theorem 1.3 now follows form observing that the solution $(\boldsymbol{x}^k,\boldsymbol{y}^k)$ has an objective value at least $(1-2\epsilon)$ times the number of constraints while by Theorem 4.1 no integral solution satisfies more than $\frac{1+\epsilon}{q}$ fraction of the constraints.

5 Other results

In this section, we prove Theorem 1.4 and Theorem 1.5 and prove integrality gaps for linear programs for k-CSP$_q$ and the vertex cover problem obtained via CG rounding.

5.1 Integrality Gap for k-CSP$_q$

In the k-CSP$_q$ problem, we are given variables $x_1,\ldots,x_n$ which take values from a finite field $\mathbb{F}_q$ and constraints $C_1,\ldots,C_m$ each of which is a k-ary boolean function applied to some k-tuple of variables. The task is find an assignment for each of the variables which maximizes the number of satisfied constraints. A constraint is satisfied when it attains a value of 1. Let V denote the set of variables. Each constraint then corresponds to a hyperedge over the set of variables. We let E denote the set of hyperedges corresponding to the constraints. Observe that for each hyperedge $e\in E$, we have $|E|=k$ and C_e defines a constraint (with $C_e(\alpha)=1$ for a satisfying assignment, 0 otherwise) where α denotes an assignment of the variables. We consider the linear program for k-CSP$_q$ given in Figure 4.

The following result is implicit in Tulsiani [38].

Theorem 5.1 ([38]) *For any $\epsilon > 0$, prime number q, there exists a $c > 0$ such that for all sufficiently large n, there is a max k-CSP$_q$ instance on n variables such that a) any assignment satisfies at most $(\frac{q^k}{kq(q-1)-q(q-2)} - \epsilon)^{-1}$ fraction of the constraints, but b) for any set S of at most $t = cn$ variables, there is a distribution $\mathcal{D}_S$ over assignments Λ_S of labels to these vertices such that the following hold*

Satisfaction: *For every constraint (e, C_e) such that $e \subseteq S$, every assignment in the support of $\mathcal{D}_S$ satisfies (e, C_e).*

Consistency: *For all $T \subseteq S$, for all $\alpha \in \mathbb{F}_q^{|T|}$, $\Pr_{\Lambda_S \sim \mathcal{D}_S}[\Lambda_S(T) = \alpha] = \Pr_{\Lambda \sim \mathcal{D}_T}[\Lambda(T) = \alpha]$.*

We can "mix" each of these distributions $\mathcal{D}_S$ with a uniform distribution over assignments to derive the following corollary.

Corollary 5.1 *For any $\epsilon > 0$, prime number q, there exists a $c > 0$ such that for all sufficiently large n, there is a max k-CSP$_q$ instance on n variables such that a) any assignment satisfies at most $(\frac{q^k}{kq(q-1)-q(q-2)} - \epsilon)^{-1}$ fraction of the constraints, but b) for any set S of at most $t = cn$ variables, there is a distribution $\mathcal{D}_S$ over assignments Λ_S of labels to these vertices such that the following hold*

Satisfaction: *For every constraint (e, C_e) such that $e \subseteq S$, $\Pr_{\Lambda_S \sim \mathcal{D}_S}[\Lambda_S$ satisfies $(e, C_e)] \geq 1 - \epsilon$.*

Consistency: *For all $T \subseteq S$, for all $\alpha \in \mathbb{F}_q^{|T|}$, $\Pr_{\Lambda_S \sim \mathcal{D}_S}[\Lambda_S(T) = \alpha] = \Pr_{\Lambda \sim \mathcal{D}_T}[\Lambda(T) = \alpha]$.*

Entropy: *For every constraint (e, C_e), for every $\alpha \in \mathbb{F}_q^k$, $\Pr_{\Lambda \sim \mathcal{D}_e}[\Lambda(e) = \alpha] \geq \epsilon/q^k$.*

Let $(\boldsymbol{x}^p, \boldsymbol{y}^p)$ be defined as follows: $\boldsymbol{y}^p(u, i) = \Pr_{\Lambda \sim \mathcal{D}_{\{u\}}}[\Lambda(u) = i]$, and $\boldsymbol{x}^p(e, \alpha) = \Pr_{\Lambda \sim \mathcal{D}_e}[\Lambda(e) = \alpha] - \frac{kp}{t}$. The entropy property above guarantees that for $p \leq \frac{t\epsilon}{kq^k}$, $\boldsymbol{x}^p$ satisfies the non-negativity constraints. The Satisfaction property guarantees that the value of the fractional solution $(\boldsymbol{x}^p, \boldsymbol{y}^p)$ is $m(1 - 2\epsilon)$ for $p \leq \frac{t\epsilon}{kq^k}$.

Theorem 1.4 now follows from the following lemma.

Lemma 5.2 *The solution $(\mathbf{x}^p, \mathbf{y}^p)$ defined above satisfies all constraints in $P^{(p)}$.*

Proof: The proof is by induction on p. For $p = 0$ the claim is immediate from corollary 5.1. Suppose that the lemma holds for $p - 1$, i.e. $(\boldsymbol{x}^{p-1}, \boldsymbol{y}^{p-1}) \in P^{(p-1)}$. Let $\boldsymbol{a}^T \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{y} \leq c$ be a constraint in $P^{(p)}$, where $\boldsymbol{a} \geq 0$. Let $F = \{e : a_{e,\alpha} > 0 \text{ for some } \alpha \in [q]^k\}$ and let $S = \cup_{e \in F} e$. Clearly $|S| \leq k|F|$.

We consider two cases. If $|S| \leq t$, then there is a distribution $\mathcal{D}_S$ over partial assignments Λ_S to variables in S. For every $u \notin S$, let $\mathcal{D}_u$ be the distribution over assignments to u given by corollary 5.1. Let $\mathcal{D}$ be the product distribution $\mathcal{D}_S \times \Pi_{u \notin S} \mathcal{D}_u$. Let $(\boldsymbol{x}_\Lambda, \boldsymbol{y}_\Lambda)$ denote the integer solution corresponding to an assignment Λ and let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ denote the expectation of $(\boldsymbol{x}_\Lambda, \boldsymbol{y}_\Lambda)$ when Λ is drawn from the product distribution $\mathcal{D}$ above. Since $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a convex combination of integer solutions, it is in $P^{(p)}$ so that $\boldsymbol{a}^T \boldsymbol{x}^* + \boldsymbol{b}^T \boldsymbol{y}^* \leq c$. But by the consistency condition, $\boldsymbol{y}^p = \boldsymbol{y}^*$. Moreover, for any $e \in F$, and α, $\boldsymbol{x}^0(e, \alpha) = \boldsymbol{x}^*(e, \alpha)$ so that $\boldsymbol{a}^T \boldsymbol{x}^0 = \boldsymbol{a}^T \boldsymbol{x}^*$. Since $\boldsymbol{x}^p \leq \boldsymbol{x}^0$, the constraint $\boldsymbol{a}^T \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{y} \leq c$ is satisfied for $(\boldsymbol{x}^p, \boldsymbol{y}^p)$.

Now suppose that $|S| > t$. By the definition of $P^{(p)}$, $\boldsymbol{a}^T \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{y} \leq c + 1$ for $(\boldsymbol{x}^{p-1}, \boldsymbol{y}^{p-1})$. But

$$\boldsymbol{a}^T \boldsymbol{x}^p + \boldsymbol{b}^T \boldsymbol{y}^p = \boldsymbol{a}^T \boldsymbol{x}^{p-1} + \boldsymbol{b}^T \boldsymbol{y}^{p-1} - \|\boldsymbol{a}\|_1 \frac{k}{t} \leq c + 1 - \frac{\|\boldsymbol{a}\|_0 k}{t} \leq c$$

since $\boldsymbol{a} \geq 0$ and $\|\boldsymbol{a}\|_0 = |F| \geq |S|/k$. The claim follows. $\square$

$$\begin{array}{lll} & \min & \sum_{v \in V} x_v \\ \text{s.t.} & & \\ x_u + x_v & \geq 1 & \forall \{u,v\} \in E \\ x_v & \geq 0 & \forall \ v \in V \end{array}$$

Figure 5: Linear program for the Vertex Cover Problem

The unique label cover problem is a special case of a 2-CSP_q. Using this approach, one can also show a gap for the analogous linear program from unique games (which is different from the one studied in section 4). The proof uses the results of Khot and Saket [28], Raghavendra and Steurer [35], and implies a $(1-\eta, \eta)$ gap for $O_\eta((\log\log n)^{\frac{1}{4}})$ rounds of C-G, for any $\eta > 0$. We omit the details from this abstract.

5.2 Integrality Gaps for Vertex Cover

Proof of Theorem 1.5: We now prove integrality gap for the vertex cover problem. We denote P to be the polytope of all feasible solutions of the linear program in Figure 5. We use the following theorem which gives integrality gap for Sherali-Adams hierarchy.

Theorem 5.3 ([10]) *For any $\epsilon > 0$, there exists a $\gamma > 0$ such that there exists a graph $G = (V, E)$ such that any integral vertex cover has $(1 - \frac{\epsilon}{8})n$ vertices but the fractional solution $x_v^0 = \frac{1}{2} + \frac{\epsilon}{16}$ for each $v \in V$, is in P_{SA}^t for $t = \frac{16n^\gamma}{\epsilon}$ rounds. Therefore, for every subset $S \subset V$ of size at most t, there exists a distribution $\mathcal{D}$ over solutions in P such that (i) the expected value of the solution from distribution $\mathcal{D}$ equals x^0 and (ii) each solution with non-zero probability in $\mathcal{D}$ is integral on S.*

Let G be the graph given by Theorem 5.3. We prove the following lemma.

Lemma 5.4 *Let $x_v^k = (\frac{1}{2} + \frac{\epsilon}{16} + \frac{k}{t})$ for each $v \in V$ for any nonnegative integer k. Then the fractional solution $x^k \in P^{(k)}$ for each $0 \leq k \leq n^\gamma$.*

Theorem 1.5 follows simply from Lemma 5.4 since for the cost function which is uniformly 1, $\boldsymbol{x}^k$ for $k = n^\gamma$ is a feasible solution in $P^{(k)}$ of cost at most $\left(\frac{1}{2} + \frac{\epsilon}{8}\right)n$ while the integral optimum is at least $\left(1 - \frac{\epsilon}{8}\right)n$

We now prove Lemma 5.4 by showing $\boldsymbol{x}^k \in P^{(k)}$ by induction on k. For $k = 0$, the claim is trivially true. Suppose that the claim is true for $k - 1 \geq 0$; we prove the claim holds for k if $k \leq r = n^\gamma$.

Let $\boldsymbol{a}^T\boldsymbol{x} \geq b$ be a CG cut for $P^{(k)}$. First suppose that $\|\boldsymbol{a}\|_0 \leq t$. Let S denote the support of a and let $\mathcal{D}$ denote the distribution given by Theorem 5.3 over solutions which are integral over S. Extend each of these solutions integrally over $V \setminus S$ (by setting x_v to 1 outside S) and let x^* denote the expectation of these solutions under $\mathcal{D}$. Since $x^* \in P_I$, we have $\boldsymbol{a}^T\boldsymbol{x}^* \geq b$. But $\boldsymbol{a}^T\boldsymbol{x}^k \geq \boldsymbol{a}^T\boldsymbol{x}^*$ proving that $\boldsymbol{x}^k$ satisfies the constraint.

Now, suppose that $\|\boldsymbol{a}\|_0 > t$. Since $\boldsymbol{a}^T\boldsymbol{x} \geq b$ is valid for $P^{(k)}$, we must have $\min\{\boldsymbol{a}^T\boldsymbol{x} : \boldsymbol{x} \in P^{(k-1)}\} > b - 1$. But we have $\boldsymbol{x}^{k-1} \in P^{(k-1)}$. Thus $\boldsymbol{a}^T\boldsymbol{x}^k = \boldsymbol{a}^T\boldsymbol{x}^{k-1} + \frac{1}{t}\boldsymbol{a}^T\boldsymbol{1} > b - 1 + 1 \geq b$. Therefore $\boldsymbol{x}^k \in P^{(k)}$. $\square$

6 Open Problems

Our negative results suggest that the connection between SA and C-G integrality gaps may extend to a fairly general class of linear programs. While this class would have to exclude hypergraph matching due to our negative result, it may include other interesting problems such as the sparsest cut. It also seems natural to investigate whether combining the various cut generation procedures improves integrality gaps when they individually do not.

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