Approximation Algorithms COMP 760 **Final Exam**
Date: April 15, 2010 **Due:** April 20, 2010

All questions carry equal points. You are allowed to use the course notes but no external resource (web or research papers). No discussion is allowed.

Read all questions carefully.

1. **Partition Max Coverage** Recall the max-coverage problem where we given a set U and a collection of subsets $S_1, \ldots, S_m$ where $S_i \subset U$, integer k and the task is to find a subcollection $I \subset \{1, \ldots, m\}$ with $|I| = k$ which maximizes $| \cup_{i \in I} S_i|$. In the partition max-coverage problem, we are also given a partition $P_1, \ldots, P_k$ of $\{1, \ldots, m\}$ and the task is to find I such that $|I \cap P_j| = 1$ for each $1 \leq j \leq k$ which maximizes $| \cup_{i \in I} S_i|$. Give an approximation algorithm for the partition max-coverage problem.

2. **Steiner Tree** In the Steiner tree problem where given a edge-weighted graph $G = (V, E)$ and set of terminals R, the task is to find a tree spanning R of minimum weight. Consider the natural greedy algorithm where we order the terminal set in an arbitrary order. When the i^{th} terminal arrives it joins itself to the nearest terminal which has arrived before it. We will show that this algorithm is a $O(\log n)$-approximation.

   (a) Prove the following tree-path lemma. Given any tree T and any subset S of the vertices spanned by T where S is even, there is a pairing of the vertices in S such that the paths in T between the paired vertices are all edge disjoint.

   (b) Use the Tree-Path Lemma recursively to show that the greedy algorithm is a $O(\log n)$-approximation.

3. **3DM** In an instance of the 3-dimensional matching, we are given disjoint sets A, B, C and a collectin of 3-dimensional edges $E \subseteq A \times B \times C$. The task is to pick a subcollection $F \subseteq E$ of maximum cardinality such that each element in $A \cup B \cup C$ appears in no more than one triple in F.

   (a) Write a linear program for the problem.

   (b) Show that there is always a triple e such that $x_e \geq \frac{1}{2}$.

   (c) Show that one can pick this triple and recurse to obtain a ~~2-approximation~~ $\frac{1}{2}$-approximation. (Caution: Be very careful about recursing since no element can appear in more than one triple).

4. **SDP and L_2 embedding**. Given a metric d on vertices V, an l_2 embedding of metric d is a mapping $\pi : V \to R^n$ for some integer n and $d_\pi(u, v) = ||\pi(u) - \pi(v)||_2$ is the Euclidean distance between the mapped points. The embedding has distortion α if for all $u, v \in V$

$$d(u, v) \leq d_\pi(u, v) \leq \alpha d(u, v)$$

Write a SDP for finding the minimum distortion l_2-embedding. What approximation factor can you achieve for such an embedding?

5. **Exactly One Set Cover** In the exactly-one set cover problem we are given a universe U and collection of sets $S_1, \ldots S_m$ and goal is to find a subcollection $I \subset \{1, \ldots, m\}$ which maximizes the number of elements which are covered exactly once, i.e. the objective function is $|\{e : |\{i \in I : e \in S_i\}| = 1\}|$.

   (a) Give a constant factor randomized algorithm when each element appears in exactly f sets.

   (b) Give a constant factor randomized algorithm when each elements appears in at least f sets and at most $2f$ sets.

   (c) Give a ~~$O(\log m)$-approximation~~ $\Omega(\frac{1}{\log m})$-approximation for the general instance. (Hint: Use the previous part).

   (d) Give a ~~$O(\log n)$-approximation~~ $\Omega(\frac{1}{\log m})$-approximation for the general instance. (Hint: Use the previous part).