Computers in Engineering COMP 208

Initial Value Problems

Michael A. Hawker

Differential Equations

- Differential Equations are based on an unknown function involving one or more variables
- An Ordinary Differential Equation is a DE that depends on one unknown variable and is usually in the form

  $F(x, f(x), f'(x), .., f^n(x)) = 0$

Ordinary Differential Equations

- In many areas of application we can measure how things change in some process
- From these measurements we would like to find the function that describes the process
- Examples:
  - Change in concentrations during chemical reaction
  - Heating or cooling of objects
  - Current flow in electrical circuits
  - Population dynamics

Simple ODE

$$dy / dx = x$$

- We want to find the original function
- We need to change things around and integrate:

$$dy = dx * x$$

$$y = x^2/2 + C$$

C is a some unknown constant

Initial Value Problems

- We need to have some way to determine the constant C
- We call this the initial value problem since we must know the value of the function, y_0, at some initial point x_0
- Then we can determine the actual function

Example Cont.

- $y(x) = x^2/2 + C$
- Extra Data: $y(4) = 3$
- $3 = 4^2/2 + C$
- $3 - 8 = C$
- $C = -5$
- $y(x) = x^2/2 - 5$

ODE’s

- We let y(x) be the function we would like to study; however, we are only able to observe the rate of growth of the function
- This leads to equations of the form

```
dy/dx = f(x,y)
```

- As we have just seen we can formulate a solution

Analytic Solutions

Some ODE's have analytic solutions

$dy/dx = x + y - 1$

Has the solution

$y(x) = e^x - x$

Others have no analytic solution. For example:

$dy/dx = x^2 + y^2$

What do we do?

- If we can't find a closed form of the function
- We turn back to numerical methods to approximate the solution

The Euler Method

- We “grow” the function from the starting value one step at a time
- Think of the `dy/dx` in terms of discrete steps, `delta_y` and `delta_x`
- Then the derivative approximates the ratio of these two values for small vaues of delta_x

The Euler Method

- Multipling `dy/dx` by `delta_x` gives an approximate value for `delta_y`
- The Euler method increments the independent variable by one stepsize, `delta_x`, at a time
- Using the derivative, we approximate `delta_y` and then the value of the function at the next step

Euler Method

We want to find an approximate solution to:

```
dy/dx = f(x,y)
y(x0) = y0
```

Now `f(x0,y0)` is the slope of the function at `(x0,y0)`

Approximate the function value at `x0+h` by `y0 + h*f(x0,y0)`

Repeat this process so that

```
x_(n+1) = x_n + h
y_(n+1) = y_n + hf(x_n,y_n)
```

A Simple Example

```
#include <stdio.h>
#include <math.h>
#define H    0.1
int main() {
   double x,newx;
   double y,newy;
   int i,steps = 10;

   x = 0;
   y = 1;
   for (i = 0; i < steps; i++)
   {
        newx = x + H;
        newy = y + H * (x + y -1);
        x = newx; y = newy;
        printf ("  %f  %f  %f\n", newx, newy,exp(x)-x);
   }
   return 0;
}
```

#define

- After the `#include` commands, we can also define names using `#define`
- In the program, the name is replaced by the expression
- Note that the name is not a variable (there is no memory location for it)
- The name is just an alias for the value
  - i.e. Find and Replace

#define

- Syntax
  - No ; at end
  - No = between name and value
- This can cause problems if complex expressions are used

#define macros

- We can create simple functions to replace basic concepts
- i.e. adding 5 to a number
    - #define ADD_5(x) x+5
- Syntax:

  #define MACRO_NAME(parameters) exp

#define Example

- #define ADD_5(x) x+5

  …6+ADD_5(7)*4…
- Becomes: 6+7+5*4

  Which Outputs: 33 not 54 as expected
- We need parenthesis for correct behavior

  #define ADD_5(x)   ((x) + 5)
- Why the extra parenthesis around x?

Example Revisited

```
void main(){
   double x, y;
   FILE * myfile = fopen("test2.csv","w+");
   int i,steps = 100;
   if (myfile){
     x = 0;
     y = 1;
     for (i = 0; i < steps; i++){
       fprintf (myfile," %f, %f, %f\n", x, y,exp(x)-x);
       Euler_step(H,x,&y,func);
       x += H;
     }
     fclose(myfile);
   }
   else printf("Could not open the file");
   return;
}
```

Writing to a File – A Review

- We declare it to be of type FILE and use fopen to open it.

```
FILE * myfile = fopen("test2.csv","w+");
```

- Then we can write to the file using fprintf

```
fprintf (myfile," %f, %f, %f\n",
         x, y,exp(x)-x);
```

- When we are finished we close the file

```
fclose(myfile);
```

Writing to a File

- When we open a file, the filename is assigned a nonzero value if the operation is successful and a value of zero if it fails.

```
if (myfile) {
  …
}
else printf("Could not open the file");
```

Function Pointers – A Review

- We can pass a function as an argument (via a pointer) to a function that computes the next iteration from the previous one.
- First give a name (fun) to the **type** of function we want to use

```
typedef double (*fun) (double,double);
```

Computing the Next Value

- Then define a function that uses this argument to compute the next value.
- We compute a new value for y, so we pass a pointer to it

```
void Euler_step(double h, double x,
                double *y, fun f){
  *y += h * f(x,*y);
  return;
}
```

Computing the Next Value

- We can define a function that we want to pass to the Euler function:

```
double func(double x, double y){
  return x+y-1;
}
```

- Functions are stored in memory at a specific address. We can pass a pointer to our function by using just its name (as we did with arrays):

```
    Euler_step(H,x,&y,func);
```

The Final Program (a)

```
#include <stdio.h>
#include <math.h>

#define H    0.1

typedef double (*fun) (double,double);

double func(double x, double y){
   return x+y-1;
}

void Euler_step(double h, double x, double *y, fun f){
   *y += h * f(x,*y);
   return;
}
```

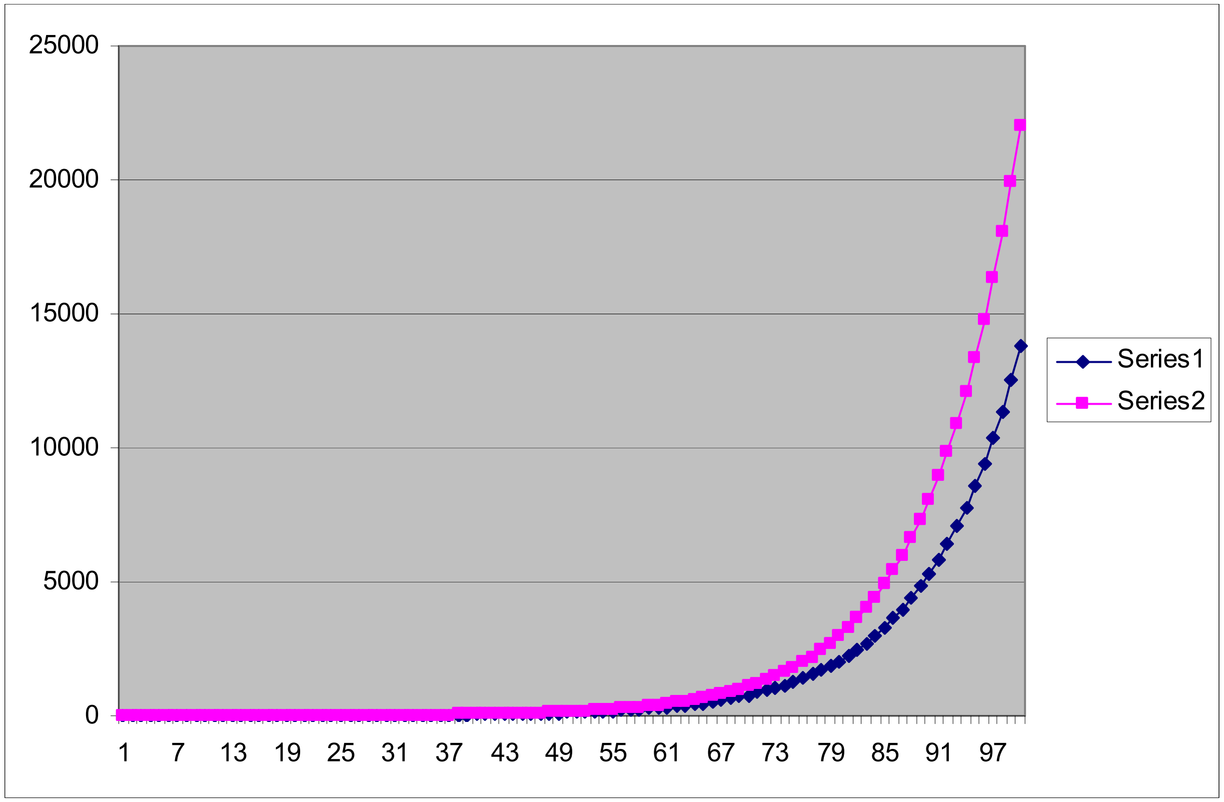
The Final Program (b)

```
int main(){
  double x, y;
  FILE * myfile = fopen("test2.csv","w+");
  int i,steps = 100;
    if (myfile){
      x = 0; y = 1;
      for (i = 0; i < steps; i++){
        fprintf (myfile," %f, %f, %f\n", x, y,exp(x)-x);
       Euler_step(H,x,&y,func);
        x += H;
      }
      fclose(myfile);
    }
  else printf("Could not open the file");
  return 0;
}
```

Part of the Output File

```
  0            1           1
0.1         1.01    1.005171
0.2        1.031    1.021403
0.3       1.0641    1.049859
0.4      1.11051    1.091825
0.5     1.171561    1.148721
0.6     1.248717    1.222119
0.7     1.343589    1.313753
0.8     1.457948    1.425541
0.9     1.593742    1.559603
  1     1.753117    1.718282
1.1     1.938428    1.904166
1.2     2.152271    2.120117
1.3     2.397498    2.369297
1.4     2.677248      2.6552
1.5     2.994973    2.981689
```

Excel Generated Graph



Runge-Kutta Method

The Euler method is not very accurate since the error tends to keep growing.

In the (fourth-order) Runge_Kutta method the derivative is evalutated four times

1. At the initial point
2. Twice at a trial midpoint
3. At a trial endpoint

Runge-Kutta Formula

Use the following to compute the next step

```
    k_1 = f(x_n, y_n)
    k_2 = f(x_n+h/2, y_n+k_1/2)
    k+3 = f(x_n+h/2, y_n+k_2/2)
    k_4 = f(x_n+h,y_n+k_3)

y_(n+1) = y_n
          + (k_1+2*k_2+2*k_3+k_4)*h/6
```

Implementing Runge-Kutta

```
void RK_step(double h, double x, double *y,
             fun f){
  double k1, k2, k3, k4, half = h/2.0;

  k1 = f(x, *y);
  k2 = f(x + half, *y + half * k1);
  k3 = f(x + half, *y + half * k2);
  k4 = f(x + h, *y + h*k3);

  *y += (h/6.0) * (k1 + 2.0*k2 + 2.0*k3 + k4);

}
```

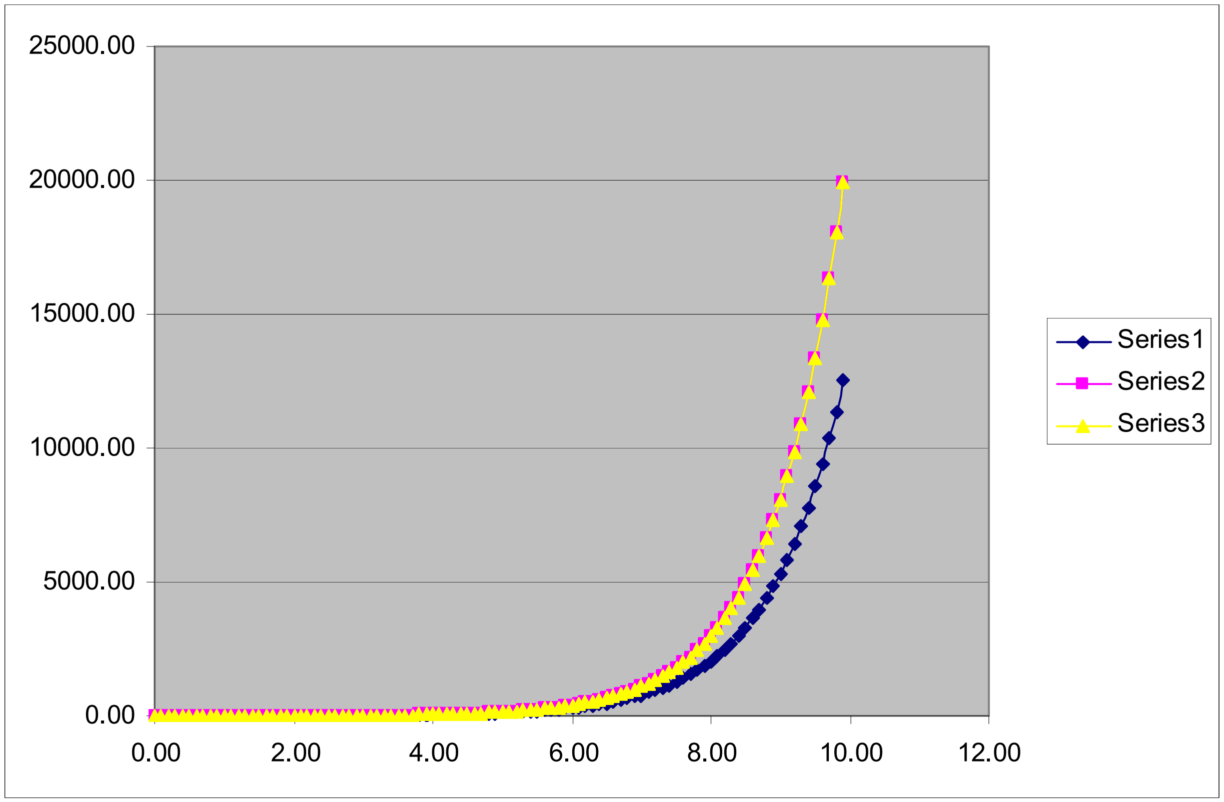
Comparing Runge-Kutta and Euler

```
int main(){
   double a, b, x, y
   FILE * myfile = fopen("RKtest.csv","w+");
   int i,steps = 100;
   if (myfile){
     x = 0; a = 0;
     y = 1; b = 1;
     for (i = 0; i <= steps; i++){
       fprintf (myfile," %f, %f, %f, %f\n", x, y, b, exp(x)-x);
       Euler_step(H,x,&y,func);
       RK_step(H,a,&b,func);
       x += H;
       a += H;
     }
     fclose(myfile);
   }
   else printf("Could not open the file");
   return 0;
}
```

Part of the Output File

```
3.100000, 16.094342, 19.097899, 19.097951
3.200000, 17.913777, 21.332470, 21.332530
3.300000, 19.925154, 23.812570, 23.812639
3.400000, 22.147670, 26.564022, 26.564100
3.500000, 24.602437, 29.615363, 29.615452
3.600000, 27.312681, 32.998133, 32.998234
3.700000, 30.303949, 36.747190, 36.747304
3.800000, 33.604343, 40.901054, 40.901184
3.900000, 37.244778, 45.502301, 45.502449
4.000000, 41.259256, 50.597983, 50.598150
4.100000, 45.685181, 56.240098, 56.240288
4.200000, 50.563699, 62.486116, 62.486331
4.300000, 55.940069, 69.399551, 69.399794
4.400000, 61.864076, 77.050594, 77.050869
4.500000, 68.390484, 85.516821, 85.517131
4.600000, 75.579532, 94.883965, 94.884316
4.700000, 83.497485, 105.246776, 105.247172
4.800000, 92.217234, 116.709970, 116.710418
4.900000, 101.818957, 129.389275, 129.389780
5.000000, 112.390853, 143.412590, 143.413159
5.100000, 124.029938, 158.921266, 158.921907
5.200000, 136.842932, 176.071519, 176.072242
5.300000, 150.947225, 195.035996, 195.036810
5.400000, 166.471948, 216.005499, 216.006416
5.500000, 183.559142, 239.190900, 239.191932
5.600000, 202.365057, 264.825246, 264.826407
5.700000, 223.061562, 293.166095, 293.167401
5.800000, 245.837719, 324.498091, 324.499560
5.900000, 270.901490, 359.135816, 359.137468
6.000000, 298.481640, 397.426937, 397.428793
6.100000, 328.829803, 439.755685, 439.757770
6.200000, 362.222784, 486.546699, 486.549041
```

Excel Generated Graph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