CS 760 Graph Theory	**Scribe: Matt Drescher**
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add 1.10, hereditary.
EXAM MARCH 1ST

Path: walk with no repeated vertices.
Cycle: closed walk with *one* repeated vertex.
$d(v)$: the degree of v in G.
$d^+(v)$: the out degree of v in G.
$d^-(v)$: the in degree of v in G.
$G := (V, E)$
$D := (V, A)$
$G[S] :=$ *Induced subgraph of G by vertex set S*
$G - S := G[V - S]$
$G - w := G - \{w\}$
$\bar{G} :=$ *the compliment of G*
component : *:maximal connected piece*
Block: *:maximal 2 connected piece*
$n := |V|$
$m := |E|$

(NEAR) ALMOST PERFECT MATCHING: A matching with $\frac{|V|}{2} - 1$ edges.

1 Konningsberg

see hand out and first chapter of text.

1.1 Necessary and Sufficient conditions for G to have a closed walk using each edge exactly once.(Circuit)

1. Every vertex must have even degree .
2. Graph must be connected except for possibly some isolated vertices.

By # 1, $E(G)$ decomposes into a set of cycles. show that a graph satisfying #1 has a cycle. rip it out, and the result follows by induction.
Now take the set of cycles. If they all intersect, then we are done. If some cycles is disjoint then the Graph is not connected.

G is *Eulerian* iff every vertex has even degree. $\Leftrightarrow$ we can orient the edges of G so that $\forall v \in G, d^+(v) = d^-(v)$.

this is called an *Eulerian Orientation*. We can find this in Linear time by just picking a vertex and following a path orienting along until we cant, then pick a different starting point.

Theorem 1 *1.Given a graph G, how many edges do we need to delete to make G Eulerian?*
2.Given a graph G, how many edges do we need to add to make G Eulerian?
3.Given an orientation of an Eulerian G, how many edges do we need to reverse to obtain an Eulerian orientation?

#3:
$V = V_{odd} \cup V_{even}$ ($|V_{odd}|$ is even).
Need to add(delete) $\geq \frac{|V_{odd}|}{2}$ edges.
Can delete(add) exactly this many to get an Eulerian graph iff $\exists$ a **perfect matching** in the subgraph of G induced by V_{odd}. Actually want a minimum size $T - join$ for $T = V_{odd}$.

$T - Join$- Subgraph of G in which $\forall v \in T$, $d_H(v)$ is odd. $\forall v \notin T$, $d_H(v)$ is even.
See 552 notes. $T - joins$ are not important in this class.

****
However this is important:
****
#3:
D^{rev} : reverse every edge of D.
Choose (some subset of edges from)a subgraph H of D^{rev} So that the net indegree of each vertex of H is given by its label.

Flow from s to t in D^{rev} whose volume is $\sum_{sv \in E} cap(sv)$. All the edges of D^{rev} have capacity 1.

So finding an eulerian orientation is trivial, given an orientation finding the closest Eurlerian orientation is an application of maxflow.

2 Counting the number of Eulerian Orientations of an Eulerian graph G

There exists a one to one correspondence between between Eulerian orientations of an (Eulerian) graph G, and perfect matchings in an auxiliary bipartite graph : $(E(G), d(v)/2\ copies\ of\ each\ vertex\ v\ of\ G)$.
$e = uv \in E(G)$ is joined to every copy of u and every copy of v.

one-to-one correspondence: e is joined to v in M iff e is directed towards v in the orientation. expected to understand this correspondence!

G is $k - edge - connected$ if there exists k internally disjoint $s - t$ paths between every two vertices of G.
D is $k - strongly - edge - connected$ if $\exists k$ internally disjoint directed paths from s to t $\forall (s, t) \in V$.

Theorem 2 *Any $2k - connected$ Eulerian graph has a $k - strongly - connected$ orientation.(any Eulerian orientation will do).*

Can ask same questions about *Acyclic Orientations*(look up definition) as we are asking about *Eulerian*. Any graph G gas acyclic orientation. How do we test if an orientation is acyclic? How many edges do we need to reverse to make an orientation acyclic?? NP complete(Feedback Arc Set).

Approx Counting Acyclic Orientations –open.

2.1 Eulerian Cycles

we have seen

Theorem 1 *G is Eulerian iff $E(G)$ can be partitioned into cycles iff every vertex degree is even.*

Theorem 2 *G has an Eulerian cycle iff G is Eulerian and connected up to isolated vertices.*

Theorem 3 *how many edges do we need to reverse?*

solved this by considering a min cost flow problem in auxiliary network with unit capacity 1 and cost 1. and adding source and sink.

It is easy to see that for every perfect matching we have $n/2$ near perfect matchings, thus the number of near perfect matchings $\geq$ # of perfect matchings $\cdot \frac{n}{2}$.

Assume G is Eulerian.

Perfect matchings correspond to Eulerian orientations of G.

Near perfect matchings correspond to orientations where edge e is not directed. One vertex v has in degree $d(v)/2 - 1$ all other w have degree $\frac{d(w)}{2}$.

Orienting e we either get an Eulerian orientation or an orientation where $d^-(v) = \frac{d(v)}{2} - 1$ $d^-(u) = \frac{d(u)}{2} + 1$

$\leq |E(t)||E(G)| + \sum_{u,v} AE(u,v)(\frac{d(u)}{2} + 1)$

Prove $\forall u, v$, $\frac{|AE(u,v)|}{|EO(G)|} \leq 1$

$(AE(u,v) \times decomposition\ of\ directed\ graph) \leq (EO(u,v) \times decomposition)$
decomposition of $E(G)$ into set of directed closed walks disjoint from $u \to v$. plus walks from u to v directed v to u internally , u to u disjoint, v to v from $\{u, v\}$.

for every orientation we have

$$\prod_{w \in V-u-v} (\frac{d(w)}{2}!)$$

join a to B if we can set a from b by reversing a $b \to u$ path of the decomposition. See paper on web page. Don't need to understand probabilistic stuff.

3 Hamilton Cycles

If G is a bipartite graph (A, B) with $|A| + |B|$ odd, or more generally $|A| \neq |B|$ then it does not have a perfect matching. It is NP complete to check if G is Hamiltonian.

Dirac 1 *Every vertex has degree at least* $\frac{n}{2}$ *G is Hamiltonian .*

Proof. G is connected. Take a longest path p in G with end points u and v. $N(v) \subseteq p, N(u) \subseteq p$ (otherwise wouldn't be longest). If there is a cycle through p then $V(p) = V(G)$ or p is not maximum, therefore G is Hamiltonian. ∎

Ore 1 *if for every pair of non adjacent vertices* u, v $d(u) + d(v) \geq n \Rightarrow$ *Himilton cycle*

Posa 1 $\forall 1 \leq k \leq \frac{n}{2}$

$$|\{v|d(v) \leq k\}| < k$$

then G has a Hamilton cycle.

3.1 Posa's Extension-Rotation Technique

Count [neighbours of u on path $\geq \frac{n}{2}$]
and [Vertices immediately following a neighbor of v on the path $\geq d(v)$]. These sets are disjoint and neither contains u. Therefore $|V(G)| \geq n + 1$

Proof. G is connected. Take p so that $deg(v) + deg(u)$ is as large as possible. claim, $d(v) \geq \frac{n}{2}, d(u) \geq \frac{n}{2}$. w_i are nodes on p such that we have edge pw_i. $\forall i d(w_i) \leq k$ by our choice of p.

4 Euler's Formula and Planar Graphs

$$|V| + |F| = |E| + 2$$

(cube, tetrahedron, dodecahedron, octahedron, icosahedron) Archimedian solids can only have $4, 6, 8, 12, 20$ sides. See text.

In a regular polyhedra, every vertex has degree d, and every face has $d \geq 3, s \geq 3$ s sides.

$$d|V| = 2|E|$$

$$s|F| = 2|E|$$

therefore

$$\frac{2}{d}|E| + 2\frac{|E|}{s} = |E| + 2$$

$$(\frac{2}{d} + \frac{2}{s})|E| = |E| + 2$$

So one of d or s is 3, and the other is 4 or 5.

$$3, 3 : \frac{4}{3}|E| = |E| + 2, |E| = 6$$

$$3,4: \frac{7}{6}|E| = |E| + 2, |E| = 12$$

$$3,5: \frac{16}{15}|E| = |E| + 2, |E| = 30$$

$$d = 3, s = 4 - -|V| = 8, |F| = 6$$

$$d = 4, s = 3 - -|V| = 6, |F| = 8$$

$$d = 3, s = 5 - -|V| = 20, |F| = 12$$

$$d = 5, s = 3 - -|V| = 12, |F| = 20$$

4.1 Drawing of a graph G

Drawing of a graph Vertices are distinct points in R^2. Edge e with endpoints x, y is a simple arc from x to y. These arcs are disjoint except at common endpoints.

Face of a drawing: Connected region when edges and vertices are removed.

Embedding of a Graph: Set of face boundaries. For two connected graphs, this is a set of directed cycles using each edge exactly once in each direction. if this holds and $|E| + |V| = |E| + 2$ then we have an embedding in the plane.

If G is connected then every face is a closed walk(possibly with repeated edges) . Every edge appears twice in face boundaries. If G is $2 - connected$ then face boundary is a cycle. Every edge is in exactly 2 faces.

Euler 1 *If G is planar then*

$$|V| + |F| = |E| + 1 + \#\ of\ components$$

Proof. Spanning forest–spanning tree of each component of G.
If G has k components $(V_1, ..., V_k$ and $n_i := |V[V_i]|$

$$|V| = \sum_{i=1}^{k} n_i$$

$$|E| = \sum_{i=1} (n_i - 1) = |V| - k$$

\# of faces is 1.

add remaining edges one by one. every time we do this 1 face splits into two and it continues to hold. ■

4.2 Which graphs are Planar?

If an induced cycle C of G is not a face then $G - V[C]$ can not be connected. In a clique with 4 vertices , every triangle is a face. $K_{3,3}, K_5$ are not planar.

Subdivision–add vertices to edges. Any graph containing a subdivision of $K_{3,3}$ or K_5 is not planar. This condition is necessary and sufficient.

Kuratowski 1 *G is planar iff G contains neither $K_{3,3}$ nor K_5 as a subdivision.*

Proof. Take a minimal counterexample M. M is connected(otherwise not minimal). Suppose we have a *cut vertex*, a vertex who's removal disconnects the graph(figure 8). then we can put v in the infinite face of each embedding. Same is true for edge cut. If there is a 3 cut which is a triangle we get a $K_{3,3}$ subdivision. If we have a 3 cut that is not a triangle we induct on the number of non edges. So it must be 4 connected. Contract an edge xy to obtain G^*. Need to show 1.that G^* has no $K_{3,3}$ or K_5 and is therefore planar. 2. From Every planar embedding of G^* we can construct one of
(a) a $K_{3,3}$ or K_5 subdivision in G.
(b) a planar embedding of G.

■

SEE HANDOUT INTRODUCTION TO ROUTING FOR COMPLETE PROOF

4.3 How many embedding does a planar graph have?

Embedding: Set of directed cycles which are face boundaries (2-connected graphs).

Whitney's 1 *A $2-$conencted planar graph has a unique embedding precisely if it is a subdivision of a 3 connected graph or a cycle, or the subgraph $x \rightarrow_{p1} y, x \rightarrow_{p2} y, x \rightarrow_{p3} y$ where pi is a path. (Faces are planar dual of triangle)*

Need to show $3-$*connected* graphs have unique embeddings. **Proof.** see notes.

If H has a vertex of degree 4, no subcubic($d(v) \leq 3 \forall v \in V$) graph has a subdivision of H.

G has a *minor* if we can obtain from G via a sequence of edge contractions, edge deletions and vertex deletions.

$e := xy$

Tutte 1 *G is connected. the number of spanning trees of $G = \#$spanning trees of $G - xy + \#"G^*_{xy}$*

a *model* of H in G is a function from $v(H) \rightarrow disjoint\ trees\ of\ G$
We have a quasi-order if G has F as a minor and F has H as a minor then G has H as a minor.

G has no $H_{minor} \forall H \in O \Rightarrow G$ has a nice structure.
G has no K_l minor $\Rightarrow$ """"
Because for $l = |V(H)|$, if G has no H minor then G has no K_l minor. Every graph with no K_5 minor is a subgraph of a graph obtained from planar graphs and an octagon with 4 diagonals by pasting together on cliques (of size ≤ 3).

Robertson -Seymore perhaps the most important theorem in graph theory

Theorem 3 *If G has no K_l minor then it is a subgraph of a graph* ***nearly embeddable*** *on a surface which K_l cannot be embedded by pasting together on (small) clique cutsets*

$nearly-embeddable$: bounded extension.

F is a minor-closed family if $\forall G \in F$, H a minor of $G \Rightarrow H \in F$. $O_F = \{H|H \in F\ but\ not\ all\ minors\ of\ H\ ar$
$G \in F$ iff G has no O_F as a minor.

Wagner's Conjecture 1 *(proved by Robertson-symore) In any infinite sequence $G_1, G_2, ...$ $\exists i \neq j$ such that G_i is a minor of G_j*

They also proved that for all H there exists a poly-time algorithm to test if G has a minor.

There is a polytime membership test for any minor closed family.

4.4 Routed Routing

Given $S := \{s_1, ..., s_k\}$, $T := \{t, ..., t_k\}$

Menger(undirected) 1 *Either $\exists k$ vertex disjoint $S - T$ paths in G, or $\exists X \subset V, |X| < k$ such that there is no $S - T$ path in $G - X$.*

Proved as an application of max flow min cut directing edges in both directions.

Poly Time Algorithm for finding v vertex disjoint paths from s to t

Given $S := \{s_1, ..., s_k\}$, $T := \{t, ..., t_k\}$ Determine if $\exists k$ vertex disjoint paths $p_1, ..., p_k$ such that s_i, t_i are the end points of p_i.

PTA algorithm for $k - DRP$ for fixed k.

Can use this algorithm to test if G contains H as a subdivision, and hence as a minor. $\forall H, \exists Z(H)$ such that G has H as a minor iff $\exists F \in Z(H)\ s.t.\ G\ has\ a\ subdivision$
F has H as a minor but does not contain a subgraph which contains H as a minor, and is not a subdivision of a graph which contains H as a minor. model of H in F every edge of F is an edge-image or an edge of a vertex image. Vertex image contains a vertex of degree 2 $im(v) \leq d(v)$ leafs, no vertex of degree 2 $\Rightarrow \leq 2d(v)\ vertices$.

Responsible for the one chapter of the text, and up to page 36.

the minor relation $H <_m G$ if H is a minor of G.

if H is cubic then $H = Z(H)$. G has at least one of $K_{3,3}$ or K_5 as a minor iff G has a subdivision of $K_{3,3}$ or K_5.

Lemma 4 *If G is obtained by identifying a clique of G_1 with a clique of G_2 then*

$$max\{l|K_l <_m G\} = max(max(l|K_l <_m G_1), max(l|K_l <_m G_2))$$

l is known as $Had(G)$ or Hadwiger number of G. **Proof.** Case 1: $Had(G) = |C_1|$.

Case 2: $Had(G) > |C_1|$, in this case there exists a vertex image disjoint from $G_1 \cap G_2$. By symmetry say it is contained in $G_1 - C_1$, this implies all vertex images disjoint from C_1 in $G_1 - C_1$. We claim that restricting each vertex to its intersection with G, yields a clique minor of the same order. ∎

Theorem 5 *G has no K_5 minor iff G is a subgraph of a graph which can be built up from planar graphs and* ***octagon with*** *4* ***diagonals*** *by pasting graphs together on cliques*

Proof. we have $\Leftarrow$ by the above lemma. If G has no K_5 minor, consider a minimal counter example M. M is 3 connected. It has no $3-cut$ X such that $G-X$ has ≥ 3 components. Add triangles to each component... in notes. ∎

Lemma 6 *If G contains L as a minor but no K_5 minor then it is not 3 connected.*

Lemma 7 *If G contains $K_{3,3}$ as a minor but $K_5$is not $<_m G$, L is not $<_m G$. Then there exists a cutset X of size 3 in G such that $G-X$ has at least 3 components.*

these lemmas help in the proof of the above theorem.

5 2-DRP

Theorem 8 *given G with disjoint $\{s_1, t_1, s_2, t_2\}$ Paths exists iff there is a K_5 minor attached to $A = \{s_1, t_1, s_2, t_2\}$ in G'.*

If $X \subseteq V$ satisfies $|X| \leq 3$ then for any K_5 model(minor) . There exists a component of $G-X$ which completely contains a vertex image. Large-component of $G-X$. The K_5 is attached to A if A intersects the large component of $G-X$ $\forall X \subseteq V$ *eith* $|X| \geq 3$

Lemma 1 *Suppose G is l connected, C is a clique cutset of size at least l in G and U is a component of $G-C$ then $G-C$ is l connected.*

Proof. $\Rightarrow$ rr.
$\Leftarrow$
Consider a minimal counter example.(K_5 minor attached, paths do not exist, $|V|$ minimum). Connected, 2-connected. 3 connected. No 3 cuts with one component disjoint from A. No 3 cuts separating A either.

Every $4-cut$ X in G' satisfies one of
(i) $X = A$
(ii) there exists exactly one component u of $G-X$ disjoint from A and $|U| = 1$ **Proof.** Suppose not, take a bad X(X is a 4-cut in G not satisfying $\exists 4$ paths from A to X because either (i),(ii)). G' is $3-connected.$ –in the notes ∎ ∎

6 Coloring

6.1 basic definitions

coloring,edge coloring, total coloring, list coloring. total coloring partitions the graph into stable sets and matchings such that the vertices in the matching are disjoint from the vertices in the stable set.

6.2 2 coloring

Build a DFS tree, take a pre-order.For any connected G and any v_n there exists an ordering $v_1, ..., v_n$ of V such that $\forall i < n$, $v_i, v_n \in E$ for some $j > i$.

6.3 3 coloring

NP- complete.

6.4 bounding the chromatic number

$\chi \geq \omega$
$\chi \leq \Delta + 1$ using greedy coloring.

Both bounds are tight if G is a clique.

Brook's 1 *$\chi \leq \Delta$ unless $\Delta = 2$ and some component is an odd cycle or some component of G is a $\Delta + 1$ clique.*

Proof. Consider a minimum counter example G. G is connected(or a component is a smaller counter example). If its one connected we can label 1 vertex separator and then paste the components to it.

Lemma 2 *In any 2 connected graph G which is not a clique and has maximum degree 3 $\exists x, y, z$ such that $x \sim y, x \sim z$*
$xy, xz \in E(G), yz \notin E(G)$, and $G - y - z$ is connected.

Proof. homework ■

we have $\Delta(G) \geq 3$
we have x, y, z as in the lemma. Order $G - y - z$ as

$$v_1, ..., v_n = x$$

so that $\forall i < n$ there exists $j > i$ such that $v_i, v_j \in E$. and then we can "'intelligently greedy color"' ■

For a 'typical' graph on n vertices, $w(G) \approx 2logn,\ \chi \approx \frac{n}{2logn}, \Delta(G) \approx \frac{n}{2}$

6.5 'Intelligent Greedy coloring'

Let v_i be min degree vertex in the subgraph of G induced by $\{v_1, ..., v_i\}$

$$\chi(e) \leq max_{1 \leq i}\{\delta(H_i) + 1\} = max_{H \subseteq G}\{\delta(H) + 1\}$$

For any H let i_H be maximum i such that $v_i \in H$, $\delta(H) \leq \delta_H(v_{i_H}) \leq d_{Hi_H}(v_{i_H}) = \delta(H_{i_H})$
See early chapter of coloring book..

Tait 1 *4 vertex(face) coloring of every planar triangulations equivalent to 3 edge coloring of every bridge less cubic planar graph.*

Theorem 4 *Every Hamilton cubic graph is* $3-edge\ colorable$

Proof.
Even number of vertices. 2-edge-color cycle, then rest is matching. ■

Tutte proved that every 4 connected planar graph is Hamiltonian.

Barnette conjectured that every cubic $3-edge$ connected bipartite planar graph is Hamiltonian– still open.

Conjecture: Every 2 edge connected graph without the Peterson graph as a minor has a nowhere zero 4-flow

7 Factoring Graphs

$r-factor = r-regular\ graph$ Partitioning an $r-factor$ into r $1-factors.$ $\chi_e(G)$

7.1 Motivations

Every bridgeless planar 3-factor can be partitioned into 3 one factors . Equivalent to $4CC$.

Peterson 1 *Factors in several graphs. Given an* $r-factor$ F *does there exist set* $s+t=r$ *and disjoint factor* F_s $t-factor$, $F=F_s+F_t$

on assignment we might want to show that every $2r$ factor can be partitioned into r $2-factors$
Konig: Every bipartite $r-factor$ can be partitioned into r 1 factors

Theorem 5 *A bipartite graph* $G=(A,B)$ *. Has a matching using all the vertices of* A *iff there does not exist* $X \subseteq A$ *such that* $N|(X)| < |X|$*. Halls theorem equivelent to Mengers*

$def(G)$**:** deficiency of G. $|V|-2|M|$ for a maximum matching M in G.
For bipartite graph (A,B), $def(A)=|A|-|M|$ for maximum matching M.

Theorem 6 *Let* $G(A,B)$ *be a bipartite graph. Then*

$$def(A)=max_{X\subseteq A}(|X|-|N(X)|,0)$$

$$def(A)\geq max_{X\subseteq A}(|X|-|N(X)|,0)$$

Proof. by Induction: Take a maximum matching in G, and a vertex $x\in A$ missed by M. Take a maximum augmenting path tree.

$$|A\cap T|=|B\cap T|+1$$

All edges from $A\cap T$ go to $B\cap T$. ■

Theorem 7 *Let* G *be a graph , then*

$$def(G)=max_{Z\subseteq v}(\#\ of\ odd\ components\ of\ G-Z)-|Z|)$$

Proof. Induction on $|E|$ rip out an edge $e = xy$. Consider Z demonstrating the theorem in $G - e$ chosen with $|Z|$ maximum. Every component of $G - Z$ is odd. $\forall X \in V(U_i)$, $U_i - X$ has a perfect matching. –take a minimal counter example rip it out, then the components are like 'super vertices' and we are basically in the bipartite case since each has a near perfect mathing. ■

Vizing 1 *Every graph G with $\Delta(G) \leq k$ satisfies $\chi_e(G) \leq k+1$*

Proof. by induction $e = vw_1$ color $G - e$ with $k+1$ colors. $k+1 > \Delta$ therefore a color α misses at v. There exists a color B_1 missing at w_1. For $i \geq 2$, vw_i has color B_{i-1}, B_i is missing at w_i. Distinct $B_1, ..., B_k$ uses Kempe Chains ■

8 Ideas from Linear Programming

(P)

$$max\ cx$$

$$s.t.\ Ax \leq b$$

$$x \geq 0$$

(D)

$$min\ bx$$

$$yA \geq c$$

$$y \geq 0$$

8.1 s-t Paths

$P_{s,t} = P_{s,t}(G)$: set of paths from s to t.

$$max \sum_{p \in P_{s,t}} x_p$$

$$s.t.\ \forall v \in V \sum_{p \ni v} x_p \leq 1$$

$$x_p \geq 0, x_p \in Z$$

8.2 Coloring

$S(G)$- set of stable sets of G

$$min \sum_{s \in S(G)} x_s$$

$$s.t. \forall v \in V, \sum_{v \in S} x_s \geq 1$$

$$x_s \geq 0\ x_s \in Z$$

Dual

$$max \sum_{v \in V} y_v$$

$$s.t.\ \forall s \in S(G) : \sum_{v \in S} \leq 1$$

$$y_v \geq 0, y_v \in Z$$

Can we solve the LPs efficiently? if we can separate we can use ellipsoid.

Integrality gap: optimal primal IP = optimal dual IP ?
optimal dual IP $\leq$ function of primal LP.

The Erdos Posa property: $\exists f\ s.t.\ F - cover\ number \leq f(F - packing\ number)$

Fractional $V.C$ NP complete to approximate.

$$\chi^f(G) \leq \chi(G) \leq log(n)\chi^f(G)$$

how do we get a $\Delta + 3$ total coloring? take a $\Delta + 1$ edge coloring $M_1, ..., M_{\Delta+1}$ (which we can by Vizing), take a $\Delta + 3$ vertex coloring $S_1, ..., S_{\Delta+3}$. So for $1 \leq i \leq \Delta + 1$, $1 \leq j \leq \Delta + 3$, $T_{i,j} = M_j - \{e | e \cap S_j \neq 0\} \cup S_j$

8.3 s-t Paths

there exists an optimum solution to the fractional matching problem where each $x_e \in \{0, 1/2, 1\}$

$$max \sum_{e \in E} x_e$$

$$s.t.\ \forall v \in V\ \sum_{e \ni v} x_e \leq 1$$

$$x_e \geq 0$$

Theorem 8 *For Bipartite graphs, max matching = min cover, so have integer optimum*

G' is bipartite, size of max fractional matching in G' is equal to the max fractional matching in G. $G' := \forall v_i v_j \in E(G)$, $v_i v'_j, v_j v'_i \in E(G)$.

8.4 Fractional edge coloring in P

$$min \sum_{M \in M(G)} x_M$$

$$s.t.\ \forall e \in E\ \sum_{e \in M} x_M = 1$$

$$x_M \geq 0$$

G has a fractional $C - coloring$ iff $(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}, ..., \frac{1}{c})$ is a convex combination of incidence vectors of matchings.

Edmond's Characterization of MPP(G) 1 *(Convex combination of incidence perfect matchings)*
$x \in MPP(G)$ *iff* $\forall v \in V\ \sum_{e \ni v} x_e = 1$
$\forall S \subseteq V, |S|$ *is odd,* $\sum_{e \in E(S,V-S)} x_e \geq 1$

Proof. Consider a minimal counter example (G, x) we have $\forall e \in E(G), x_e > 0$ Claim 1: $\exists$ a perfect matching M in G–easy consequence of when a graph does not have a perfect matching.
Claim 2: there exists a tight non-trivial odd cut $(S, V - S)$. ... See Vempala ∎

9 Matchings

Tutte-Berge Formula 1

$$def(G) = max_{Z \subseteq V}\{\#of\ odd\ components\ of\ G - Z - |Z|\}$$

Bipartite $G := (A, B)$
max size of matching = min size of a cover.

$$def(A) = max\{|x| - |N(X)|\}\ X \subseteq A$$

if $\Delta(G) \leq k$ then $\chi_e(G) \leq k + 1$.

9.1 Consequences

Theorem 9 *For any set* $S \subset V$ *there is a matching* M *saturating* S $\exists Z \subseteq V$ *such that* $G - Z$ *contains more then* $|Z|$ *odd components completely contained in* S.

$f : V \to Z^{\geq 0}$ we want to find a subgraph H of G such that $\forall v \in V$, $d_H(v) = f(v)$
G' has a perfect matching M' iff G has a matching M saturating S.

Tutte's f-factor theorem 1 *G has no $f - factor$ precisely if disjoint $X, Y \subseteq V$ such that*

$$\sum_{v \in X} f(v) + \sum_{v \in Y}(d(v) - f(v)) < |E(X,Y)| + \#\ of components\ K\ of\ G - Y - Y\ such\ that\ \sum_{v \in K} f(v) - |E(K,Y)|\ is\ odd$$

Proof. $\forall v \in V$ make $f(v)$ copies of v, $\forall e = vw \in E$ add vertices e_v, e_w with an edge between them and add an edge from e_v to every copy of v "' "' "' every copy of w has neighbours $w_1 - -w_{d(v)}$
G has an $f - factor$ H iff G_f has a perfect matching M. $e = vw \in H$ iff $e_v e_w \notin M$.
G has no $f - factor$ iff $\exists Z \subseteq V(G)$ such that # of odd components of $G_f - Z > |Z|$
Choose Z minimal with this property. Therefore if one copy of v is in Z, every copy of v is in Z. Therefore if e_v is in Z for some edge e containing v, then f_v is in Z for every edge f containing v. and every copy of v is not in Z.

$$|Z| = \sum_{v \in X} f(v) + \sum_{v \in Y} d(v)$$

$$X = \{v|all\ copies\ of\ v\ in\ Z\},\ Y = \{v|every\ e_v\ is\ in\ Z\}$$

$$Z = all\ copies\ of\ v \in X\ plus\ all\ e_v\ v \in Y$$

Cycle Basis 1 *If G is connected with n vertices and $m$edges we need $m-n+1$ fundamental set of cycles.*

proof is in text.

rip out the edges in G. Subgraph H^* of G^* with

$$|E(H^*)| - |V(H^*)| + 1 = r$$

then the number of components increases by r.

Matroid 1 *Ground set X, A family I of independent subsets of X :*
(A) $I \neq \emptyset$
(B) $S \in I, I \subseteq S \Rightarrow T \in I$
(C) $X, Y \subseteq I, |X| = |Y| + 1, \exists x\ inX - Y\ s.t. Y \cup \{x | isinI\}$
therefore all independent sets have the same size.

uniform matroid: $I = S \subseteq X, |S| \leq r, r > 1, x \neq \emptyset$

Set of acyclic subgraphs of a non-empty undirected graphs. *Graphic Matroids*.

MATROID DUALITY: *Circuit:* minimal non-independent set. Dual m^* of m: the circuits of m^* are the minimum sets hitting all maximal independent sets of G.

remember cycle basis not matroids...

10 Perfect Graphs

G is perfect if and only if $\forall H \subseteq_i G\ \chi(H) = \omega(H)$. if and only if the polytope:

$$\forall S \in S(G) \sum_{v \in S, x \geq 0} x_v \leq 1$$

has integer vertices. Proof used the *replication* lemma. (replication preserves perfection.) G is perfect iff $\forall H \subseteq_i G\ \exists$ a stable set meeting every maximum clique of H.

Two conjectures:

WPGC: G is perfect iff $\bar{G}$ is perfect.

SPGC: G is perfect iff C_{2k+1} *is not* $\subseteq G_i$ *or* $\bar{G}$ $k \geq 2$.

10.1 Classes of perfect Graphs

Bipartite Graphs. Comparability Graphs. Triangulated graphs: C_k *not* $\subseteq_i G$ $k \geq 4$

Lemma 3 *Every triangulated graph contains a simplicial vertex.*

10.2 Proof of the Weak perfect graph conjecture(Lovasz 72)

Need to show if G is perfect $\forall H \subseteq_i \bar{G}$. There exists a stable set meeting all the maximum cliques of H.

Enough to show: $(F = \bar{H})$ F is perfect $\Rightarrow$ there exists a clique of F meeting all the maximum stable sets of F.

Let T be the number of $\alpha(G)$ stable sets, and t_x be the number of $\alpha(F)$ stable sets containing x $(\forall x \in V)$. Obtain F' by replicating $x \in V$, $t_x - 1$ times. F' is perfect(replication lemma)

$$|V(F')| = T|\alpha(G)|$$

$$T \leq \chi(F') \leq T \textit{ in other words omega}(F') = \chi(F') = T$$

shrink it down to a clique of F

■

G is called *Berge* if C_{2k+1} *not* $\subseteq G,$ *or* $\bar{G}$ $k \geq 2$

G is $i-triangulted$ iff every odd cycle of G has 2 non-crossing chords $\Leftrightarrow_i G$ $k \geq 2$. No $C_5 \Rightarrow \bar{C}_5$ No $P_5 \Rightarrow$ no $\bar{P}_5$

hereditary Clique-Separable

Every $H \subseteq_i G$ either H has a clique cutset or H is one of two base classes: Complete multipartite graphs:$S_1, ..., S_k$ stable all edges fromS_i to S_j for $i \neq j$

Set of universal vertices(vertices that see all other vertices of G) added to a bipartite graph.

$i - triangulated$ graphs are clique separable and hence perfect.

10.3 Polynomial time algorithm for determining and finding if G gas a clique cutset(Sue Whitesides)

First find either a clique cutset of G or an induced $C_k, k \geq 4$. can assume that there is a v that is not universal. let U be a component of $G - v - N(v)$. If $\{x | x \in N(v), \exists y \in U \ s.t. \ xy \in E(G)\}$ is a clique C, C is a clique cutset. Set H_1 to be this subgraph. Iteration 1: $H_i \subseteq_i G$ with no clique cutset which is not a clique. If $H_i = G$, we are done(no clique cutset in G). (Otherwise) let U be a component of $G - V(H_i)$. If the attachments of U in $V(H_i)$ induce a clique C then C is a clique cutset, separating U from $H_i - C$. Otherwise add a shortest path p through U between two non-adjacent attachments. Consider any clique C of $H_i + P$. $C \cap P$ is an edge or vertex of P. and all of $P - C$ is in the same component of $H_{i+1} - C$ as x or y. therefore $H_i - C$ has only one component.

this gives us an On^3 algorithm for testing if G has a clique cutset. Rooted tree- every node labeled by a subgraph of G. internal nodes also labeled by cliques. G is clique separable iff leafs are in one of the base classes for a clique cutset.

11 Clique separable graphs. Clique cutset trees.

Comparability graphs. G is a comp graph if it can be given an orientation which is transitive.

12 Comp Graphs

Determine if G gas an orientation without $a \to b \to c$

ab directed forces cb ba directed forces bc cb directed forces ab bc directed forces ba

Auxiliary digraph G'

$$V(G') = \{\vec{ab} | (a, b) \in E(G)\}$$

$(\vec{ab}, \vec{cd}) \in E(G) iff\ ab\ directly\ forces\ \vec{cd}$ E.C. are connected components.

Determine if G gas an orientation without $a \to b \to c$. If not then it is not a comp. graph. If yes then check if it contains a directed triangle. If not we are done, if yes decompose. A homogeneous set(module) in a graph G is a subset H of V such that $\forall x \in V - H$, either $\{xy \in E\ \forall y \in E\}\ or\ \{xy \notin E \forall y \in E\}$ with $2 \leq |H| < |V|$. If G has a homogeneous set H then for any $v \in H$, G is a comp graph iff $G[H]$ is a comp graph and $G - (H - v)$ is a comp graph.

Lemma 4 *If G has an orientation with no $a \to b \to c, a \to c$ but have a directed triangle then it has a homogeneous set.*

Proof. No $\forall e V - x - y - z$ sees only one of $\{x, y, z\}$. If $\exists v$ seeing exactly 2 of $\{x, y, z\}$ we have a homogeneous set or $V = \{x, y, z\}$. By symmetry can assume there exists a v with $vx, vy \in E(G), vz \notin E(G)$ Let F be the set of vertices which see both x and y. Let H be the connected component of $G[F]$. Algorithm
Build a homogeneous set decomposition for G. Claim H is a homogeneous set. Otherwise there exists a vertex v of $G - F$ and an edge sz of $G\bar{[}H]$ such that t sees s but not z. ∎

Furthermore there is a polytime algorithm to find such a set.

Algorithm

Check if there is a orientation with no $\to\to$. No $\Rightarrow$ G not comp. Yes $\Rightarrow$ check if triangle:No $\Rightarrow$ Done. Yes$\Rightarrow$ find a homogeneous set H . Solve recursively on $G[H]$ and on $G - (H - v)$.

Homogeneous set tree: $G[H] \leftarrow (G, H) \to G - (H - v)$. dont decompose if $|V| = 2$. $\forall G$ with $|V| \geq 2$ have at most $|V| - 2$ nodes in a $h.s$ decomposition tree with atleast 2 nodes.

G is perfect iff the leaves of homogeneous set decomposition tree are. $\equiv$ No min graph has a homogeneous set.

Star Cutset Cut set C containing a center v such that $vx \in E \forall v \in C - x$.

13 Tree Decompositions

starting with chapter 9 in hand hout.

13.1 Helly Property for Trees

If F is a family of subtrees of a tree T s.t. $, s' \in F, S \cap S' \neq \emptyset$ then

$$\cap_{S \in F} S \emptyset$$

Proof. By Induction. Let l be a leaf of T. If $l = N(T)$ then we are done. If $\exists S \in F$ such that $N(S) = \{l\}$ then we are done. Otherwise $\forall S \subseteq F, S - l$ is a subtree of $T - l$. Furthermore if $l \in S \cap S'$ for $S, S' \in F$, then $(S - l) \cap (S' - l)$ contains the unique neighbor of l in T and hence is non empty. ∎

13.2 S.I.R

A subtree intersection representation for G is a tree T and a family $\{S_v|v \in V(G)\}$ of subtrees of T such that $S_u \cap S_v \neq \emptyset \Leftrightarrow uv \in E(G)$

Theorem 10 *G has a S.I.R iff G is chordal (triangulated).*

Proof. $\Leftarrow$ if G is a clique use a 1 node tree T. Otherwise G has a clique cutset G_A has a $S.I.R$ $\{T^1, \{S_v^1|v \in A \cup C\}\}$, $\{T^2\{S_v^2|c \in B \cup C\}\}$ ...proof in handout. ■

Width of a tree decomposition: $max_{t \in N(t)}\{S_v|t \in S_v\} - 1 = \omega(G) - 1$.

13.3 Subtree Decompositions

A subtree decomposition for G consists of a tree T and a family $\{S - [v|v \in V\}$ of subtrees of T such that $uv \in E \Rightarrow S_u \cap S_v \neq \emptyset$.

Every graph has a subtree decomposition where T has one node and $S_v = T \; \forall v \in V$.

$$width = max_{t \in N(t)}|W_t| - 1 = min_{H \supseteq G, H-chordal} = tree\ width\ of\ t\ min\ width\ of\ a\ tree\ decomp$$

where $W_t = \{v|S_v \ni t\}$

Minimum Weight Stable set problem: $\quad$ *Graph $G, W(v) \in Z^+ \forall v \in V$ find a stable set*

S maximizing $\sum_{v \in S} W(v)$. Root the tree at r. $\forall t \in T \; T - t$ is the subtree rooted at t. we use dynamic programming. See handout(wendsday feb 8th).

Min wight stable set for a chordal graph G(weights $f(v)$) S.I.R $[T, \{S_v|v \in V\}]$ root T at r. T_t as before. $G_t = G[\{v|S_v \cap T_t \neq \emptyset\} \; \forall t \in T, v \in W_t$ we will compute $F(t, v) = max(f(S)|S\ is\ stable, S \subseteq G_t, S \cap W_t = v\} - F(t, \emptyset)$.

traverse in post order: If t is a leaf, $F(t, v) = f(v), F(t, \emptyset) = 0$ if t is not a eaf

$$F(t, v) = f(v) + \sum_{C\ a\ child\ of\ T, v \in W_c \cap W_t} (F(c, v) - f(v))$$

$$+ \sum_{c\ a\ of\ t\ v \notin W_c \cap W_t} max(max(F(c, w)|w \in W_c - W_t)F(c, \emptyset)$$

14 Subtree Decompositions

$$[T, \{S_v|v \in V(G)\}]_{uv \in E \Rightarrow S_u \cap S_v \neq \emptyset}$$

$$w_t = \{v|t \in S_v\} \Leftrightarrow [T, \{w_t|t \in M_t\}]$$

Each edge corresponds to a cutset separating G 'like' the edge separates the tree.

$\forall H \subseteq G$, H connected then $S_H = \cup_{v \in H} S_v$ is a subtree of T. Width of $[T, \{S_v|v \in V\}] = max_{t \in N(t)}|W_t| - 1$. Tree width of G $TW(G)$ is min width of a tree decomp of G. $= max_{H \supseteq G, H\ chordal} \omega(H) - 1$.

14.1 MWSS on Graphs of BTW

Root T at some r, T_t =maximal subtree rooted at t.

$$G_t = G[\{v|S_v \cap T_t \neq \emptyset\}]$$

$$\forall S^* \subseteq W_t,\ F(S^*, T) = max_{v\in S}\{\sum f(v)|S\ stable, S\cap W_t = S^*\}$$

If t is a leaf $F(S^*, T) = \sum_{v\in S^*} f(v)$ Otherwise

$$\sum_{v\in S^*} f(v) + \sum_{c\ a\ child\ of\ v} max(F(S', c) - \sum_{v\in S"'\cap S^*} f(v))$$

$\frac{\text{Time Taken(if } G \text{ has tree width } k)}{\sum_{t\in T} 2^{k+1}(\sum_{c\ is\ a\ child\ of\ T} O(2^{k+1}k))}$:

$$= \sum_{c} \sum_{t|c\ is\ a\ child\ of\ t} O(2^{2k+2}k)$$

$$= O(2^{2k+2}k|V|)$$

because can choose T so

$$|N(T)| \leq 2|V(G)|$$

coloring as an example might be on exam...

Bramble B: Set of cennected subgraphs of G every two of which touch: intersect or are joined by an edge. node images in a K_l model. For any $S \subseteq V$

$$B_S = \{H \subseteq G|H\ connected, |H \cap S| > \frac{1}{2}|S|\}$$

Order of B = min size of a hitting set for B: $H \subseteq V s.t.\forall b \in B\ H \cap b \neq \emptyset$
node images in a K_l model.

$$BN(G) := max\ order\ of\ a\ bramble\ in\ G$$

Theorem 9 $TW(G) = BN(G) - 1$

Proof. of $TW(G) \geq BM(G) - 1$ (easy direction) For any tree decomposition $[T, \{S_v|v \in V\}]$ of G and bramble B of G, $\exists$ a node t of T such that w_t is a hitting set for B.

$$BN(G) = ord(B) \leq |w_t| \leq width\ of\ tree\ dec + 1 = TW(G) + 1$$

One proof: $\forall b \in B, S_b\ is\ a\ subtree\ of\ T$. $\forall b_1, b_2 \in B\ S_{b1} \cap S_{b2} \neq \emptyset$ because b_1, b_2 touch. By Helly property there exists t such that $t \in \cap_{b\in B} S_b$

Other direction in 10.2 ■

Theorem 10 $TW(G) \leq 3BN(G) + 2$

Algorithmic **Proof.** ($4BN(G)+3$)

$$TW(G) \leq 3(max_{S \subseteq V} order(B_S)) + 2$$

Theorem 11 *If* $\forall S \subseteq V, \exists X \subseteq V, |X| \leq k$ *such that every component* U *of* $G - X$ *satisfies*

$$|U \cap S| \leq \frac{1}{2}|S|$$

Then G *has a tree decomposition of width* $\leq 3k - 1$

X is a hitting set for B_S iff every component U of $G - X$ satisfies

$$|U \cap S| \leq \frac{1}{2}|S|$$

15 tree decompositions

$$[T, \{S_v | v \in V\}]_{w \in E \Rightarrow S_u \cap S_v \neq 0}$$

$width = max|W_t| - 1 (w_t = \{v | t \in S_v\})$

Bramble: Set of B connected subgraphs every two of which touch.

Order of B = min size of a hitting set for B. max order of a bramble in G = min width of a tree decomposition for $G + 1$.

We Proved:
$BN(G) \leq$ tree width of $G \leq 4\ max_{S \subseteq V} order(B_S)$

$$B_S = \{H \subseteq G, H\ connected\ |H \cap S| > \frac{1}{2}|S|\}$$

gave a linear time algorithm to find tree decomposition of width k for graphs of tree width $k(k - fixed)$. = using this decomposition can optimize on these graphs in linear time.

$$h(G) = max\{k | G\ has\ k \times k\ grid\ minor\}$$

$$h(G) \leq TW(G) \leq 2^{20h(G)^5}$$

Does G gave a planar graph H as a minor??
Erdos - Posa property holds for cycles.
Either k vertex disjoint cycles OR $f(k)$ vertices which intersect all cycles.

True if $TW(G) > w = 2^{20(2k)^5}$ **Proof.** by induction for graphs with tree width $\leq w$. $f(k) = 3^k w$

16 Random Graphs and the Probabilistic Method

$$G_{n,1/2} : uniformly\ chosen\ graph\ on\ V = \{1, ..., n\}$$

$P = 2^{-\binom{n}{2}}$

$$E(\#\ of\ edges) = p\binom{n}{2}$$

$$E(\#\ of\ triangles) = p^3\binom{n}{3} = \sum_{x \neq y \neq z \subseteq V} \prod_{\Delta(x,y,z)}$$

Theorem 12 *Markov: if* $X \geq 0$ *then* $P(X \geq tE(X)) \leq \frac{1}{t}$

Theorem 13 *Chebyshev*
$P(|Z - EX(Z)| \geq t) \leq (\frac{t^2}{E(Z^2) - E(Z)^2})^{-1}$

$BIN(n, p) := sum\ of\ n\ independent\ 0, 1\ random\ variables$

Theorem 14 *Chernoff Bounds*
$P(|BIN(n, p) - pn| \geq t) \leq e^{-\frac{t^t}{3np}}$

$$E(\#\ of\ k\ cliques\ in\ G_{n,\frac{1}{2}}) = \binom{n}{k}2^{-\binom{k}{2}}$$

assume $k \leq 3logn$

$$\approx (1 + o(1))\frac{n^k}{k!}2^{-\binom{k}{2}}$$

$$= (1 + o(1))2^{klog(n)+klogk-log_2ek+(1/2)logk+o(1)-k(k-1)/2}$$
$$= 2^{k(logn+logk-log_2e+1/2+o(1/k)-(k-1)/2)}thisgoesbelow$$

1.

$$k - 1 = 2logn + 2logk - 2log_2e + 1 + O(1/k)$$

$$k = 2logn + 2loglogn + O(1)$$

For n large enough, Almost every graph on n vertices satisfies

$$\alpha(G) \leq 3logn, \omega(G) \leq 3logn$$

$$\Rightarrow \chi(G) \geq \frac{n}{3log(n)}$$

Hajos's Conjecture:

$$\chi(G) \geq l \Rightarrow$$

G contains a subdivision of K_l.

asymptotically almost surely $\chi(G_{n,1/2}) \geq \frac{n}{3logn}$ a.a.s $G_{n,1/2}$ does not contain a $K_{10\sqrt{n}}$ subdivision.

$G_{n,1/2}$ is a.a.s. Hamiltonian

$G_{n,1/2}$ a.s.s. satisfies (1) $\forall u, v\ N(u) \cup N(v) \geq \frac{2n}{3}$
(2)$\forall u\ N(v) \geq \frac{n}{3} + 4$

$$E(\#of\ vertices\ of\ degree \leq \frac{n}{3} + 4) \rightarrow 0\ as\ n \rightarrow \infty$$

$$= nP(1\ has\ degree \leq \frac{n}{3} + 4) \rightarrow 0\ as\ n \rightarrow \infty$$

$$\leq nP(|BIN(n-1, \frac{1}{2}) - \frac{n-1}{2}| \geq \frac{n}{6} - \frac{9}{2}$$

$$\leq ne^{blah}$$

$$E(\#\ of\ bad\ pairs) \leq \binom{n}{2} P(|BIN(n-2, 3/4) - 3/4(n-2)| \geq n/12 - 3/2)$$

$G_{n,1/2}$ a.a.s has chromatic index Δ

$G_{n,1/2}$ a.a.s. has a unique vertex of maximum degree.

$$E(degree = \frac{n-1}{2})$$

$$P(\exists\ a\ vertex\ of\ degree \geq \frac{n}{2}\sqrt{n}logn$$

$$\leq E(\#\ of\ vertices"`)$$

$$\leq n(P(1"')$$

$$\leq nP(|BIN(n-1, 1/2) - \frac{n-1}{2}|) \geq \sqrt{n}logn$$

$$\leq nn2e^{-\frac{(\sqrt{n}logn)^2}{something}}$$

Markov's Inequality: $X \geq 0, P(X \geq t) \leq \frac{E(X)}{t}$
In particular if X is integer valued $P(X \neq 0) \leq E(X)$.

Chernoff's Bound: $P(|BIN(n,p) - np| \geq t) \leq 2e^{-\frac{t^2}{3np}}$

$$\exists k_n = 2logn - 2loglogn + O(1)$$

such that $E(\# \ of \ cliques \ of \ size \ k_n) < 1$ but the expected number of cliques of size $k_n - 3 > n$ $\Rightarrow a.a.s. \omega(G), \alpha(G) \leq 2logn \Rightarrow a.a.s. \ \chi(G) \geq \frac{n}{2logn}$

a.a.s *not exist*$C \subseteq V$ does not contain s.t. $|E(G[C]| \geq 48n$

$$|C| = \lceil 10\sqrt{n} \rceil$$

therefore a.a.s. G contains no $K_{10\sqrt{n}}$ subdivision. So almost every graph is a counter example to Hajos' conjecture.

$\exists$ Triangle free graph with arbitrarily large chromatic number $\forall k$ there exists a graph with no K_3 and no stable set of size $\geq \frac{|V|}{k}$.

Proof.
$G_{n,n^{\frac{-2}{3}}}$ has $\leq \frac{n}{2}$ triangles (in book) and no stable set of size $\frac{n}{2k}$
expected number of triangles in $G_{n,n^{\frac{-2}{3}}}$ is $\binom{n}{3}(n^{-2/3})^3 \leq \frac{n}{6}$.

$$P(G_{n,n^{\frac{-2}{3}}} \ has > n/2 \ triangles) \leq \frac{1}{3}$$

$$P(G_{n,n^{\frac{-2}{3}}} \ has \ a \ stable \ set \ of \ size \ \frac{n}{2k})$$

$$\leq E(\# \ of \ stable \ sets \ of \ size \ \lceil \frac{n}{2k} \rceil \ in \ G_{n,n^{\frac{-2}{3}}})$$

$$= \binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}} \leq 2^n e^{-p\frac{n^2}{16k^2}} \ for \ n \geq 2 \ large = 2^n e^{-n^{4/3}/(16k^2)} \ if \ n > (16k^2)^3 < 2^n e^{-n} < \frac{1}{3} \ for \ n \ large \ enough$$

see book ■

Hadwiger's Conjecture: $\chi(G) \geq l \Rightarrow K_l \ minor$

We will show

$$(G) \geq 100l \cdot log(l) \Rightarrow col(G) \geq 100l \cdot log(l)$$

$$col = max \ _{H \subseteq G}(\delta(H) + 1)$$

$$min(deg(G)) \geq 100l \cdot log(l) \Rightarrow K_l \ minor$$

$$avg \ degree(G) \geq 100 \ l \cdot log(l) \Rightarrow K_l \ minor$$

see chapter 6 .

16.1 Maximum degree of $G_{n,\frac{1}{2}}$

Chebychev: $P(|Z - E(Z)| \geq t) = P((Z - E(Z))^2 \geq t^2) \leq \frac{(E(Z^2) - E(Z)^2)}{t^2}$

Suppose Z is the sum of symmetric indicator variables in some set U of events.

$$\sum_{\{X,Y,Z\}\subseteq V} \prod_{\Delta(X,Y,Z)}$$

$$Z^2 = \sum_{outcomes \; X} \sum_{A\in U} \sum_{B\in U} P(X) \prod_A \prod_B$$

$$= \sum_{A\in U} \sum_{B\in U} \sum_{outcomes \; X} p(X) \prod_A \prod_B = \sum_{a\in U} \sum_{B\in U} P(A\cap B)$$

$$= \sum_{A\in U} \sum_{B\in U} P_A P(B|A)$$

$$=_{A\in U} P(A)(\sum_{B\in U} P(B|A))$$

$$= \sum_{A\in U} P(A)(E(Z|A))$$

$$\sum_{A\in U} P(A)E(Z|A_0)$$

$A_0 := specific \; event \; in \; U$
$E(Z)E(Z|A_0)$ if $E(Z|A)$ is $(1+o(1))E(Z)$.