Combinatorics	**Scribe: Matt Drescher**
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1 Counting Spanning Trees

Problem 1 (number of labeled spanning trees) *Given n labeled vertices $v_1, ..., v_n$ How man different spanning trees are there?*

Theorem 1 (Cayley) *there are n^{n-2} labeled trees on n vertices.*

Proof. (Bijection) Let T_n be the set of labeled trees on n vertices, and $C_n(n-2)$ be the set of $(n-2)$ element sequences with alphabet $\{1, 2, ..., n\}$. We give bijection $C : T_n \leftrightarrow C_n(n-2)$. Given $t \in T_n$ create a code $C(t) \in C_n(n-2)$ as follows:

1. let $t_1 = t$
2. for $i = 1$ to $n-2$
   (a) Let v be the largest leaf in t_i.
   (b) let (u, v) be the edge in t_i
   (c) set $c_i = u$
   (d) set $t_{i+1} = t_i - (u, v) - v$

We show this is a bijection by giving its inverse. Given $C \in C_n(n-2)$ let $[n] := \{1, 2, ..., n\}$

1. For $j = 1$ to $n-2$
   (a) let $l_j = max\{[n] - \{l_1, ..., l_{j-1}, c_j, c_{j+1}, ..., c_{n-2}\}\}$
   (b) let (l_j, c_j) be edge of T
2. the last edge is (v, c_{n-2}) where $v = [n] - \{l_1, ..., l_{n-2}, c_{n-2}\}$

Claim 1 *1. Any number not in $(c_i, c_{i+1}, ..., c_{n-2})$ is a leaf in $t_1, t_2, ..., t_i$*

2. Any number in $(c_i, ..., c_{n-2})$ is an interior vertex of t_i.

Proof.

- By construction c_j in $C(T)$ is an internal vertex in t_i
- Any interior node in t_i will appear in $(c_i, ..., c_{n-2})$ as it has degree 0 or 1 in T_{n-2}. It had $deg \geq 2$ in t_i so we must remove a (leaf) neighbor later on. In other words the number of times a node of t_i appears in $(c_i, ..., c_{n-2})$ is equal to its degree minus one. ■

Therefore $l_1 = max\{[n] - \{c_1, ..., c_n\}\}$, and more generally

$$l_i = max\{[n] - \{\{c_i, ..., c_n\} \cup \{l_1, ..., l_{i-1}\}\}\}$$

Thus given $C(T)$ we can uniquely determine l_i. In other words our function is reversible. ■

Proof. (Generating Functions) Let $t(n; d_1, ..., d_n)$ be the number of labeled trees in which vertex v_i has degree d_i.

$$\gamma_n = \sum_{d_1,...,d_n} t(n; d_1, ..., d_n)$$

We can assume $d_1 \geq d_2 \geq ... \geq d_n = 1$. So $d_n = 1$ and so v_n has a neighbor. It could be any of the other vertices. Removing v_n and conditioning on the possible neighbor we have:

1. $t(n, d_1, ..., d_n) = \sum_{i=1}^{n-1} t(n-1, d_1, ..., d_i - 1, ..., d_{n-1})$
2. Now consider the multinomial coefficients $\binom{m}{a_1,a_2,...,a_k}$ The number of ways to pick k disjoint subsets $S_1, ..., S_k$ from $[n]$ of sizes $a_1, ..., a_k$
3. We know **Multinomial Theorem:** $(x_1 + x_2 + ... + x_k)^m = \sum_{a_1,...,a_k} \binom{m}{a_1...a_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$
4. Since $(x_1 + x_2 + ... + x_k)^m = (x_1 + x_2 + ... + x_k)^{m-1}(x_1 + x_2 + ... + x_k)$ it follows that $\binom{m}{a_1...a_k} = \sum_{i=1}^{k} \binom{m-1}{a_1,...,a_i-1,...,a_k}$
5. Induction $t(n, d_1, ..., d_n) = \binom{n-2}{d_1-1,d_2-1,...,d_n-1}$ or let the children of each node v denote a set S_v. Since there are 2 leaves in any tree the claim follows.
6. Base case $n = 3$ works.

recurrences 1,4 are the same so the claim is true by induction. Finally setting $x_i = 1, k = n, m = n-2, a_i = d_i - 1$ we have

$$n^{n-2} = \sum_{d-1} \binom{n-2}{d_1 - 1, ..., d_n - 1} = \sum_{d} t(n; d_1, ..., d_n) = \gamma(n)$$

■

Proof. (double counting) Consider a more complicated problem. Let $F_{n,k} = \#$ forests with k rooted trees. $|F_{n,1}| = n|T_n|$ Take Forest $F_{n,k} \in F_{n,k}$ direct edge away from roots. We say F_i contains F_j if F_i contains F_j as a directed subgraph. We say $F_1, ..., F_k$ is a refining sequence if $F_i \in F_{n,k}$ and F_i contains $F_{i+1} \forall i$. Fix a forest $F_k \in F_{n,k}$ set

1. $N(F_k) = \#$ rooted spanning trees containing F_k
2. $N^*(F_k) = \#$ refining sequences in in F_k

We count $N(F_k)$ in 2 ways.

1. Start at spanning tree. Suppose $F_1 \in F_n$ contains F_k. $F_1 - F_k$ contains $k-1$ edges. We can remove them in any order to get a refining sequence from F_1 to F_k $N^*(F_k) = (k-1)!N(F_k)$
2. Start at F_k to get an F_{k-1} from F_k.

3. Pick any v add an arc from v to one of the other $k-1$ roots.

Can do this in $n(k-1)$ ways. So

$$N^*(F_k) = n(k-1)n(k-2)...n(1)$$
$$= n^{k-1}(k-1)!$$
$$n^{k-1}(k-1) = (k-1)!N(F_k)$$

For $k = n$ $N(F_n) = n^{n-1}$ So F_n = set of n singleton vertices. So $N(F_n) = \#$ of rooted spanning trees $= n^{n-1}$ therefore $\gamma = n^{n-2}$ ■

2 Enumerative Combinatorics

The main question in enumerative combinatorics is to count the number of objects in a set. Often we have an infinite collection $S_1, ...$ of sets and we try to count the number of items $f(i)$ in S_i(err $|S_i|$?) simultatiously for all i. Counting can be done in many ways.

1. Closed formula(nicest way but rare). e.g.
   - $f(n) = |power.set[n]| = 2^n$
   - $f(n) = |T_n| = n^{n-2}$ where $T_n :=$ labled trees on n vertices.
   - $f(n) = \#0,1$ matrices such that each row sum and, each col sum $= 3$ this is $\frac{1}{6}\frac{\sum_{a,b,c:a+b+c=n}(-1)^3 n!^2(b+3c)!2^a3^b}{a!b!c!6^c}$
   - $f(n)$ = number of ways a postman can deliver n letters tp all the wrong houses. $= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$ *A derangement is a set of non empty cycles*
2. By recurrence: A recurrence formula often allows us to find $f(n)$. e.g. $g(n) = \#$ of subsets of $[n]$ that dont contain two consecutive integers. $g(n) = g(n-1) + g(n-2)$ (consider n)
3. Asymptotic Formula: We say $f(n) \approx g(n)$ if $lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$. This gives an estimate $g(n)$ of $f(n)$. e.g. $f(n) \approx e^{-2}36^{-n}(3n)!$ if $f(n) = \#0,1$ matrices such that each row sum and, each col sum $= 3$
4. **Generating Functions:** this is the most useful way. We count an object using a formal power series.

2.1 Generating functions

Definition 1 (Ordinary GF) $F(x) = \sum_{n\geq 0} f(n)x^n$

Definition 2 (Exponential GF) $F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$

The major advantage with generating functions is that we can perform many *combinatorial* operations on them. e.g. addition multiplication, convolution, calculus.

From this we can extract information:

1. find exact formulas
2. find recurrences
3. find asymptotics
4. statistical properties
5. prove unimodal/convex properties
6. **proving combinatorial identities**
7. Allows us to tackle much harder problems.

EXAMPLE: consider

$$e^x \cdot e^{-x} = 1$$

$$1 = \sum_{n\geq 0} \frac{x^n}{n!} \cdot \sum_{n\geq 0} (-1)^n \frac{x^n}{n!}$$

$$= \sum_{n\geq 0} (\sum_{r=0}^{n} \frac{(-1)^r}{r!n-r!}) x^n$$

$$= \sum_{n\geq 0} [\sum_{r=0}^{n} (-1)^r \binom{n}{r}] \frac{x^n}{n!} = 1$$

$$\Rightarrow \sum_{r=0}^{n} (-1)^r \binom{n}{r} = 1 \ if \ n \ = 0, \ 0 \ otherwise$$

We have shown that the number of even sized subsets is equal to the number of odd sized subsets of $[n]$.

3 Compositions

A *composition* of an integer n is an expression of that integer as a sum of positive integers. e.g.

$$3 = 1 + 1 + 1, 2 + 1, 1 + 2, 3$$

$$4 = 1 + 1 + 1 + 1, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2, 3 + 1, 1 + 3, 2 + 2, 4$$

A composition of n has k pars if $n = x_1 + x_2 + ... + x_k$ Let $g(n) = \#$ compositions of n into parts all of value 1 or 2. Let $G(x)$ be the ordinary generating function for g(O.G.F). We have

$$G(x) = \sum_{n\geq 0} g(n)x^n = \sum_{k\geq 0} (x + x^2)^k$$

More generally if $f(n) = \#$ of compositions of n into parts that belong to a set A of integers, then

$$F(x) = \sum_{k\geq 0} (\sum_{a\in A} x^a)^k$$

e.g. if $A := \{2, 3, ...\}$ we have $F(x) = \frac{1-x}{1-x-x^2} = 1 + x^2 \cdot \frac{1}{1-x-x^2}$

$$G(x) = \sum_{k\geq 0}(x+x^2)^k = 1 + (x+x^2)\sum_{k\geq 0}(x+x^2)^k$$

$$G(x) = 1 + (x+x^2)G(x)$$

so

$$F(x) = 1 + x^2\frac{1}{1-x-x^2} = 1 + x^2G(x) = 1 + x^2\sum_{n\geq 0} g(n)x^n$$

$$= 1 + \sum_{n\geq 2} g(n-2)x^n = \sum_{n\geq 0} f(n)x^n$$

thus

$$f(n) = g(n-2)$$

Theorem 2 *# of compositions of n into parts greater then 1 equals the number of compositions of $n-2$ into parts of value 1 or 2*

Exercise: Try to give a combinatorial proof
Remark:$g(n)$ *satisfies the Fibinoci recurrence*

$$g(n) = g(n-1)_{1st\ part=1} + g(n-2)_{1st\ part\ =2}$$

Let $A :=$ set of odd integers. $h(n) = \#$comps into parts with odd value.

$$H(x) = \sum_{n\geq 0} h(n)x^n = \sum_{k\geq 0}(\sum_{i\ odd} x^i)^k$$

$$= \frac{1}{1-\sum_{i\ odd} x^i} = \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2}$$
$$= 1 + \frac{x}{1-x-x^2}$$
$$= 1 + xG(x)$$
$$\Rightarrow h(n) = g(n-1)$$

Theorem 3 *# of compositions of h into n odd parts = # of compositions of $n-1$ into parts 1 or 2.*

Exercise: Try to give a combinatorial proof

4 Elementary Counting

Given an $n - set$ X we let $\binom{n}{k}$ denote the number of subsets of size k of X. Let n_k denote the number of ordered $k - subsets$. So

$$n_k = n \cdot n - 1 \cdots (n - k + 1)$$

we could also write

$$n_k = \binom{n}{k} k!$$

$$\binom{n}{k} = \frac{n_k}{k!} = \frac{n \cdot n - 1 \cdots (n - k + 1)}{k!}$$

Remark this formulation is better then $\frac{n!}{k!(n-k)!}$ as it allows to evaluate $\binom{n}{k}$ when n is negative or complex!. Recall:

Theorem 4 (Binomial)

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

therefore

- $x := 1$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

- $x := -1$

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

- Differentiate

$$n(1 + x)^{n-1} = \sum_{k=1}^{k} k \binom{n}{k} x^{k-1}$$

set $x := 1$

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

this shows that the number of ways to choose an element and a subset on the remaining elements is equal to the number of ways to choose a subset and then choose an element of the subset.

4.1 Compositions

How are binomial coefficients related to compositions? let $c(n,k) :=$ *number of compositions of* n *into exactly* k *parts(order matters)*

Lemma 1 $c(n,k) = \binom{n-1}{k-1}$

Proof. Draw n dots in a line.

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

there are $n-1$ spaces. We need to choose $k-1$ of them. so there are $\binom{n-1}{k-1}$ ways to do this. ∎

Corollary 1 *the number of solutions to* $\sum_{i=1}^{n} x_i = n$ *into non negative solutions is* $\binom{n+k-1}{k-1}$

Proof. Add 1 to all x_i. Get the number of solutions to $\sum_{i=1}^{n} y_i = n+k$ in positive solutions. ∎

4.2 Multisets

A k *subset* of a n set X does not allow repitions of elements. What if elements of X can be chosen multiple times? We denote the number of ways by $\binom{n}{k}_M$.

Theorem 5 $\binom{n}{k}_M = \binom{n+k-1}{k}$

A multiset of $X := \{x_1, ..., x_n\}$ has the form $\{x_1^{a_1}, ..., x_n^{a_n}\}$ where a_i is the number of copies of x_i in the multiset. So the number of ways is equal to the number of non negative solutions to $a_1 + ... + a_n = k$ this is $\binom{k+n-1}{n-1} = \binom{n+k-1}{k}$ ∎

Proof. (2) Or let $1 \leq s_1 < s_2 ... < s_k \leq n+k-1$ be a subset of $[n+k-1]$ Let $t_i := s_i + 1 - i$. then

$$1 \leq t_1 \leq ... \leq t_k \leq n$$

is a k multiset of $[n]$. This is a bijection. ∎

4.2.1 Multisets and GFs

consider the GF

$$(1 + x_1 + x_1^2 + ...)(1 + x_2 + x_2^2 + ...) \cdots (1 + x_n + x_n^2 + ...)$$

$$= \sum_{a:X \to \mathbf{N}^n} \prod_{x_i \in X} x_i$$

$$= \sum_{a_1,...,a_n} \prod_{x_i \in X} x_i$$

Set $x_i = x \forall i$

$$(1 + x + x^2 + ...)^n = \sum_{a_1,...,a_n} x^{a_1 + ... + x_n} = \sum_{H \text{ is multiset}} x^{|H|}$$

$$= \sum_{k \geq 0} \binom{n}{k}_M x^k$$

But

$$(1+x+x^2....)^n = \frac{1}{(1-x)^n} = (1-x)^{-n}$$

Now define

$$(1-x)^{-n} := \sum_{k\geq 0} \binom{-n}{k} (-1)^k x^k$$

So

$$\binom{n}{k}_M = (-1)^k \binom{-n}{k}$$

$$= \frac{-n(-n-1)(-n-2)\cdots(-n-k+1)}{k!}(-1)^k$$

$$= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \binom{n+k-1}{k}$$

4.2.2 example

$$F(x) = \prod_{n\geq 1} \sum_{i\geq 0} \binom{\frac{\mu(n)}{n}}{i}_M x^{in}$$

$$= \prod_{n\geq 1} (1-x^n)^{-\frac{\mu(n)}{n}}$$

where μ is the *Mobius function*

$$G(x) = \lg[F(x)] = \sum_{n\geq 1} \lg(1-x^n)^{\frac{-\mu(n)}{n}}$$

$$= -\sum_{n\geq 1} \frac{\mu(n)}{n} \lg(1-x^n)$$

$$= -\sum_{n\geq 1} \frac{\mu(n)}{n} (\sum_{i\geq 1} \frac{-x^{in}}{i})$$

$$G(x) = \sum_{n\geq 1} \frac{\mu(n)}{n} (\sum_{i\geq 1} \frac{x^{in}}{i})$$

What is the coefficient of x^m in $G(x)$? $\frac{1}{m}\sum_{d|m}\mu(d)$

but $\sum_{d|m}\mu(d) = 1\ if\ m=1, =0\ if\ m\neq 1$, $m = p_1^{a_1}\cdots p_k^{a_k}$ Subsets of $p_1\cdots p_k$ that don't give 0. Even subset give $\mu(d)=1$ odd give $\mu(d)=-1$. $\#even = \#odd$. So $\sum_{d|m}\mu(d) =\ 0_{m\neq 1}\ = 1_{m=1}$
$G(x) = x\ F(x) = e^x$

4.3 Multinomial Coefficients

The binomial coefficient $\binom{n}{k}$ can be interpreted as splitting X into 2 sets. This generalises. Let $\binom{n}{a_1a_2...a_k}$ = *# of ways to split X into k sets of sizes $a_1...a_k$ resp.* Equiv place place n balls into k boxes such that box i has a_i balls. Take a_i balls of color i. How many ways can we arrange the balls in a row (in a distinguishable manner). There are $n!$ orderings there are $\frac{n!}{a_1!\cdots a_k!}$ distinguishable arrangements. There are $\binom{n}{a_1a_2...a_k}$ arrangements as the positions of balls of color i give a subset X_i of X. Ho many ways to partition $[n]$ into b_i subsets of size i when $\sum_{i=1}^{k} i - b_i = n$.

Partition $[n]$ into unordered sets. Apply above method, but the collections of subsets of size i can themselves be permuted. So

$$\frac{1}{b_1!b_2!\cdots b_k!}\frac{n!}{(1!)^{b_1}\cdots(k!)^{b_k}}$$

Problem 2 *How many sequences $A_1, A_2, ..., A_k$ of subsets of $[n]$ are there such that $\cup_{i=1}A_i = [n]$?*

ANSWERE:$(2^k - 1)^n$

5 Inclusion-Exclusion

Notice: if A_1, A_2 are sets, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. Take a collection of sets $\{A_i : i \in I\}$

Theorem 6 *I-E*

$$|\cup_{i\in I} A_i|$$

$= \sum_{S\subseteq I:|S|=1} |\cap_{i\in} A_i| - \sum_{S\subseteq I:|S|=2} |\cap_{i\in S} A_i| + ... + (-1)^{|I|-1}|\cap_{i\in I} A_i|$

Proof. If x is in k of the A_i. How many times is x counted by RHS?

$$k - \binom{k}{2} + \binom{k}{3} - ... + (-1)^{k-1}\binom{k}{k}$$

By the binomial theorem$_{variable:=1}$

$$0 = \sum_{i=0}(-1)^i\binom{k}{i}$$

■

5.1 Examples

- **Derangements:** A permutation π such that $\pi_i \neq i$. Let A_i be the set of π such that $\pi_i = i$

$$|\cup_{i=1}^{n} A_i| = n \cdot (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!... + (-1)^{n-1}\binom{n}{n}(n-n)!$$

$$= n! - \frac{n!}{2} + \frac{n!}{3!}.... + (-1)^{n-1}\binom{n}{n}(n-n)!$$

so

$$d_n = n! - |\cup_{i=1}^n A_i| = n!(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + ...)$$
$$= n!\sum_{i=0}^{n} \frac{-1^i}{i!} \approx \frac{n!}{e}$$

- Can also derive this using generating functions. **aside:** $S_n = e^C$ so $D_n = e^{C-x}$
  Let $D(x) := \sum_{n\geq 0} d_n \frac{x^n}{n!}$ be exponential GF. Now

$$e^x \cdot D(x) = (\sum_{r\geq 0} \frac{x^r}{r!})(\sum_{s\geq 0} d_s \frac{x^s}{s!})$$

$$= \sum_{k\geq 0} x^k \sum_{i=0}^{k} \frac{d_i}{i!} \cdot \frac{1}{(k-i)!}$$

$$= \sum_{n\geq 0} \frac{x^k}{k!} \sum_{i=0}^{k} d_i \binom{k}{i} = \sum x^k = \frac{1}{1-x}$$

$$\Rightarrow D(x) = \frac{e^{-x}}{1-x}$$

$$D(x) = (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}...)(1 + x + x^2 + ...)$$

So

$$\frac{d_n}{n!} = \sum_{i=0}^{n} \frac{-1^i}{i!}$$

- **Binomial Coeffs**

$$\binom{m}{k} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i}$$

Assume $m \geq k$. For the LHS there are $\binom{m}{k}$ ways to pick k blue balls from m blue balls. For the RHS add n red balls $r_1, ..., r_n$. Let A_j be a collection of k subsets of $R \cup B$ that contains r_j. The number of ways to pick a blue $k - set$ from $R \cup B$ is then

$$\binom{n+m}{k} - (\sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \binom{m+n-i}{k-i})$$

pick i of $r_1, ..., r_n$ and we have $k - i$ choices for the other balls

- **Euler Function**: let $n := p_1^{a_1} \cdots p_k^{a_k}$ be prime decomposition of n. Let $\phi(n) = \#$ *of integers coprime with n and less then n.* Set A_i =set of integers divisible by p_i. Set $A_i :=$ *set of integers divisible by p_i.* So

$$\phi(n) = n - [\sum_{i=1}^{k} \frac{n}{p_i} - \sum_{1\leq i_1 < i_2 \leq r} \frac{n}{p_{i_1} p_{i_2}} \ldots + (-1)^r \sum_{1\leq i_1 \leq \ldots i_r \leq r} \frac{n}{p_1 \cdots p_r}]$$

$$= n\prod_{i=1}^{k}(1-\frac{1}{p_i})$$

Theorem 7 *Euler* ϕ $n = \sum_{d|n}\phi(d)$

Proof. The number of integers m such that $gcd(m,n) = d$ is $\phi(\frac{n}{d})$. $[m = m, d, n = n, d]$ So $\sum_{d|n}\phi(\frac{n}{d}) = \sum_{d|n}\phi(d) = n$ ■

- This relates to the Mobius function: $\mu(1) = 1, \mu(n) = 1$ if n is a product of even number of distinct primes. $\mu(n) - 1$ if n is a product of odd number of distinct primes, $\mu(n) = 0$ if n is not square free.

Theorem 8

$$\sum_{d|n}\mu(d) = 1\ if\ n = 1\ 0\ otherwise$$

Proof. $n = 1$, $n = p_1^{a_1}\cdots p_k^{a_k}$ then

$$\sum_{d|n}\mu(d) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}$$

$$= (1-1)^k = 0$$

■

Corollary 2

$$\frac{\phi(n)}{n} = \sum_{d|n}\frac{\mu(d)}{d}$$

Proof.
$\frac{\phi(n)}{n} = 1 - \sum_{i=1}^{k}\frac{1}{p_i} + \sum_{1\le i_1<i_2\le k}\frac{1}{p_{i_1}p_{i_2}}... = \mu(1) + \sum_{d|n}\mu(d)\frac{1}{d} + \sum_{d|n}\mu(d)\frac{1}{d} + ... = \sum_{d|n}\frac{\mu(d)}{d}$
$d = some\ p_i$ ■

6 Mobius Inversion

Theorem 9 *Let* $f(n) = \sum_{d|n} g(d)$ *then* $g(n) = \sum_{d|n}\mu(d)f(\frac{n}{d})$

Proof.

$$\sum_{d|n}\mu(d)\cdot f(\frac{n}{d}) = \sum_{d|n}\mu(\frac{n}{d})f(d)$$

$$= \sum_{d|n}\mu(\frac{n}{d})\sum_{d'|d}g(d') = \sum_{d'|n}g(d')\sum_{d''|\frac{n}{d'}}\mu(d'') = g(n)$$

which is equal to 0 or 1, and is only 1 when $\frac{n}{d'} = 1$. i.e. when $d' = n$. ■

Example: Let $H_n =$ the number of circular $0-1$ sequences of size n, sequences are distinct if not rotations of each other. Let $\mu(d) = \#$ sequences of period d.

$$H_n = \sum_{d|n} \mu(d)$$

we know

$$\sum_{d|n} d \cdot \mu(d) = 2^n$$

Let $f(n) = 2^n$, $g(n) = \sum_{d|n} \mu(d) 2^{\frac{n}{d}} = n \cdot \mu(n)$ in the Mobius Inversion Formula. So $H_n = \sum_{d|n} \mu(d) = \sum_{d|n} \frac{1}{d} \sum_{d'|d} \mu(d') 2^{\frac{d}{d'}}$

$$= \sum_{d|n} \frac{1}{d} \sum_{l|d} \mu(\frac{d}{l}) 2^l$$

$$= \sum_{l|n} \sum_{k|\frac{n}{l}} \frac{2^l}{l} \cdot \frac{\mu(k)}{k}$$

$$= \sum_{l|n} \frac{2^l}{l} \sum_{k|\frac{n}{l}} \frac{\mu(k)}{k}$$

$$\Rightarrow H_n = \frac{1}{n} \sum_{l|n} \phi(\frac{n}{l}) 2^l$$

■

7 Stirling numbers of the first kind

Let $\pi := \pi_1, \pi_2, ..., \pi_n \in S_n$ be a permutation of $[n]$. We can view π as a collection of disjoint cycles. So we can write π as a set of disjoint cycles:

- $\pi := (a_1, ..., a_k), (b_1, ..., b_r), ...$
- Each cycle starts with its largest element.
- The cycles are written in increasing order of their largest element.

for example (726)(84513)(9).

Now we can forget about the parenthesis. Let $f(\pi) = \pi -$ *parenthesis.* For example

$$f(368517249) = f((726)(84513)(9)) = 726845139$$

Theorem 10 (surprizing) *f is a bijection.*

Proof. To see this take $f(\pi)$ and we insert (before every left-right maxima. ■

Corollary 3 *$\pi \in S_n$ has k cycles iff $f(\pi)$ has k left, right maxima.*

Now given $\pi \in S_n$ let $c_i := c_i(\pi) = \#\ of\ cycles\ of\ length\ \pi$ We say π has type $(c_1, c_2, ..., c_n)$

Lemma 1 *number of permutations of type* $(c_1, c_2, ..., c_n)$ *is*

$$\frac{n!}{1^{c_1} \cdot 2^{c_2} \cdots n^{c_n} c_1! \cdot c_2! \cdots c_n!}$$

Proof. Write π in a non-standard cycle form. Place cycles of length 1 first, then cycles of length 2, etc... We can order the $-cycles$ in $c_i!$ ways and we can pick thier first element in i^{c_i} ways. ■

Definition 3 (Stirling number of the first kind) *Let* $\bar{s}(n,k) = \#$ *permutations of* $[n]$ *with* k *cycles.* $s(n,k) = (-1)^{n-k}\bar{s}(n,k)$

Lemma 2 $\bar{s}(n,k)$ *satisfy:*

$$\bar{s}(n,k) = (n-1)\bar{s}(n-1,k) + \bar{s}(n-1,k-1)$$

Proof. Given $\pi \in S_{n-1}$ with $k-1$ cycles we a $\pi \in S_n$ with k cycles by setting $\pi_n = n$. Given $\pi \in S_{n-1}$ with k cycles then we have $n-1$ 'slots' to insert n. (add n as a midpoint in one of the $n-1$ edges of the cycles) ■

Theorem 11 $\sum_{k=0}^{n} \bar{s}(n,k)x^k = (x+n-1)_n$

Proof. Set $F_n(x) = (x+n-1)_n = (x+n-1)(x+n-2)\cdots(x+1)-x$. We can write this as

$$\sum_{k=0}^{n} b(n,k)x^k$$

and determine what b is. Now $b(0,0) = 1$, and set $b(n,k) = 0$ if $n < 0$ or $k < 0$.

$$F_n(x) = (x+n-1) \cdot F_{n-1}(x) = xF_{n-1}(x) + (n-1)F_{n-1}(x)$$

$$= \sum_{k=0}^{n-1} xb(n-1,k)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k$$

So

$$\sum_{n=0}^{n} b(n,k)x^k = \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k$$

So $b(n,k) = b(n-1,k-1) + (n-1)b(n-1,k)$. The base cases for $\bar{s}(n,k)$ are the same so

$$\bar{s}(n,k) = b(n,k)$$

■

We can prove combinatorial via a bijection show coeffs on the LHS equal coeff on RHS. Instead we give a different type of combinatorial proof.

- **Two polynomials are the same if they agree on sufficiently many values of the variable** x

Using this we give a second proof of the theorem. **Proof. (2)** We show

- *
    - $\sum \bar{s}(n,k)x^k = (x+n-1)_n$

for all positive integers x. Let $C(\pi)$ be the set of cycles of π. The LHS counts pairs (π, f) with $f : C(\pi) \to [x]^k$ The RHS counts integer sequences $(_1, b_2, ..., b_n)$ where $1 \leq b_i \leq x+n-i$.

Given sequence $(b_1, ..., b_n)$ find a bijection to (π, f)

1. Write down n and assume it starts cycle C_1. Let $f(C_1) = b_n$
2. Given $n, n-1, ..., n-i+1$ have been inserted into cycles
    (a) if $1 \leq b_{n-i} \leq x$ start a new cycle C_j with $(n-i)$ to the left of previous elements. Set $f(C_j) = b_{n-1}$
    (b) If $b_{n-i} = x+p$ insert $(n-i)$ into an odd cycle such that it is to the right of p elements(it doesnt start a cycle)

7.1 Example

$$(b_1 \dots b_9) = (596186352)\ n = 9\ x = 4$$

- $b_9 = 2$ (9) so $f(C_1) = 2$
- $b_8 = 4+1$ $p = 1$ (98)
- $b_7 = 3$ (7)(98) $f(C_2) = 3$
- (7)(968)
- (7)(9685)
- (4)(7)(9685)
- (4)(73)(9685)
- (4)(73)(96285)
- (41)(73)(96285)

this is in standard form by construction. given (π, f) for example $(41)_{color=1}(73)_{col=3}(96285)_{col=2}$

1. 1 has one element to the left $\Rightarrow p = 1$ cross off 1 $(5 \dots)$
2. 2 now has 5 elements to the left $\Rightarrow p = 5$ $(5, 9 \dots)$
3. 3 has 2 elements to the left so $p = 2$ $(5, 9, 6)$
4. ....
5. $(5, 9, 6, 1, 8, 6, 3, 5, 2)$

e.g. set $x := 1$

Corollary 4 *the number of integer sequences* $(b_1, ..., b_n)$ *such that* $1 \leq b_i \leq n+1-i$ *with exactly* k *of the* $b_i = 1$ *is* $\bar{s}(n,k)$.

Corollary 5

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n$$

Proof. Take $\sum \bar{s}(n,k)x^k = (x+n-1)_n$ set $y = -x$.

$$(-1)^n \sum \bar{s}(n,k)(-1)^k y^k = (n-1-y)_n(-1)^n$$

$$\begin{aligned}\sum \bar{s}(n,k)(-1)^{n+k}y^k &= (-1)^n(n-1-y)(n-2-y)\cdots(1-y)-y \\ &= y(y-1)\cdots(y-(n-1)) \\ &= \sum \bar{s}(n,k)(-1)^{n-k}y^k \\ \sum s(n,k)y^k &= (y)_n\end{aligned}$$

■

8 Stirling numbers of the second kind

Definition 4 (Partition) *A* ***partition*** *of* $[n]$ *is an unordered collection of subsets(blocks)* $B_1, ..., B_k$ *such that*

- $B_i \neq 0$
- $B_i \cap B_j = \emptyset$
- $B_1 \cup B_2 \ldots \cup B_k = [n]$

Definition 5 (second kind) *Let* $S(n,k) := \#$ *of partitions of* $[n]$ *into exactly* k *blocks. We say* $S(n,k)$ *is a Stirling number of the second kind.*

By convention $S(0,0) = 1$. We have

1. $S(n,k) = 0$ if $k > n$
2. $S(n,0) = 0$ for $n > 0$
3. $S(n,1) = 1$
4. $S(n,2) = 2^{n-1} - 1$
5. $S(n,n-1) = \binom{n}{2}$
6. $S(n,n) = 1$

We have the following recurrence

Lemma 3 $S(n,k) = kS(n-1,k) + S(n-1,k-1)$

Proof. Look at element n. We can add it to any block in a $k-block$ partition of $[n-1]$ or we can put it in a block of its own. We have a $(k-1)-block$ partition of $[n-1]$ ■

Theorem 12 $\sum_{k=0}^{n} S(n,k)(x)_k = x^n$

Proof. Let X be a set of size x. The RHS is the number of functions $f:[n] \to X$. Each function is a surjection onto a unique subset $Y \subseteq X$. Fix $|y| = k$. There are $k!S(n,k)$ such surjections. There are $\binom{x}{k}$ choices for Y. Thus

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k} = \sum_{k=0}^{n} S(n,k)(x)_k$$

■

Recall $\sum_{k=0}^{n} s(n,k)x^k = (x)_n$.

Theorem 13
- $\sum_{k=r}^{n} S(n,k)s(k,r) = 0_{r\neq n}$
- $\sum_{k=r}^{n} S(n,k)s(k,r) = 1_{r=n}$

Proof. $x^n = \sum_{k=0}^{n} S(n,k)(x)_k = \sum_{k=0}^{n} S(n,k) \cdot \sum_{r=0}^{k} s(k,r)x^r$

$$= \sum_{r=0}^{n} x^r (\sum_{k=r}^{n} S(n,k)s(k,r)) = x^n$$

■

8.0.1 Interpritation

- Let s be ∞ matrix with ij entry $s(i,j)$
- Let S be ∞ matrix with ij entry $S(i,j)$
- Then the theorem implies that S and s are inverses

8.0.2 Example

$B := \{1, x^1, x^2, ...\}$ is a basis for the complex vector space defined by polynomials with complex coefficients. But the $B_2 := \{1, (x)_1, (x)_2, ...\}$ is also a basis as S is transition matrix for bases B_2 to B_1. **Remark:** The equations

1. $(x)_n = \sum s(n,k)x^k$
2. $x^n = \sum S(n,k)(x)_k$

Are important in the theory of "calculus of finite differences".

8.1 Generating Functions

Let

$$F_k(x) := \sum_{n\geq k} S(n,k)\frac{x^n}{n!}$$

So

$$F_k(x) = k\sum_{n\geq k} S(n-1,k)\frac{x^n}{n!} + \sum_{n\geq k} S(n-1,k-1)\frac{x^n}{n!}$$

$$\begin{aligned}F'_k(x) &= k\sum_{n-1\geq k} S(n-1,k)\frac{x^{n-1}}{(n-1)!} + \sum_{n\geq k} S(n-1,k-1)\frac{x^{n-1}}{(n-1)!} \\ &= k\sum_{n\geq k} S(n,k)\frac{x^n}{n!} + \sum_{n-1\geq k-1} S(n-1,k-1)\frac{x^{n-1}}{(n-1)!} \\ &= kF_k(x) + F_{k-1}(x)\end{aligned}$$

Lemma 4 $\sum_{n\geq k} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$

Proof. Induction:

$$S(n,1) = \sum_{n\geq 1}\frac{x^n}{n!} = e^x - 1$$

then

$$F'_k(x) = kF_k(x) + \frac{1}{(k-1)!}(e^x-1)^{k-1}$$

has solution

$$F_k(x) = \frac{1}{k!}(e^x-1)^k$$

unique since the coefficient of x^k is $\frac{1}{k!}$ ■

Corollary 6 $S(n,k) = \frac{1}{n!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$

Proof. Coeff of x^n in $\sum S(n,k)\frac{x^n}{n!}$ is the coefficient of x^n in

$$\frac{1}{k!}(e^x-1)^k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{ix}$$

$$= \frac{1}{k!}\sum(-1)^{k-i}\binom{k}{i}(\sum_{r\geq 0}\frac{(ix)^r}{r!})$$

$$\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{i^n}{n!} = \frac{S(n,k)}{n!}$$

■

8.1.1 Bell Numbers

The total number $B(n)$ of partitions of an $n-set$ is called the Bell number. So $B(n) = \sum_{k=0}^{n} S(n,k)$

Corollary 7 $B(x) := \sum_{n\geq 0} B(n)\frac{x^n}{n!} = exp(e^x - 1)$

Proof.

$$\sum B(n)\frac{x^n}{n!} = \sum_{n\geq 0}(\sum_{k=0}^{n} S(n,k))\frac{x^n}{n!}$$

$$= \sum_{k\geq 0}\sum_{n\geq k} S(n,k)\frac{x^n}{n!}$$

$$= \sum_{k\geq 0}\frac{1}{k!}(e^x - 1)^k = exp(e^x - 1)$$

■

So lets extract some information

$$B(x) = exp(e^x - 1)$$

$$B'(x) = e^x exp(e^x - 1)$$

$$B'(x) = \sum B(n)\frac{x^{n-1}}{(n-1)!}$$

$$\sum B(n+1)\frac{x^n}{n!} =$$

$$B(n+1) = \sum_{k=0}^{n}\binom{n}{k}B(k)$$

Consider the exponential generating function for the FIRST KIND

8.1.2 SNOTFK

Theorem 14 $\sum_{n\geq k} s(n,k)\frac{x^n}{n!} = \frac{1}{k!}(\lg(1+x))^k$

Proof.

$$(1+y)^x = exp(x\log(1+y)) = \sum_{k\geq 0}\frac{1}{k!}[\lg(1+y)]^k x^k$$

Also

$$(1+y)^x = \sum_{n\geq 0}\binom{x}{n}y^n$$

$$= \sum_{n\geq 0}\frac{1}{n!}(x)_n y^n$$

$$= \sum_{n\geq 0}\frac{y^n}{n!}(\sum_{k=0}^{n} s(n,k)x^k)$$

$$\sum_{k\geq 0} x^k \sum_{n\geq k}\frac{y^k}{n!} s(n,k)$$

■

9 Catalan Numbers

How man sequences of n $'+'$ signs and n $'-'$ signs are there such that each partial sum is non-negative? This is the same as the number of paths from $(0,0)$ to $(2n,0)$ using arcs $(1,1)$ if we are not allowed to go below the $x-axis$. Such waks are called **byck** walks. The number of these walks is equal to the catalan number

Theorem 15 $C_n := \frac{1}{n+1}\binom{2n}{n}$

Proof.

- Clearly the number of walks from $(0,0)$ to $(2n,0)$ is $\binom{2n}{n}$ if there is no restriction of staying non negative
- Any path that goes below the $x - axis$ hits $y = -1$. These are "bad" walks. Count the number of bad walks.

  Let P go below the $x - axis$, therefore it hits $y = -1$ for the first time at $(x', -1)$. Say this divides P into P_1 and P_2. i.e. P_1 goes from $(0,0)$; $(x', -1)$. Let $\bar{P}$ be the reflection of P_1 in $y = -1$. So (P_1', P_2) is a walk from $(0,-2)$ to $(2n,0)$. This is a one to one mapping as any walk from $(0,-2)$ to $(2n,0)$ crosses $y = -1$ from below at least once. There are $\binom{2n}{n+1}$ of these walks.
- So the number of **Dyck** paths is

$$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n}\frac{1}{n+1} = C_n$$

■

Lets see more examples of C_n

- **Plane Trees on** $n+1$ **vertices**. Consider a caterpiller(i.e. DFS order) walking around the tree. If he goes up we have a $+$, if he goes down we write a $-$.
- **Planted trivalent trees on** $2n+2$ **vertices**.

$$T = x(1 + T^2)$$

  solve using quadratic equation.
- **Decompositions of** $(n+2)$**-gon into** n **triangles using** $n-1$ **non intersection diagonals.** Outer face $= root + leaves....$
- **Linear extensions(full ordering such that** $x \leq y$ **if** $x \leq y$**) of the poset** $2 \times n$**.** For each odd number we have a $+$ and for each even number we have a $-$. For example

$$\pi := 1\ 3\ 5\ 2\ 4\ 7\ 6\ 8 \Rightarrow +\ +\ +\ -\ -\ +\ -\ -$$

- **Covering non-comparable intervals** on $\{1, ..., n\}$

$$(1,2)(3,4,5,6) + (2,3,4)$$

covers $(1,2,3,4,5,6)$.

Construct a Lattice:

$$123$$

$$12 \ 23$$

$$1 \ 2 \ 3$$

any maximal antichain in this lattice defines a linear extension and gives us a corresponding Dyck path.

- **Binary Bracketing**. A recursive partition of a non associative product $x_1, x_2, ..., x_{n+1}$ into products of 2 non empty products. Bijection with Dyck paths by sending '$(' \Rightarrow +, ')' \Rightarrow -$ and reading from left to right.

9.1 Catalan GF

let $f(n) = \#$ of binary bracketings of $(1, ..., n)$. then $f_n = C_{n-1}$ Clearly

$$f(n) = \sum_{j=1}^{n-1} f(j)f(n-j)$$

Let

$$F(x) = \sum_{n\geq 1} f(n)x^n = x + \sum_{n\geq 2} f(n)x^n$$

$$= x + \sum_{n\geq 2} (\sum_{j=1}^{n-2} f(j)f(n-j))x^n$$

$$= x + F(x)^2$$

so

$$F(x) = x + F(x)^2$$

thus

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2}$$

How can we go backwards? we have

$$F(0) = 0$$

so we have

$$F(x) = \frac{1-\sqrt{1-4x}}{2} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

$$= \binom{\frac{1}{2}}{0} + \sum_{n\geq 1} (-4)^n x^n \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{-(2n-3)}{2}}{n!}$$

$$= 1 - \sum_{n\geq 1} 4^n x^n \frac{(2n-3)\cdots 5\cdot 3\cdot 1}{2^n n!}$$

$$= 1 - \sum 4^n x^n \frac{(2n-2)!}{2^{n-1}(n-1)!2^n n!}$$

$$= 1 - \sum_{n\geq 1} \frac{2}{n}\binom{2n-2}{n-1} x^n$$

So

$$F(x) = \frac{1}{2}\sum \frac{2}{n}\binom{2n-2}{n-1} x^n$$

$$= \sum_{n\geq 1} C_{n-1} x^n$$

10 Partitions of an integer

We say that $p(n) = \#$ of ordered partitions of n. e.g. $p(5) =$ they are

$$5,\ 4+1\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$$

We often represent a partition $\lambda \perp n$ by a Ferrers diagram. We obtain the conjugate λ' of λ by transposing the rows and cols. A partition is self conjugate if $\lambda = \lambda'$.

Theorem 16 *The number of partitions of n into at most k parts is equal to the number of partitions of $n+k$ into exactly k parts. i.e.*

$$\sum_{j=1}^{k} p_k(n) = p_k(n+k)$$

Proof. take a partition of $n+k$ into exactly k parts and remove the first column. ■

Theorem 17 *the number of partitions of n into distinct odd parts is equal to the number of self conjugate partitions of n*

Proof. Take the 'hooks' i.e. $first\ column + first\ row$, ...$2nd\ col\ + 2nd\ row$ ..etc ■

10.1 Generating Functions

Consider $\prod_{i\geq 1} \frac{1}{1-x_i}$ and take a term $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ setting x_i to x^i we have $x^{\alpha_1} x^{2\alpha_2} \cdots x^{r\alpha_r}$. It follows that the coefficient of x^n in

$$\prod_{i\geq 1} \frac{1}{1-x^i}$$

is $p(n)$. Similarly we have generating functions for

1. Partitions into distinct parts: $D(x) = \prod_{i\geq 1}(1+x^i)$.
2. Partition into odd parts: $O(x) = \prod_{i\geq 1} \frac{1}{1-x^{2i-1}}$

3. Partitions into size at most r

$$R(x) = \prod_{r \geq i \geq 1} \frac{1}{1 - x^i}$$

Theorem 18 *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof.

$$\begin{aligned} D(x) = \prod_{k \geq 1} (1 + x^k) &= \prod_{k \geq 1} \frac{(1 + x^k)(1 - x^k)}{(1 - x^k)} \\ &= \prod_{k \geq 1} \frac{(1 - x^{2k})}{(1 - x^k)} \\ &= \prod_{i \geq 1} \frac{1}{1 - x^{2k-1}} = O(x) \end{aligned}$$

■

Lets investigate $P(x)$ more. Consider the inverse of $P(x)$ i.e. $P(x) \cdot P(x)^{-1} = 1$. Clearly $P(x)^{-1} = \prod_{k \geq 1} (1 - x^k)$. This looks like $D(x)$! It follows that in the expansion of $P(x)^{-1}$

- any partition of n into an even number of distinct parts contributes 1 to the coefficient of x^n
- any partition of n into an odd number of distinct parts contributes -1 to the coefficient of x^n
- $\hat{e}(n) :=$ number of partitions of n into distinct parts with even number of parts
- $\hat{o}(n) :=$ number of partitions of n into distinct parts with odd number of parts

Lemma 5

$$P(x)^{-1} = 1 + \sum_{n \geq 1} [\hat{e}(n) - \hat{o}(n)] x^n$$

What is $\hat{e}(n) - \hat{o}(n)$? Set up a bijection between partition of $\hat{e}(n)$ and $\hat{o}(n)$. Take a partition $\lambda \perp n$. Set $s(\lambda) :=$ *size of smallest part.* $d(\lambda) :=$ length of $45°$ angle starting at top right.

We give a transformation $\lambda \to \lambda'$ as follows

1. `If` $s(\lambda) \leq d(\lambda)$ `then` $\lambda \to \lambda'$ `by moving the smallest part to the far right diagonal.`
2. `If` $s(\lambda) > d(\lambda)$ `then move diagonal to the bottom row.`

These transformations keep parts distinct but change the parity of # parts. Also case 1 changes $s(\lambda) \leq d(\lambda)$ to $d(\lambda') < s(\lambda')$. Similarly for case 2. Are we done ? no its not a bijection! We have 2 problems.

- `in Case 1`: what if $s(\lambda) = d(\lambda)$ and the row and diagonal intersect? This is not a valid diagonal.

- `in Case 2`: what if $s(\lambda) = d(\lambda) + 1$ and they intersect?

These are the only two problems . So we "nearly" have a bijection. Moreover the bad examples in case 1 satisfy

$$n = s(\lambda) + (s(\lambda) - 1) + (s(\lambda) + 2) + ... + (s(\lambda) + s(\lambda) - 1)$$
$$= s + (s + 1) + ...(2s - 1)$$

n is of the form

$$n^2 + \frac{1}{2}n(n-1) = \frac{1}{2}n(3n-1)$$

The **Case** 2 bad example has $n = s + (s + 1) \dots (s + s - 2)$ thus n is of the form $\frac{1}{2}m(3m + 1)$. Remarkably we have shown :

Theorem 19 $\hat{e}(n) - \hat{o}(n) =$

- $(-1)^m$ *if* $n = \frac{1}{2}m(3m \pm 1)$
- 0 *otherwise*

for some $m \geq 1$

For example

$$P(x)^{-1} = 1 - (x - x^2)_{m=1} + (x^5 + x^7)_{m=2} - (x^{12} - x^{15})_{m=3} + (x^{22} + x^{26})_{m=4} + ...$$

We remark that the numbers $1, 5, 12, 22, ...$ are the "pentagonal numbers". For $2, 7, 15, ...$ add one dot per pentagon. Observe:

$$(1 + p_1 + p_2x^2...)(1 - x - x^2 + x^5 + x^7...) = 1$$

So we can recursively compute $p(n)$ then

$$p(n) = p(n-1) + p(n-2) - p(n-5) - ...$$

10.2 Asymptotics

First 2 terms are $p(n) = p(n + 1) + p(n - 2)$ But Fib grows much quicker than $p(n)$.

Lemma 6 $p(n) \leq \frac{\pi}{\sqrt{6(n-1)}} e^{\pi \cdot \sqrt{2/3}\sqrt{n}}$

Proof.

$$\lg(P(x)) = - \lg \prod_{k \geq 1} (1 - x^k)^{-1}$$
$$= - \sum_{k \geq 1} \lg(1 - x^k)$$
$$= \sum_{k \geq 1} \sum_{j \geq 1} \frac{(x^k)^j}{j} = \sum_{j \geq 1} \frac{1}{j} \sum_{k \geq 1} x^{jk}$$
$$= \sum_{j \geq 1} \frac{1}{j} \frac{x^j}{1 - x^j}$$

... ∎

11 Inversions

Definition 6 (Inversion) *Given a permutation $\pi := \pi_1, \pi_2, ..., \pi_n$ we say a pair (i, j) forms an inversion if*

- $i < j$
- $\pi_i > \pi_j$

We say $i(\pi) :=$ the number of inversions in π.

Lemma 7 *There is a one to one mapping between permutations and sequences $b := (b_1, ..., b_n)$ such that $0 \leq b_i \leq n - i$*

Proof. Take $\pi \in S_n$ let b_c be the number of elements to the left of c in π that form an inversion with c. Given b we can recover π by first writting n, then $n-1$,... using b. ■

Definition 7 (q-factorial) $(k)! := (k)(k-1)(k-2)...(2)(1)$ *where* $(j) := 1 + q + q^2 + ... + q^{j-1}$.

Notice when $q = 1$ we have $(k)! = k!$.

Theorem 20 $\sum_{\pi \in S_n} q^{i(\pi)} = (n)!$

Proof. Construct b as before such that $i(\pi) = \sum_{j=1}^{n} b_j$. We have

$$\sum_{\pi \in S_n} a^{i(\pi)} = \sum_{b_1=0}^{n-1} \sum_{b_2=0}^{n-2} \cdots \sum_{b_n=0}^{0} q^{b_1+b_2+...+b_n}$$

$$= \sum_{b_1=0}^{n-1} q^{b_1} \sum_{b_2=0}^{n-2} q^{b_2} \cdots \sum_{b_n=0}^{0} q^{b_n}$$

$$= (1 + q + q^2 + ... + q^{n-1})(1 + ... + q^{n-2}) \ldots (1 + q)1 = (n)!$$

■

This generalizes to permutations of multisets. Take $M := \{1^{\alpha_1}, 2^{\alpha_2}, ..., m^{\alpha_m}\}$ let $\pi \in S(M)$. Again an inversion is a pair $i < j$ with $\pi_i > \pi_j$

Definition 8 (q-multinomial coeff)

$$\binom{n}{a_1, ..., a_m} = \frac{(n)!}{(a_1)!(a_2)!...(a_m)!}$$

11.1 remarks

1.

$$\binom{n}{a_1, ..., a_m}$$

is polynomial in q.

2.

$$\binom{n}{a_1, ..., a_n} = \binom{n}{a_1}\binom{n-a_1}{a_2}...$$

$$\binom{n}{k} = \binom{n-1}{k} + q^{n-k}\binom{n-1}{k-1}$$

$$\binom{n}{0} = 1$$

Theorem 21 *For $M = \{1^{a_1}, ...m^{a_m}\}$*

$$\sum_{\pi \in S(M)} q^{i(\pi)} = \binom{n}{a_1, ..., a_m}$$

Proof. Define a map $\phi : S(M) \times S_{a_1} \times S_{a_2} \times ... \times S_{a_m}$ as follows. Given $\pi \in S(M)$ and $\pi^1 \in S_{a_1}, ..., \pi^n \in S_{a_n}$. Convert the a_i $i's$ to the numbers $a_1+a_2+....+a_{i-1}+1$,$a_1+a_2+....+a_{i-1}+2$,... $a_1 + a_2 + ... + a_{i-1} + a_i$. Let $\hat{\pi} \in S_n$. Place these numbers in π in the order created by $\pi^1, ..., \pi^m$ FOR EXAMPLE

$$(\pi \in S(m), \pi^1, \pi^2, \pi^3)$$

$$(21331223, 21, 231, 312)$$

$$21331223$$

$$4\mathbf{2}86\mathbf{1}537 = \hat{\pi} \in S_8$$

This is a bijection. Moreover $i(\hat{\pi}) = i(\pi^1) + i(\pi^2) + ... + i(\pi^m) + i(\pi)$ So

$$\sum_{\hat{\pi}} q^{i(\hat{\pi})} = \sum_{\pi \in S(M)} q^{i(\pi)} \prod_{j=1}^{m} \sum_{\pi^j \in S_{a_j}} q^{i(\pi^j)}$$

$$(n)! = \sum_{\pi \in S(m)} q^{i(\pi)}(a_1)! \dots (a_m)!$$

by the previous theorem. ■

12 Vector Spaces

Let q be prime and F_q be a finite field with q elements. Let $V_n(q)$ be the n dimensional vector space

$$F_q^n := \{(\alpha_1, ..., \alpha_n) | \alpha_i \in F_q\}$$

Theorem 22 *The number of k dimensional subspaces of $V_k(q)$ $\binom{n}{k}$*

Proof. Let this number be $S(n,k)$. Let $N(n,k) :=$ the number of ordered k tuples $(v_1, ..., v_k)$ of linearly independent vectors in $V_n(q)$. We can choose

- v_1 in $q^n - 1$ ways
- v_2 in $q^n - q$ ways
- v_3 in $q^n - q^2$ ways
- ....

So $N(n,k) = (q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$. On the other hand we choose $(v_1, ..., v_k)$ by first choosing a k dimensional subspace in $S(n,k)$ ways then choosing v_1 in $(q^k - 1)$ ways. ■

13 Posets

Definition 9 (Poset) *A partially ordered set $\mathcal{P}$ is a set P with a binary relation $\leq$ such that*

1. *Reflexivity: $a \leq a \; \forall a \in P$*
2. *Transitivity: $a \leq b, \; b \leq c \Rightarrow a \leq c$*
3. *Anti-symmetry: $a \leq b, b \leq a \Rightarrow a = b$*

A partial order is a total order(linear order) if all pairs are comparable. **Hasse Diagram:** *draw covering relations* .

Definition 10 (Chain) *$C \subseteq P$ is a chain if it is totally ordered.*

Definition 11 (Anti-chain) *$A \subseteq P$ is an antichain if all pairs in A are not comparable.*

Theorem 23 (Dilworth) *In a poset $\mathcal{P}$ the maximum size of an antichain is equal to the minimum number of chains needed to cover the elements of $\mathcal{P}$.*

Proof. A chain covers at most one element in an antichain thus *max antichain* $\leq$ *min chain cover*. We prove the other direction for finite $|P|$ but this result is also true in the infinite case. We induct on $|P|$ if $|P| = 1$ the statement is satisfied. Take $|P| \geq 2$ with *max antichain* $= k$. Pick a max chain C in P and a max antichain A in $P - C$. If $|A| = k - 1$ we are done by induction. Create 2 new posets $P^+ := \{x : x \geq q_i \textit{ some } i\}$, $P^- := \{x : x \leq a_i \textit{ some } i\}$. Let y be max element in C, z min element. We have $y \notin P^-, z \notin P^+$ or C is not maximal. So by induction P^- or P^+ can be covered by k chains. We claim $a_1, ..., a_k$ are minimal elements of the chains for P^+ and max elements for the chains in P^-.

Every element is in P^- or P^+, otherwise we get an antichain of size $k+1$. So if claim is true we can put the chains together to cover P. Consider P^+ and suppose there is some x in some chain such that $x \leq a_1$. This cant happen since $x \in P^+$ so $x \geq a_i$. So a_1, a_i are comparable by transitivity. Similar for P^- ∎

There is a dual result:

Theorem 24 *Max size of a chain = minimum number of antichains needed to cover $\mathcal{P}$.*

Proof. *min* $\geq$ *max* is obvious. To prove *max* $\leq$ *min* use induction on size k of maximum chain. If $k = 1$ all elements are incomparable. If $k \geq 2$ let A_{max} := *set of maximal elements in poset.* Clearly A_{max} is an antichain. The maximum chain in $P - A_{max}$ has size at least $k-1$. So we are done by induction. ∎

Theorem 25 *Let $S_1, ..., S_m$ be subsets of an $n-set$ such that $S_i \neq S_j$, $i \neq j$. Then*

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Proof. Consider Poset of subsets of $\{1, 2, ..., n\}$ with $S_i \leq S_j$ iff $S_i \subseteq S_j$. We want the maximum antichain in P.

Let a be an antichain of size $\sum_{k=0}^{n} n_k$ where n_k := *the number of subsets in a.* There are $n!$ chains from $\emptyset$ to $\{1, 2, ..., n\}$. Exactly $k!(n-k)!$ of the chains intersect a particular $k - subset$. No chain contains more than 1 element of a. So the number of chains containing some element of a is

$$\sum_{k=0}^{n} (n_k \cdot k!(n-k)!) \leq n!$$

$$\sum_{k=0}^{n} \frac{n_k}{\binom{n}{k}} \leq 1$$

since

$$\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

We have

$$\sum n_k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

∎

Here is a less obvious application of Dilworth

Theorem 26 *In any n^2+1 sequences of numbers there is either a non decreasing subsequence of $n+1$ numbers or a non increasing*

Proof. Take $x_1, x_2, ..., x_{n^2+1} = x$. we create a poset on x. For $i < j : x_i \leq_p x_j \ if \ x_i \leq x_j, x_i \triangle x_j \ if \ x_i > x_j$. A chain of size $n+1$ gives a non-decreasing subsequence of size $n+1$. If there is no such chain then we can cover P with n antichains. Therefore there is an antichain of size $\lceil \frac{n^2+1}{n} \rceil = n+1$. This gives decreasing subsequence of size $n+1$. ∎

13.1 Graph Theory

We will be considering bipartite graphs $G := (X, Y)$

Definition 12 (Matching) *A matching $M \subseteq E$ is a set of vertex disjoint edges. A matching M is complete if every vertex in X is adjacent to an edge in M.*

Theorem 27 $|\Gamma(S)| \geq |S| \forall S \subseteq X$ *iff there is a complete matching.*

Proof. If $|\Gamma(S)| < |S|$ you can't match up S. Let $|X| = n_1$, $n_2 := |Y| \geq |\Gamma(X)| \geq |X| = n_1$. Create a poset $\mathcal{P}$ by setting $x_i \leq y_i$ if there is an edge (x_i, y_j). Take a maximum antichain a of size k. $a = \{x_1, x_2, ..., x_r, y_1, ..., y_s\}$. Since $\Gamma(x_1, ..., x_r) \subseteq Y - \{y_1, ..., y_s\}$. So $n_2 - s \geq r$. Hence $n_2 \geq r + s = k$. y is antichain so $n_2 \leq k$. By Dilworth there are k chains that cover P. Antichain has size 1 or 2, so the k chains consist of k_1 vertices plus k_2 edges. The k_2 edges form a matching (or we don't need one of them). Thus $k = (n_1 - k_2) + (n_2 - k_2) + k_2$ and $k_1 = (n_1 - k_2) + (n_2 - k_2)$, so $k_2 = n_1 + n_2 - k = n_1$ ■

One last example

Theorem 28 *Let $\mathcal{S} := \{\mathcal{S}_\infty, ..., \mathcal{S}_\Updownarrow\}$ be pairwise intersecting k-subsets of an $n - set$. Then $m \leq \binom{n-1}{k-1}$.*

Proof. We can obtain this bound. Take all $k - subsets$ that contain element 1. Draw a'Drum' labled from $1, .., n$. Let $\{F_1, F_2, ..., F_n\} = \mathcal{F}$ be the k-subsets induced by this ordering starting at $1, 2, ..., n$ respectively. $|\mathcal{S} \cap \mathcal{F}| \leq \|$. The left points differ by at most $k - 1$ if they intersect. The same statement holds for any π on the drum. Let $\mathcal{F}^\pi = (\mathcal{F}_{\pi_\infty}, ..., \mathcal{F}_{\pi_\backslash})$ Let $\bar{m} = \sum_{\pi \in S_n} |\mathcal{S} \cap \mathcal{F}^\pi| \leq \| \cdot \backslash!$. We can count $\bar{m}$ in another way. Fix $S_i \in \mathcal{S}$ and see how many π have S_i as an interval. There are $k!(n-k)!$ ways to start a permutation with S_i. There are n ways to pick the start position on the drum. So $\bar{m} = \sum_{j=1}^{m} n \cdot k!(n-k)! = m \cdot nk!(n-k)! \leq kn!$ So $m \leq \binom{n-1}{k-1}$ ■

14 Posets

A poset $\mathcal{P}$ has a maximum element $\hat{1}$ if there exists element $x = \hat{1}$ such that $y \leq x \forall y \in \mathcal{P}$. We define $\hat{0}$ similarly. A poset is *graded* if every maximal saturated chain has the same length. A graded poset has a rank function $\rho : P \to \{0, 1, ..., n\}$ such that $\rho(x) = i$ if every maximal chain from a minimal element to x has length i.

Definition 13 (boolean algebra) *$B_n :=$ the boolean algebra of set $[n]$.*

B_n is a graded poset. We let $r_i :=$ *the number of nodes of rank i in a graded poset.*

Definition 14 (Combining Posets) *Given posets P, Q*

- *Difect sum $P + Q$: $x \leq y$ in $P + Q$ if*
  - *$x \leq y$ in P*
  - *or $x \leq y$ in Q*
- *Direct Producr $P \times Q$: $(x, y) \leq (x', y')$ in $P \times Q$ if*

– $x \leq x'$ *in* P *and*

– $y \leq y'$ *in* Q

Definition 15 (Sperners Property) *We say that a graded poset has the Sperner Property if the size of a maximum antichain is equal to* r_i *for some level* i*(any level is an antichain)*

Note that $B_n, D_n, \mathbf{n}$ are *Sperner*. Where $\mathbf{n}$ is a chain, D_n is the divisor poset.

Definition 16 (Rank Symmetric) *A graded poset is rank-symmetric if* $r_i = r_{n-i}$ $\forall i$*. A chain is symmetric if it starts at level* i *and finishes at level* $n-i$*.*

Definition 17 (Sym Chain Decomp) *A poset has a symmetric chain decomposition(SCD) if it can be covered by disjoint symmetric chains*

Lemma 8 *If* P *has a SCD then it is Sperner*

Proof. Take a SCD and suppose it has k chains. Then a a maximum antichain has size at most k. Each chain covers the middle row or middle two rows if n is odd. By symmetry δ middle levels has antichain of size k. ■

EX:THE CONVERSE IS NOT TRUE, FIND A COUNTER EXAMPLE.

Lemma 9 B_n *has a* SCD*.*

Proof. Observe $B_n = \mathbf{2} \times \mathbf{2} \times ... \times \mathbf{2}$ i.e. a $0, 1$ vector corresponds to a subset. Clearly $\mathbf{2}$ (along with chains in general) has a SCD i.e. itself. The lemma will follow from the observation that if P and Q have SCDs then so does $P \times Q$. To see this let $c_1, ..., c_r$ be SCD of P and $c'_1, ..., c'_r$ be SCD of Q. Now $c_i \times c'_j$ has SCD obtained by taking *'hooks'*.

■

14.1 Order Ideals

Definition 18 (Order Ideals) *An order ideal* I *of* P *is a subset such that if* $x \in I$ *then* $y \leq x \Rightarrow y \in I$*.*

If P is finite then there is a one to one correspondence between antichains and order ideals. The maximal elements in I are an antichain. We write $I =< a_1, ..., a_k >$ if I is generated by antichains $a_1, ..., a_k$. The order ideals form a poset $J(P)$ when ordered by inclusion.

Remarks

1. the number of elements in $J(P)$ that 'cover' exactly k elements is equal to the number of k element antichains in P. [Remove a generator for $< a_1, ..., a_k >$ to get another order ideal.

2. the number of elements of rank k in $J(P)$ is the number of order ideals of rank k in P.

Theorem 29 *Let* P *be a finite poset. Then the number of surjective order preserving maps* $\gamma : P \to [k]$ *is equal to the number of chains* $\hat{0} = I_0 < I_1 < \ldots < I_k = \hat{1}$ *of length* k *in* $J(P)$*.*

Proof. Construct a bijection between chains and such surjection. Given valid $\gamma : P \to [k]$ set $I_j = \cup_{r=1}^{j} \gamma^{-1}(r)$ ■

Of particular interest is the special case $k = n := |P|$. Then the number of bijective order preserving maps $\gamma : P \to [n]$ is equal to the number of saturated maximal chains in $J(P)$. This is called a linear extension of P. The number of such extensions is denoted $e(P)$ and is perhaps the most useful measure of the complexity of a poset. So finding $e(P)$ is equivalent to counting lattice paths from $\hat{0}$ to $\hat{1}$ in $J(P)$.(permutation π such that $\pi_1, ..., \pi_i$ is an order ideal $\forall i = 1, ..., n$).

If $P := P_1 + P_2 + ... + P_k$ where $n_i = |P_i|$ then $e(P) = \binom{n_1 + ... + n_k}{n_1, n_2, ..., n_k} \cdot e(P_1) \cdot e(P_2) \cdots e(P_k)$ the multinomial coefficient picks image points of $P_1, ..., P_k$ then the $e(P_i)$ points can be ordered.

15 Posets Mobius Inversion

An interval $I(x, y)$ of P is the induced poset formed by $\{z \in P : x \leq z \leq y\}$. Consider all functions $f : Int(P) \to \mathbf{C}$. Where multiplication(convolution) is defined by :

$$fg(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y)$$

The following functions are of interest

1. *identity* : $\delta(x, y) = 1(x, y) = 1 \ if \ x = y \, 0 \ otherwise$
2. zeta function $\zeta(x, y) = 1 \ if \ x \leq y \ 0 \ otherwise$
3. mobius inversion: $\mu(x, y) = 1$, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \ \forall x < y$

   μ is the left inverse of ζ

$$\mu\zeta(x, y) = \sum_{x \leq z < y} \mu(x, z) \zeta(z, y)$$

$$= \mu(x, y) + \sum_{x \leq z < y} \mu(x, z) = 1 \ if \ x = y \ 0 \ otherwise$$

Lemma 10 *Left inverse = right inverse = inverse if it exists*

15.1 Examples of zeta function

1. $$\zeta^2(x, y) = \sum_{x \leq z \leq y} \zeta(x, y) \zeta(z, y) = \sum_{x \leq z \leq y} 1 = |I(x, y)|$$

2. $$\zeta^k(x, y) = \sum_{x = x_0 \leq x_1 \leq ... \leq x_k = y} 1$$

   this is the number of multichains of length k from x to y.

3. $$(\zeta - 1)(x, y) = 1 \ if \ x < y \ 0 \ otherwise$$

So

$$(\zeta-1)^k = \sum_{x=x_0\leq x_1 \ldots \leq x_k=y} (\zeta-1)(x_0,x_1)(\zeta-1)(x_1x_2)\ldots(\zeta-1)(x_{k-1}x_k)$$

$$= \sum_{x=x_0<\ldots<x_k=y} 1$$

equals the number of chains of length k starting at x and ending at y.

Lemma 11 $(2-\zeta)^{-1}(x,y) =$ *total number of chains from* x *to* y

Proof. $(2-\zeta)(x,y) = 1$ *if* $x = y$ -1 *if* $x < y$ Let l be the longest chain in $I(x,y)$. Then $(\zeta-1)^{l+1}(u,v) = 0$ $\forall x \leq u \leq v \leq y$ So $(2-3)(1+(3-1)+(3-1)^2+\ldots+(3-1)^l(u,v)$

$$(1-(\zeta-1)) = (1-(\zeta-1)^{l+1})(u,v) = l(u,v)$$

So

$$(2-3)^{-1} = (1+(3-1)\ldots(3-1)^l)$$

■

Theorem 30 (Mobius Inversion Formula) *Finite* P *and* $f,g : P \to \boldsymbol{C}$. *Then*

$$g(x) = \sum_{y\leq x} f(y)\ \forall x \in P \Leftrightarrow f(x) = \sum_{y\leq x} g(y)\mu(y,x)\ \forall x \in P$$

Proof. Follows from the fact that μ is inverse of ζ. ■$(g(x) = \sum_{y\geq x} f(x) \Leftrightarrow f(x) = \sum_{y\geq x} \mu(x,y)g(y))$

15.2 Examples

1. $P =$ *chain* $\mathbf{N}$
2. $\mu(x,x) = 1$
3. $\mu(x,y) = -\sum_{x=z<y} \mu(x,z) = -1$ $y \dot{\geq} x$ 0 *otherwise*

$$\mu(i,j) = 1 \textit{ if } i = j \ , -1 \textit{ if } i = j-1 \ 0 \textit{ otherwise}$$

$g(n) = \sum_{j=0}^{n} f(i) = f(n) = g(n) - g(n-1)$.

Theorem 31 (Product theorem) *If* $(x,y) \leq (x',y')$ *in* $P \times Q$ *then* $\mu_{P\times Q}((x,y),(x',y')) = \mu_P(x,x') \cdot \mu_Q(y,y')$

Proof. Take $(x,y) \leq (x',y')$ Then $\sum_{(x,y)\leq(u,v)\leq(x',y')} \mu_P(x,u)\mu_Q(y,v)$

$$= \sum_{x\leq u\leq x'} \mu_P(x,u) \sum_{y\leq v\leq y'} \mu_Q(y,v)$$

$$= \delta_P(x,x')\delta_Q(y,y') = \delta_{P\times Q}((x,y),(x',y'))$$

But

$$\sum_{(x,y)\le(u,v)\le(x',y')} \mu_{P\times Q}((x,y),(u,v)) = \delta_{P\times Q}((x,y),(x',y'))$$

■

Lets see where this takes us with the Boolean algebra $B_n = 2 \times 2 \times ... \times 2$ Identify B_n with subsets of an n set X.

$$\mu_{B_n}(S,T) = \mu_1(s_1,t_1)\mu_2(s_2,t_2)\ldots\mu_n(s_n,t_n)\ s_i \in \{0,1\}$$

μ_i corresponds to chain $C_i = 2$.

$$= (-1)^{|T-S|}$$

Mobius Inversion:

$$g(T) = \sum_{S\subseteq T} f(S)\ \forall t \Leftrightarrow f(t) = \sum_{S\subseteq T} (-1)^{|T-S|} g(S) \forall t$$

or

$$g(t) = \sum_{S\geq T} f(S) \forall t \Leftrightarrow f(T) = \sum_{S\geq t} (-1)^{|S-T|} g(S)\ \forall T$$

Lets interpret this . Let $\{A_i : i \in I\}$ be family of subsets of X. $I-E$ is $|\cup_{i\in I} A_i| = \sum_{\emptyset\neq S\subseteq I}(-1)^{|S|-1}|\cap_{i\in S} A_i|$ Let

- $g(T) = |\cap_{i\in T} A_i|$
- $f(T) = |\cap_{i\in T} A_i - \cup_{i\notin T} A_i| \Rightarrow get\ I-E$
- $T = \emptyset$

Consider the divisor poset D_n. D_n is the poset of divisors of n ordered by divisibility. if $n := p_1^{a_1}\cdots p_k^{a_k}$ then $D_n = a_1 + 1 \times a_2 + 1 \times \cdots \times a_k + 1$ Hence $\mu_{D_n}(x,y) = \mu_1(x,y)\cdots\mu_k(x_k,y_k) = (-1)^t\ if\ \frac{y}{x}$ is product of t distinct primes, 0 otherwise.

So $g(x) = \sum_{y\le x} f(y) \Leftrightarrow f(x) = \sum_{y\le x} g(y)\mu(yx)$

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$$

16 Lattices

Given $x, y \in P$ z is an upper bound of x , y if $z \le x, z \ge y$. z is a least upper bound or *join* if $z \le w\ \forall w\ upperbounds\ of\ x\ and\ y$.

Definition 19 (Lattice) *A lattice is a poset in which every pair x, y has a*

- *Join $x \vee y$*
- *Meet $x \wedge y$*

Clearly a finite lattice has a $\hat{0}$ and $\hat{1}$.

Definition 20 (Modular) *A finite lattice L is* ***modular*** *if it is graded and*

$$\rho(x)+\rho(y)=\rho(x\wedge y)+\rho(x\vee y)\ \forall x,y\in L$$

Lemma 12 *In a finite lattice the follow are equivalent*

1. *L is graded: $\rho(x)+\rho(y)\geq\rho(x\wedge y)+\rho(x\vee y)\ \forall x,y$*
2. *If x and y both cover $x\wedge y$ then $x\vee y$ covers x and y*

Proof. (1) $\rightarrow$ (2) trivial. For (2) $\rightarrow$(1): First show L is graded. Suppose not , take smallest interval $I(u,v)$ that is not graded. u is covered by $x_1,x_2\in I$ or I is not graded. But $I[x_1,v],I[x_2,v]$ are graded, i.e. maximal chains have the same lengths sayl_i in $I[x_i,v]$ So WLOG $l_1\neq l_2$.

By (2) $x_1\vee x_2$ covers x_1,x_2 But then $l_1=l_2$(use chains through $x_1\vee x_2$) . So L is graded. Choose $x,y\in L$ such that

$$\rho(x)+\rho(y)<\rho(x\wedge y)+\rho(x\vee y)$$

such that $l(x\wedge y,x\vee y)$ is minimized. (such that $\rho(x)+\rho(y)$ is minimized). ∎

Lemma 13 *In a finite lattice TFAE*

1. *L is graded and $\rho(x)+\rho(y)\geq\rho(x\wedge y)+\rho(x\vee y)\forall x,y$*
2. *Both x,y can not cover $x\wedge y$.*

Proof. WLOG $x>x'>x\wedge y$ by minimality

$$\rho(x')+\rho(y)\geq\rho(x'\wedge y)+\rho(x'\vee y)=\rho(x\wedge y)+\rho(x\vee y)$$

But

$$x\wedge(x'\vee y)\geq x'$$

and

$$x\vee(x'\vee y)=x\vee y$$

i.e. this pair violates the choice of x,y ∎

17 Distributive Lattices

Combinatorially the most important lattices are *distributive lattices.*

Definition 21 (Distributive Lattices) ***Distributive Lattices if satisfy***

1. $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$
2. $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$

There is a nice way to view distributive lattices

Theorem 32 (Fundamental thm for finite DL) *If L is finite D.L then there exists a unique P such that $J(P)\cong L$*

Proof. We say that $x \in L$ is join(meet) irreducible if we can't write $x = y \vee z$ where $x > y, z$. Let $P \subseteq L$ be join irreducible elements in L. Then $J(P) \cong L$. Take $x \in L$ and let $I_x := \{y \in P : y \leq x\}$. Clearly $I_x \in J(P)$. The resulting map $\phi : L \to J(P)$ is an order preserving injection. We want to show it is surjective. Take $I \in J(P)$ and set $x = \bigvee\{y : y \in I\}$. We want $I = I_x$, now $I \subseteq I_x$. Suppose $z \in I_x$ want to show that $z \in I$.

$$x = \bigvee\{y : y \in I\} \leq \bigvee\{y : y \in I_x\} = x$$

So apply $\bigwedge z$ by distributivity.

$$\bigvee\{y \wedge z : y \in I\} = \bigvee\{y \wedge z : y \in I_x\}$$
$$= z$$

So there exists $y \in I$ such that $y \wedge z = z \Rightarrow y \geq z$ since I is an order ideal this means that $z \in I$ as well.

18 Binet- Cauchy Thm

From linear algebra:

Theorem 33 (Binet-Cauchy) *Let A be $n \times m$ matrix, B $m \times n$, D a $m \times m$ diagonal matrix. Then*

$$|ADB| = \sum_S |A_S| \cdot |B^S| \prod_{i \in S} d_i$$

where the sum is over all n-subsets $S \subseteq [1, 2, ..., m]$

- *A_S is $n \times n$ submatrix of A induced by cols corresponding to S.*
- *B^S is $n \times n$ submatrix of B induced by cols corresponding to S.*

All we need is

Theorem 34 $|AA^T| = \sum_S |A_S|^2$

18.1 Example

Take graph G with labeled vertices $v_1, ..., v_n$. Let C be vertex arc adjacency matrix induced by "some" orientation of the edges. Take $C \cdot C^T$ is a square matrix with $m_{ii} = deg(v_i)$, $m_{ij} = -1 \ if(v_i, v_j) \in E, \ else \ 0.$ $= D - A$ where A is vertex -vertex adjacency matrix, and D is the diagonal matrix of degrees.

Let M_{ii} be the matrix obtained by deleting the ith row and ith column of M.

$$M_{ii} = (C - row_i)(C - row_i)^T$$

write as

$$C^{-i} \cdot (C^{-i})^T$$

Theorem 35 (Matrix Tree thm) *The number of spanning trees of G is equal to $|M_{ii}|$.*

Proof. By Binet-Cauchy
$$|M_{ii}| = \sum_B |B| \cdot |B^T| = \sum_B |B|^2$$
where B runs over $(n-1) \times (n-1)$ submatrices of C^{-i}. So if $|B| = \pm 1$ then the edges of$B$ correspond to a spanning tree, 0 otherwise. To show this suppose B does not give a spanning tree. WLOG let $i = 1$. So B induces at least 2 components . So there is a component R that does not contain v_i. But row sum of vertices in R is 0 since each edge has $+1, -1$ in rows of R.

Therefore $|B| = 0$. Take B a spanning tree. Renumber vertices s.t. $w_1 \neq v_i$ and v_1 has degree $= 1$ with respect to B. A tree has at least 2 leaves, so w_1 exists. Repeat on $B - w_1$, (let e_1 be edge of B incident to w_1. To get $w_2 \neq v_2$, e_2, w_3, e_3. By construction if $e = (w_s, w_t)$ then $s < t$. i.e. Lower triangular. Therefore ± 1 on diagonals. Therefore $|B| = \pm 1$ ■

Corollary 8 (Cayley's Theorem) *The number of spanning trees of K_n is n^{n-2}.*

Proof. $M_{11}(K_n)$ = product of $n-1$ on diagonal -1's everywhere else and Add $\sum_{i>1} r_i$ to give row 1 all ones. is upper diagonal with $n-2$ n's on diagonal.

19 Generating Functions

Let $f(n)$ be the number of objects of type f of size n. Then the basic generating function method is

- find a recurrence for $f(n)$
- multiply both sides of the recurrence by x^n and sum over values of n for which recurrence holds
- Solve the resulting equation for $F(x) := \sum_{n \geq 0} f(n)x^n$
- Use this generating function , e.g. we may using partial sums find an exact formula for $f(n)$.

for example
$$f(n) = 2f(n-1) + n - 1$$
$$\sum_{n\geq 1} f(n) = 2\sum_{n\geq 1} f(n-1)x^n + \sum_{n \geq 1}(n-1)x^n$$
$$F(x) - f(0) = 2xF(x) + x\sum_{n\geq 0}(n)x^n$$
$$F(x) - f(0) = 2xF(x) + x^2\frac{1}{(1-x)^2}$$
$$F(x) - 1 = 2xF(x) + x^2\frac{1}{(1-x)^2}$$
$$F(x) = \frac{1-2x+2x^2}{(1-2x)(1-x)^2}$$

Now we can find nicer representation using partial fractions

$$F(x) = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x} = \frac{1-2x+2x^2}{(1-2x)(1-x)^2}$$

multiply by $(1-2x)(1-x)^2$

$$A(1-2x) + B(1-x)(1-2x) + C(1-x)^2 = 1 - 2x + 2x^2$$

substitute some convenient values for x and solve for $A = -1, B = 0, C = 2$. So

$$F(x) = -1 \cdot \frac{1}{(1-x)^2} + 2 \cdot \frac{1}{1-2x}$$

$$[x^n]F(x) = f(n) = -1[x^n]\frac{1}{(1-x)^2} + 2[x^n]\frac{1}{1-2x}$$
$$= -1 \cdot (n+1) + 2^{n+1}$$

Lets look at a multi-variable example: $f(n,k) := \#$ of k-subsets of an n-set Have

$$f(n,k) = f(n-1,k) + f(n-1,k-1)$$

Let $F_n(x) = \sum_{k \geq 0} f(n,k)x^k$

$$F_n(x) = f(n,0) + \sum_{k \geq 1}(f(n-1,k)x^k + f(n-1,k-1)x^k)$$

$$= 1 + \sum_{k \geq 1} f(n-1,k)x^k + \sum_{k \geq 1} f(n-1,k-1)x^k$$
$$= 1 + (F_{n-1}(x) - 1) + F_{n-1}(x)$$

$F_0(x) = 1$ so

$$F_n(x) = (1+x)^n$$

So

$$[x^k]F_n(x) = \binom{n}{k}$$

Summing over n and multiplying by y^n

$$\sum_{n \geq 0} F_n(x)y^n = \sum_{n \geq 0}(\sum_k f(n,k)x^k)y^n$$

$$\sum_{n \geq 0}(1+x)^n y^n$$

$$= \frac{1}{1-y(1+x)}$$

For example consider

$$\sum_{n \geq 0} \binom{n}{k} y^n = \sum_{n \geq 0} [x^k]F_n(x)y^n$$

$$= [x^k] \sum_{n\geq 0} F_n(x) y^n$$

$$= [x^k] \frac{1}{1-y(1+x)}$$

$$[x^k] \frac{1}{1-y} \cdot \frac{1}{1-\frac{yx}{1-y}}$$

$$= \frac{1}{1-y} [x^k] \frac{1}{1-\frac{y}{1-y}x}$$

$$= \frac{y^k}{(1-y)^{k+1}}$$

20 Formal Power Series

Take formal power series $F(x) = \sum_{n\geq 0} f(n)x^n$. We work in the ring of $f.p.s$ here issues of convergence are non-existent(if after applying operations in the ring our series does converge then we may also apply analytic techniques). If not our algebraic work still applies. e.g. exact formulas for the series still apply.

$$F(x) = \sum_{n\geq 0} n! x^n$$

converges at $x = 0$ but is a *nice* formal power series.

We have operations on formal power series :

1. Addition

$$F(x) + G(x) = \sum_{n\geq 0} (f(n) + g(n))x^n$$

2. Multiply

$$F(x)G(x) = \sum_{n\geq 0} (\sum_{k=0}^{n} f(x)g(n-k))x^n$$

$$\sum_{n\geq 0} h(n)x^n = H(x)$$

We say that $G(x) = F^{-1}(x)$ is the multiplicative inverse or reciprocal of F if $F(x)G(x) = 1$.

Lemma 14 *A formal power series $F(x) = \sum_{n\geq 0} f(n)x^n$ has a multiplicative inverse iff $f(0) \neq 0$. if so it is unique.*

Proof. If F has a reciprocal then $f_0 + f_1 x + f_2 x^2 + ...)(g_0 + g_1 x + ...) = 1 + 0x + 0x^2 + ...$ So $f_0 \cdot g_0 = 1 \Rightarrow f_0 \neq 0$ if $f_0 \neq 0$ then $g_0 = \frac{1}{f_0}$ But then

$$f_0 g_0 = 1$$

$$f_0 g_1 + f_1 g_0 = 0$$

$$f_0 g_2 + f_1 g_1 + f_2 g_0 = 0$$

so we can find $G(x) = F(x)^{-1}$. ■

$H(x) = F(x)^{-1}$ is the compositional inverse of F if $F(H(x)) = x$

$$F(H(x)) = \sum_{n\geq 0} f(n)(H(x))^n = x^2$$

Lemma 15 *A formal power series F has compositional inverse iff $f(1) \neq 0$, $f(0) = 0$*

Proof. If F has computable inverse then

$$f(0) + f(1)H(x) + f(2)H(x)^2 + ... = x$$

$$f_0 + f_1(h_0 + h_1x + h_2x^2 + ...) + f_2(h_0 + h_1x + ...) + ... = x$$

This means $h_0 = 0$ or we need convergence which we cant do. Therefore $f_0 = 0$ therefore $f_1h_1 = 1$ so $f_1h_2 + f_2h_1 = 0...$ etc So we can find H ■For example

$$B(x) = e^{e^x - 1}$$

is a well defined formal power series. Other operations in the ring including calculus exist dont need limiting operations.

20.1 Ordinary Generating Functions

Let

$$F(x) := \sum_{n\geq 0} f(n)x^n$$

be an ordinary generating function.

1. $$sum_{n\geq 0} f_{n+k}x^n = \frac{F(x) - f_0 - f_1x - ... - f_{k-1}x^{k-1}}{x^k}$$

2. Let $xD = x\frac{x}{dx}$

3. $$xDF(x) = \sum_{n\geq 0} nf(n)x^n$$

4. $$(xD)^k F(x) = \sum_{n\geq 0} f_n x^n$$

5. $$F(x)G(x) = \sum_{n\geq 0}(\sum_{k=0}^{n} f_k g_{n-k})x^n$$

For example let $G(x) = \frac{1}{1-x}$. Then $F(x) \cdot G(x) = \sum_{n\geq 0}(\sum_{k=0}^{n} f_k)x^n$. This generalizes to more than 2 products. It is typically the product rule that determines the best choice of generating function. Here for ordinary generating functions $h_n := \sum_{k=0}^{n} f_{n-k}g_k$ means that these h structures are made up of an f object of size $n-k$ and a g object of size k. (Here elements are unlabeled). For example suppose we want to find $\sum_{n=1}^{N} n^2$. Observer

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n$$

$$(xD)^2 \frac{1}{1-x} = \sum_{n\geq 0} n^2 x^n$$

So

$$[x^n \frac{1}{1-x}(xD)^2 \frac{1}{1-x} = \sum_{j=0}^{n} j^2$$

$$= [x^n]\frac{x(1-x)}{(1-x)^4} = \binom{n-1+3}{3} - \binom{n-2+3}{3}$$

20.2 Exponential Generating functions

Let $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ be an exponential generating function. Then

1.

$$\sum_{n\geq 0} f(n+k)\frac{x^n}{n!} = D^k F(x)$$

2.

$$(xD)^k = \sum_{n\geq 0} n^k \frac{f(n)}{n!} x^n$$

3.

$$F(x)G(x) = \sum_{n\geq 0}(\sum_{k=0}^{n} \binom{n}{k} f(k)g(n-k))\frac{x^n}{n!} = H(x)$$

This is useful when the elements are labeled . An h object of size n consists of an $f-object$ of size k and a g object of size $n-k$.

For example consider derangements: $d(n)$.

$$= \sum_{k=0}^{n} \binom{n}{k} f(k)d(n-k) = \sum_{k=0}^{n} \binom{n}{k} 1d(n-k)$$

$$H(x) = \sum_{n\geq 0} n! \cdot \frac{x^n}{n!} = \frac{1}{1-x} = e^x D(x) = \frac{e^{-x}}{1-x}$$

Another example are the Bell numbers. i.e. The number of ways to partition an $n-set$. Recall

$$b(n+1) = \sum_{k=0}^{n} \binom{n}{k} b(k) \cdot 1$$

$$\sum_{n\geq 0} b(n+1)\frac{x^n}{n!} = \sum_{n\geq 0}(\sum_{k=0}^{n} \binom{n}{k} \cdot 1 \cdot b(k))\frac{x^n}{n!}$$

So

$$DB(x) = B(x) \cdot e^x$$

$$B(x) = cexp(e^x)$$

$$B(0) = 1 \Rightarrow c = \frac{1}{e}$$

therefore

$$B(x) = exp(e^x - 1)$$

20.3 Dirchlet Series GFs (DGF)

Let $F(x) = \sum_{n\geq 1} f(n)\frac{1}{n^x}$ is a d.g.f. Take $F(x) \cdot G(x) = (f_1\frac{1}{1^x} + f_2\frac{1}{2^x} + ...)(\frac{g_1}{1^x} + \frac{g_2}{2^x} + ...)$

$$= f_1g_1 + \frac{f_1g_2 + f_2g_1}{2^x} + \frac{f_1g_3 + f_3g_1}{3^x} + \dots$$

$$\sum_{n\geq 1}(\sum_{d|n} f(d)g(\frac{n}{d}))\frac{1}{n^x} = H(x)$$

What if $f(n) = 1$?

$$\sum_{n\geq 1}\frac{1}{n^x} = \zeta(x)_{Rieman\ Zeta\ Function}$$

$$\zeta(x)\zeta(x) = \sum_{n\geq 1}(\sum_{d|n} z(d)z(\frac{n}{d}))\frac{1}{n^x}$$

$$= \sum_{n\geq 1} \#divisors(n)\frac{1}{n^x} = \sum p(n)\frac{1}{n^x}$$

A number theoretic function f is multiplicative if $f(mn) = f(m)f(n)$ when m, n relatively prime. i.e.

$$f(n) = f(p^{r_1})f(p^{r_2})\cdots f(p^{r_k})$$

Theorem 36 *If f is a multiplicative function then*

$$\sum_{n\geq 1} f(n)\frac{1}{n^x} = \prod_{prime}(1 + \frac{f(p)}{p^x} + \frac{f(p^2)}{p^{2x}} + ...$$

Proof. Multiply it out... ∎

$$\zeta(x) = \prod_{p\ prime} (1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \frac{1}{p^{3x}} + \ldots)$$

$$= \prod_{p\ prime} \frac{1}{1 - \frac{1}{p^x}} = \frac{1}{\prod_{p\ prime}(1 - \frac{1}{p^x})}$$

Take the mobius function

$$\mu(p^a) = 1\ if\ a = 0,\ -1\ if\ a = 1,\ 0\ if\ a \geq 2$$

$$\mu(x) = \sum_{n \geq 1} \mu(n) \frac{1}{n^x} = \prod_{p\ prime} (1 + \frac{\mu(P)}{p^x})$$

$$= \prod_{p\ prime} (1 - \frac{1}{p^x})$$

Theorem 37 (Mobius Inversion Formula)

$$a_n = \sum_{d|n} b_d \Leftrightarrow b_n = \sum_{d|n} a(d)\mu(\frac{n}{d})$$

Proof.

$$A(x) = B(x)\zeta(x) \Leftrightarrow B(x) = A(x)\mu(x)$$

∎

21 Combinatorial Identities

Here we give a general technique for proving combinatorial identities.

- Identify free variable say n, and call function $f(n)$.
- Consider the generating function for f.
- Change order of summation
- Solve the new inner summation and the outer one.
- Equate coefficients to give $f(n)$.

21.1 Examples

Evaluate

$$\sum_{k\geq 0}\binom{k}{n-k}$$

- $f(n)=\sum_{k\geq 0}\binom{k}{n-k}$
-

$$\sum_{n\geq 0} f(n)x^n = \sum_{n\geq 0}\sum_{k\geq 0}\binom{k}{n-k}x^n$$

$$=\sum_{k\geq 0}\sum_{n\geq 0}\binom{k}{n-k}x^n$$

$$=\sum_{k\geq 0}x^k\sum_{n\geq 0}\binom{k}{n-k}x^{n-k}$$

$$=\sum_{k\geq 0}x^k(1+x)^k$$

$$=\sum_{k\geq 0}(x+x^2)^k$$

$$=\frac{1}{1-x-x^2}$$

What is $f(n)=\sum_{k\leq\frac{n}{2}}(-1)^k\binom{n-k}{k}y^{n-2k}$

$$\sum_{n\geq 0}\sum_{k\leq\frac{n}{2}}(-1)^k\binom{n-k}{k}y^{n-2k}x^n$$

$$\sum_{k}(-1)^k x^k y^{-k}\sum_{n\geq 2k}\binom{n-k}{k}(xy)^{n-k}$$

$$\sum_{k}(-1)^k(\frac{x}{y})^k\sum_{r\geq k}\binom{r}{k}(xy)^r$$

$$=\sum_{n}(-1)^k(\frac{x}{y})^k\frac{(xy)^k}{(1-xy)^{k+1}}$$

$$=\frac{1}{(1-xy)}\sum_{k}(\frac{-x^2}{1-xy})^k$$

$$\frac{1}{1-xy}\cdot\frac{1}{1+\frac{x^2}{1-xy}}=\frac{1}{1-xy+x^2}$$

Solving by partial fractions...

$$f(n) = \frac{1}{\sqrt{y^2-4}}[(\frac{y+\sqrt{y^2-4}}{2})^{n+1} - (\frac{y-\sqrt{y^2+4}}{2})^{n+1}]$$

Heres another one

$$\sum_{2k \leq n} (-1)^k \binom{n-k}{k} 2^{n-2k} = n+1$$

Another example

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k} 2^k \ m, n \geq 0$$

LHS

$$\sum_{n \geq 0} \sum_k \binom{m}{k}\binom{n+k}{m} x^n$$

$$= \sum_k \binom{m}{k} x^{-k} \sum_{n \geq 0} \binom{n+k}{m} x^{n+k}$$

$$= \sum_k \binom{m}{k} x^{-k} \frac{x^m}{(1-x)^{m+1}}$$

$$= \frac{x^m}{(1-x)^{m+1}} \sum_{k \geq 0} (\frac{1}{k})^k \binom{m}{k}$$

$$= \frac{x^m}{(1-x)^{m+1}} (1+\frac{1}{x})^m = \frac{(x+1)^m}{(1-x)^{m+1}}$$

for the RHS we have

$$\sum_{n \geq 0} \sum_k \binom{m}{k}\binom{n}{k} 2^k x^n = \sum_k 2^k \binom{m}{k} \sum_{n \geq 0} \binom{n}{k} x^n$$

$$= \sum_k 2^k \binom{m}{k} \frac{x^k}{(1-x)^{k+1}}$$

$$= \frac{1}{1-x} \sum_k \binom{m}{k} (\frac{2x}{1-x})^k = \frac{1}{1-x}(1+\frac{2x}{1-x})^m$$

$$= \frac{(1+x)^m}{(1-x)^{m+1}}$$

Heres an exponential generating function one: What is $f(n) = \sum_k s(n,k)b(k)$? where $s(n,k)$ is the Stirling number of the first kind, and $b(k)$ are the Bernoulli numbers which satisfy

$$\sum_{n \geq 0} b(n) \frac{x^n}{n!} = \frac{x}{e^x - 1}$$

$$\sum_n f(n) \frac{x^n}{n!} = \sum_n \sum_k s(n,k) b(k) \frac{x^n}{n!}$$

$$= \sum_k b(k) \sum_n s(n,) \frac{x^n}{n!}$$

$$= \sum_k b(k) \frac{(\log(\frac{1}{1-x}))^k}{k!}$$

$$= \sum_k b(k) \frac{z^k}{k!} = \frac{z}{e^z - 1}$$

So far we used the method in cases where the free variable appears once. What is

$$sum_k \binom{n}{k} \binom{2n}{n-k}$$

?? Solve

$$\sum_k \binom{n}{k} \binom{m}{r-k}$$

instead. Specialize the answer.

22 Combinatorial Interpretation of Generating Functions

Consider $F(x)G(x)$ exponential generating functions.

Lemma 16 *Take $f, g, h : N \to C$ suppose $h(\#X) = \sum_{S,T} f(\#S)g(\#T)$ where S, T are ordered partitions over a finite set X. Then $H(x) = F(x)G(x)$.*

Let $n := \#X$ there are $\binom{n}{k}$ partitions (S, T) with $\#S = k$ so $h(n) = \sum_k \binom{n}{k} f(k)g(n-k)$ ■

Interpretation: We put 2 structures f, g on a set X, then a "combined" structure $h = f \cup g$ is obtained by splitting X into two and putting an f structure on one part and a g structure on the other. If the number of structures (f or g) depends only on the set size then $h(n) = \sum_k \binom{n}{k} f(k)g(n-k)$ is the number of h structures.

e.g. Let $h(n)$ be the number of ways to partition an n set X into S, T and to linearly order S and choose a subset of T. There are $f(k) = k!$ ways to order the k set, and there are $g(k) = 2^k$ ways to pick a subset of a k set.

$$H(x) = F(x)G(x)$$

$$= \sum n! \frac{x^n}{n!} \cdot \sum 2^n \frac{x^n}{n!}$$

$$= \frac{1}{1-x} \cdot e^{2x}$$

More generally

Lemma 17 *Take $f_1, ..., f_k : N \to C$ such that $h(\#X) = \sum_{(S_1,...,S_k)} f_1(\#S_1) \cdots f(\#S_k)$ where $(S_1, ..., S_k)$ ranges over ordered partitions of X into k sets. Then $H(x) = F_1(x)F_2(x) \cdots F_k(x)$*

What is the interpretation of *composition*(algebraic)?

Theorem 38 (Compositional Formula) $f, g, h : N \to C$ $g(0) = 1, f(0) = 0$

$$h(|X|) = \sum_{B_1,...,B_k \in \Pi(X)} f(|B_1)f(|B_2|) \cdots f(|B_k|)g(k)$$

$h(0) = 1$ *where* $B_1, ..., B_k$ *is over unordered partitions of* X*. Then* $H(x) = G(F(x))$

Proof. Let $n := |X|$ and for fixed k let $h_k(|X|) = \sum_{(B_1,..,B_k)} f(|B_1|) \cdots f(|B_k|)g(k)$ Since B_i are non-empty we can order them in $k!$ ways. So by the previous lemma

$$\frac{k!}{g(k)} H_k(x) = (F(x))^k$$

$h(|X|) = \sum_{k \geq 1} h_k(|X|)$. So

$$H(x) = \sum_{k \geq 1} \frac{g(k)F(x)^k}{k!} + h(0)$$
$$= G(F(x))$$

■

Interpretation: many structures on a set, graph, poset, permutation can be considered as structures on a disjoint union of structures. More over some additional structure ordering may be put on the components themselves. Of particular interest is the case $g(k) = 1$ $\forall k$.

Theorem 39 (Exponential Formula) $f, h : N \to C$ $f(0) = 0, h(0) = 1$, $h(|X|) = \sum_{B_1,...,B_k} f(|B_1|) \cdots f(|B_k$ *then* $H(x) = e^{F(x)}$

22.1 Examples

Permutations: A permutation π is a collection of disjoint directed cycles.

$$h(|X|) = \sum_{B_1,...,B_k} f(|B_1|) \cdots f(|B_k|)$$

where $f(n) = (n-1)! = \#$ of dicycles on n set. So $H(x) = e^{F(x)}$. We have

$$\frac{1}{1-x} = e^{F(x)}$$

so

$$F(x) = \log(\frac{1}{1-x}).$$

How many labeled connected graphs $c(n)$ **are there on an** $n - set$ V **?** It is easy to count simple graphs. $h(|V| = n) = 2^{\binom{n}{2}}$. So

$$h(n) = \sum_{B_1,...,B_k} c(|B_1|) \cdots c(|B_k|)$$

So

$$H(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} = e^{C(x)}$$

$$C(x) = \log(\sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!})$$

We see that a useful operation: $xD \log$

- log simplifies RHS
- D then simplifies LHS
- x puts back power lost by D.

$$xD\log(H(x)) = xDC(x)$$

$$x\frac{H'(x)}{H(x)} = xDC(x)$$

$$xDH(x) = H(x)xDC(x)$$

$$\sum_{n\geq 1} nh(n)\frac{x^n}{n!} = H(x)\sum_{n\geq 1} nc(n)\frac{x^n}{n!}$$

$$n2^n = \sum_k \binom{n}{k} kc(k)h(n-k)$$

$$n2^n = \sum_k \binom{n}{k} kc(k)2^{\binom{n-k}{k}}$$

This is a recurrence for $c(n)$.

22.2 Exponential Formula (two variables)

Suppose we want to keep track of the number of blocks. e.g. the number of connected components , number of cycles in π... We do this with a 2 variable generating function.

$$H(x,y) := \sum_{n\geq 0}(\sum_{k\geq 0} h(n,k)y^k)\frac{x^n}{n!}$$

Its easy to evaluate $H(x,y)$ using compositional formula

Theorem 40 (two variable exponential formula) *Take $f,h : N \to C$ $h(|X|) = \sum_{B_1,\ldots,B_k} f(|B_1|)\cdots f(|B_k$ with $h(0) = 1$. Then $H(x,y) = e^{yF(x)}$*

Proof. Set $g(k) = y^k$ in the compositional formula to keep track of the number of blocks. So $G(x) = \sum_{n\geq 0} y^n \frac{x^n}{n!} = e^{xy}$. So $H(x,y) = G(F(x))$ by compositional formula.

$$= e^{(F(x)y)}$$

■

Corollary 9 $H(x,y) = (H(x))^y$

Proof. $H(x) = e^{F(x)}$ ■

22.3 Exponential Formula -cont

$H(x,y) = e^{yF(x)} = H(x)^y$

22.3.1 Examples

Let $h(n)$ be the number of permutations , we have

$$H(x,y) = H(x)^y = \frac{1}{(1-x)^y}$$

$h(n,k) = s(n,k) =$ *signless* Stirling number of the first kind. So

$$\sum_k s(n,k)y^k = [\frac{x^n}{n!}]\frac{1}{(1-x)^y}$$

$$= n![x^n]\frac{1}{(1-x)^y}$$

$$= n![x^n](1+x+x^2+...)^y$$

$$= n!\binom{n+y-1}{y-1} = n!\binom{n+y-1}{y-1} = (n+y-1)(n+y-2)\cdots(y+1)$$

$$= y(y+1)(y+2)\cdots(y+n-1)$$

Since $H(x,y) = e^{y\log(\frac{1}{1-x})}$

$$[y^k]H(x,y) = \frac{1}{k!}(\log(\frac{1}{1-x}))^k$$

$$s(n,k) = [\frac{x^n}{n!}]\frac{1}{k!}(\log(\frac{1}{1-x}))^k$$

$$= [x^n]\frac{n!}{k!}(\log(\frac{1}{1-x}))^k$$

Another example. For graphs

$$H(x,y) = H(x)^y = (\sum_{n\geq 0} 2^{\binom{n}{2}}\frac{x^n}{n!})^y$$

Lets count permutations such that $\pi^m = 1$. This means that π consists of cycles whose lengths divide m. So

$$F(x) = \sum_{d|m}(d-1)!\frac{x^d}{d!}$$

So $H(x) = \exp(\sum_{d|m}\frac{x^d}{d})$. e.g. if $m = 2$ we count *involutions*($\pi^2 = 1$). $t(n) := \#$ of involutions. $T(x) = \exp(x + \frac{1}{2}x^2)$.

Suppose we want to count $h(n)$ for specific values of n. e.g. $n \in S \subseteq N$.

Corollary 10 $H(x) = \sum_{n\in S}\frac{F(x)^n}{n!}$

Example: take $h(n) :=$ number of permutations with even number of odd cycles, and no even cycles.

$$F(x) = \sum_{n\ odd} (n-1)!\frac{x^n}{n!} = \sum_{n\ odd} \frac{x^n}{n}$$
$$= \frac{1}{2}\log(\frac{1+x}{1-x})$$

So by the corollary

$$H(x) = \sum_{n\ even} \frac{F(x)^n}{n!}$$
$$= Cosh(F(x)) = \frac{1}{2}(e^x + e^{-x})$$
$$= Cosh(\log(\sqrt{\frac{1+x}{1-x}}))$$
$$= \frac{1}{\sqrt{1-x^2}} = (1-x)^{\frac{1}{2}}$$
$$= \sum_{n\geq 0} \binom{-\frac{1}{2}}{n}(-x^2)^n$$
$$= \sum_{n\ even} \frac{1}{2^n}\binom{n}{\frac{n}{2}}x^n$$

i.e.

$$h(n) = \frac{n!}{2^n}\binom{n}{\frac{n}{2}}$$

Corollary 11 *The probability that a random π consists of an even number of odd cycles is equal to the probability that we get $\frac{n}{2}$ heads in n coin tosses.*

22.4 Combinatorial Interpretations of Generating Functions

So we know what multiplication and composition mean. What about addition, differentiation?

Lemma 18 *Let x be finite set $f, g \to N$, if $h(|X|) = f(|X|) + g(|X|)$ then $H(x) = F(x) + G(x)$.*

Interpretation: Place either f structure of g structure on X.

Lemma 19 *If $h(|X|) = |x|f(|Y|)$ where $|Y| = |X| - 1$, then*

$$H(x) = xF(x)$$

Interpretation: Pick "node" r in X. Put f structure on $x - r$.

Lemma 20 *If $h(|X|) = f(|Z|)$ when $|Z| = |X| + 1$ then $H(x) = F'(x)$*

Interpretation: Add a new element z to x , put f structure on $X \cup z$.

Lemma 21 *If $h(|X|) = |X|f(|X|)$ then $H(x) = xF'(x)$*

Interpretation: Place f structure on X , then pick root of X.

Theorem 41 (Exp Formula) $h(|X|) = \sum_{B_1,\ldots,B_k} f(|B_1|)\cdots f(|B_k|) \Rightarrow H(x) = \exp(F(x))$

$$H'(x) = F'(x)\exp F(x) = F'(x)H(x)$$

Compare coefficients of $\frac{x^n}{n!}$ to get

$$h(n+1) = \sum_{k=1}^{n} \binom{n}{k} h(k)f(n+1-k)$$

$$f(n+1) = h(n+1) - \sum_{k=1}^{n} \binom{n}{k} h(k)f(n+1-k)$$

By Lemma 3 $H'(x)$: add z to X put H structure on $X \cup z$.

$$F'(x)H(x)$$

Pick subset $S \subseteq X$. Add z to S, put f structure on $S \cup z$. Put h structure on $X - S$.

But h is a disjoint set of f structures. So these are the same things.

23 Enumeration of Trees

Labeled vertices. Let

- $t(n) :=$ the number of trees on $[n]$.
- $f(n) :=$ the number of forests on $[n]$
- $r(n) :=$ the number of rooted (planted) trees on $[n]$
- $p(n) :=$ the number of rooted (planted forests) on $[n]$

Theorem 42 $R(x) = xe^{R(x)}$

Proof. $R(x) := \sum_{n\geq 1} r(n)\frac{x^n}{n!}$ So $P(x) = e^{R(x)}$. But $xP(x)$ is a root and a P structure on the rest, i.e. a root and a forest on the rest i.e. a rooted tree. So

$$R(x) = x\exp(R(x))$$

Similarly

$$F(x) = \exp(T(x))$$

and also

$$P(x) = T'(x)$$

24 Lagrange Inversion Formula

Theorem 43 (Lagrange Inversion) *Let $G(x) = g_0 + g_1\frac{x}{1!} + g_2\frac{x^2}{2!} + \ldots$ with $g_0 \neq 0$ and let $f(x) = xG[F(x)]$. Then $n[x^n]f(x)^k = k[x^{-k}]G(x)^n$ $(k, n \in Z)$*

Proof.
Let $F(x)^{(-1)} = H(x)$ be the compositional inverse of $F(x)$ if $H(F(x)) = F(H(x)) = x$

Corollary 12 *Let $F(x) = f_1x + f_2x^22! + \ldots$, $f_i \neq 0$. then $n[x^n](F(x)^{-1})^k = k[x^{-k}]F(x)^{-n}$*

Proof. This follows as $f(x) = F^{-1}(x)$ is the same as $f(x) = xG(f(x))$ where $G(x) = \frac{x}{F(x)}$ ∎

24.1 Example

Let $r(n)$ be the number of rooted trees on $[n]$. Know $R(x) = xe^{R(x)}$ i.e. $x = R(x)e^{-R(x)}$ so $R(x) = H^{-1}(x)$ where $H(x) = xe^{-x}$

Lemma 22 *The number of rooted trees is n^{n-1}*

Proof. $R(x) = (xe^{-x})^{(-1)}$ Set $F(x) = xe^{-x}$ set $k = 1$

$$[x^n](xe^{-x})^{(-1)} = \frac{1}{n}[x^{n-1}](\frac{x}{xe^{-x}})^n$$

$$[x^n]R(x) = \frac{1}{n}[x^{n-1}]e^{nx}$$

$$= \frac{1}{n}[x^{n-1}]\sum_{t\geq 0}\frac{(nx)^t}{t!} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$$

∎

Lemma 23 *The number of k forests is $\binom{n-1}{k-1}n^{n-k}$*

Proof.

$$[x^n]R(x)^k = \frac{k}{n}[x^{n-k}](\frac{x^n}{xe^{-x}})^n = \frac{k}{n}\frac{n^{n-k}}{(n-k)!}$$

Recall from our exponential formula

$$\sum_n\sum_k p(n,k)y^k\frac{x^n}{n!} = \exp(yR(x))$$

where $p(n,k) =$ the number of planted k forests on $[n]$ So $\frac{p(n,k)}{n!} = [x^n]\frac{R(x)^k}{k!}$ i.e.

$$p(n,k) = \frac{n!}{k!}\frac{k}{n}\frac{n^{n-k}}{(n-k)!}$$

So the number of planted k forests is $\binom{no1}{k-1}x^{n-k}$ ∎

Corollary 13 *Take* $F(x) = f_1 x + f_2 \frac{x^2}{2} + ...$ *and* $H(x)$ *a Laurant series. Then*

$$n[x^n]H(F(x)^{(-1)}) = [x^{n-1}]H'(x)(\frac{x}{F(x)})^n$$

Proof. By linearity it suffices to prove for $H(x) = x^k$ This is Lagrange inversion formula as $H(x) = kx^{k-1}$. ■

another example *Find the sum of the first* n *terms in binomial expansion of* $(1 - \frac{1}{2})^{-n}$
We need to compute $f(n) = [x^{n-1}](1 - \frac{1}{2}x)^{-n}(1-x)^{-1}$. Let

$$\frac{x}{F(x)} = (1 - \frac{1}{2}x)^{-1}$$

and

$$H'(x) = \frac{1}{1-x}$$

$$n[x^n]H(F(x)^{(-1)}) = [x^{n-1}]H'(x)(\frac{x}{F(x)})^n$$

$$= [x^{n-1}](1-x)^{-1}(1 - \frac{1}{2}x)^{-n}$$

$$F(x)^{(-1)} = 1 - \sqrt{1-2x}$$

$$H(x) = -\log(1-x)$$

$$s(n) = n[x^n] - \log(1 - (1 - \sqrt{1-2x}))$$

$$= n[x^n] - \log(\sqrt{1-2x})$$

$$= \frac{n}{2}[x^n]\log(1-2x)$$

$$= \frac{n}{2}[x^n]\frac{(2x)^n}{n} = 2^{n-1}$$

25 Young Tableau

Definition 22 (Standard Young Tableau) *A standard Young Tableau is an* n *box Young diagram filled with the numbers* $1, 2, ..., n$ *such that numbers increase rightwards along rows and increase downwards along columns.*

Definition 23 (Young Tableau) *Weakly increasing along rows. Strictly increasing down columns.*

Let f^λ be the number of standard Young Tableau of shape λ. Then

Theorem 44 (Frobenius-Young)

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof.

We give a bijection between permutations and pairs (P, Q) of SYT of the same shape on n boxes. Take $\pi \in S_n$. We build P as follows:

- Given partial SYT built by $\pi_1, ..., \pi_{i-1}$.
- We insert π_i in row 1 in place of the smallest entry y greater than π_i(exists as put ∞ at the end of each row). Insert y into row z by same procedure etc. This is called **Bumping**.

e.g Take

$$2761354$$

$$2\ 27\ 26\ 16\ 13\ 135\ 134$$

$$-\ \ --\ 7\ 2\ 26\ 26\ 25$$

$$-\ \ ---\ \ -\ 7\ 7\ 7\ 6$$

$$-\ \ --\ -\ -\ -\ -\ 6$$

How do we get Q? Q has the same shape but numbers are given according to the order in which boxes are added.

Note: P is a SYT.

- Rows are fine as we insert x after smaller number in row before bigger numbers.
- Columns are fine as a number bumped down can not move further to the right in the row below.
- Q is a SYT by a similar argument. Numbers are bigger than the previous ones added.

This is a bijection. Given (P, Q) we recover π as follows. We pick boxes in reverse order based on numbering in Q. We then bump upwards the corresponding element x in P. Push x to row above replacing largest number smaller than it y. Repeat with y on row above. Repeat until pops out of the top. ■

Corollary 14

$$\sum_{\lambda \vdash n} f^\lambda = \#\ of\ involutions$$

Proof. Suppose $\pi \to (P, Q)$. Look at π^{-1}. Show $\pi^{-1} \to (Q, P)$. So involutions $\pi \to (P, P)$. ■

To count involutions:

$$a_{n+1} = a_n + na_{n-1}$$

$$\sum a_n \frac{x^n}{n!} = e^{x+\frac{1}{2}x^2}$$

Since we can view each involution is a matching with singletons

$$\#involutions = \sum_{l,k,l+2k=n} \binom{n}{l} (2k-1)(2k-3)\cdots 3\cdot 1$$

$$= \sum_k \binom{n}{n-2k} \frac{(2k)!}{2^k k!}$$

26 Schur Polynomials

Let T be a Young tableau(opposed to standard) of shape λ(weakly increasing in rows). Then

$$f(T) = \prod_{i \geq 1} x_i^{n_i}$$

and the Schur polynomial for shape λ is

$$S_\lambda(x_1, ..., x_m) = \sum_{T \text{ with shape } \lambda \text{ lables in } [m]} f(T)$$

Two important cases are: *complete symmetric polynomial*, and *elementary symmetric polynomial* . In fact Schur polynomials are symmetric for general λ. To show this we first use the Schensted bumping algorithm to define a product of tableau

$$T = U \cdot V$$

as follows:

- Repeatedly insert the first element of the last row of V into U.

Lemma 24 *This is associative.*

(can prove it using a sliding algorithm $U \cdot V = Rect[U \cdot V]$)

Lemma 25 *if $T = Y \cdot [.][.][.][.]$ then the boxes we add to U to get T are in different columns.*

(As we add numbers in increasing order the elements that get bumped move rightwards.)

Lemma 26 *If $T = U \cdot column$ then the boxes we add to U to get T are in different rows.*

As corollaries

Theorem 45 $s_\lambda(x_1...x_m)s_{row\ length\ k}(x_1...x_m) = \sum_{\hat{x}} s_{\hat{x}}(x_1...x_m)$ *where $\hat{x}$ is obtained from λ by adding k boxes in different columns.*

Theorem 46 $S_\lambda(x_1, ..., x_m)S_{column}(x_1, ..., x_m) = \sum_{\hat{x}} s_{\hat{\lambda}}(x_1, ..., x_m)$

where $\hat{\lambda}$ is obtained from λ by adding k boxes in different rows. ■

26.1 Kostka Numbers

$$K_{\lambda\mu} := \#\ of\ tableau\ of\ shape\ \lambda\ with\ \mu_1\ 1's\ \mu_2\ 2's\ etc$$

where

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq ... \geq \mu_m$$

So

$$K_{\lambda\mu} = \#\ of\ sequences\ \lambda_1 \subseteq ... \subseteq \lambda_m = \lambda$$

where $\lambda_{i+1} - \lambda_i$ has μ_{i+1} boxes in different columns.

Corollary 15

$$s_{\mu 1}s_{\mu 2}\cdots s_{\mu_m} = \sum_{\lambda} K_{\lambda\mu}s_{\lambda}(x_1...x_m)$$

$$s_{p_1}...s_{p_m} = \sum_{\lambda} K_{\lambda^T p}s_{\lambda}(x_1,...,x_m) = \sum_{\lambda} K_{\lambda p}S_{\lambda^T}(x_1,...,x_m)$$

We can order the λ lexicographically as follows

$$\lambda \leq \lambda^* \ if \ \lambda_i < \lambda_i^* \ and \ \lambda_j = \lambda_j^*$$

So

$$K_{\lambda,\mu} = 1 \ if \ \lambda = \mu \ 0 \ if \lambda > \mu$$

Corollary 16 *Schur polynomial is symmetric*

26.2 The Hooklength Formula

So the number of standard young tableau on n boxes is equal to the number of involutions. How many standard Young tableau are there for a fixed shape λ? Is there a nice formula for $f^{\lambda} = f(\lambda_1,...,\lambda_m)$ where $|\lambda| = n$.

First lets investigate $f(\lambda_1,..,\lambda_m)$ clearly

1. $f(\lambda_1,...,\lambda_m) = 0$ unless $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$
2. $f(0,0,...,0) = 1, \ f(n) = 1$
3. $f(\lambda_1,...,\lambda_m,0,0..) = f(\lambda_1,...,\lambda_m)$

We also have a recurrence

$$f(\lambda_1...\lambda_m) = f(\lambda_1 - 1,\lambda_2,...,\lambda_m) + f(\lambda_1,\lambda_2 - 1,...,\lambda_m) + \cdots + f(\lambda_1,\lambda_2,...,\lambda_m) - 1$$

if $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$. (Look at where $\#n$ goes). We will relate f^{λ} to the Vandermode Determinate $V(\alpha_1,...,\alpha_m) = \prod_{i<j}(\alpha_i - \alpha_j)$.

Lemma 27 *if* $g(\alpha_1,...,\alpha_m;\Delta) = \sum_i \alpha_i V(\alpha_1,...\alpha_i+\Delta,...\alpha_m)$*. Then* $g(\alpha_1,...,\alpha_m;\Delta) = (\alpha_1+\alpha_2+...+\alpha_m+\binom{n}{2}\Delta)V(\alpha_1,...,\alpha_m)$

Proof. g is a homogeneous polynomial of degree $1 + degV(\alpha_1,...,\alpha_m)$ moreover if we interchange α_i and α_j then g changes sign. So $(\alpha_i - \alpha_j)$ divides g and so is $\prod_{i<j}(\alpha_i - \alpha_j)$

If $\Delta = 0$ the result is trivial. So if $\Delta \neq 0$ what is its coefficient?

$$\alpha_1 V(\alpha_1 + D, \alpha_2,...,\alpha_m)$$

$$= \alpha_1 \prod_{j\geq 2}(\Delta + (\alpha_1 - \alpha_j)) \prod_{2\leq i<j}\prod(\alpha_i - \alpha_j)$$

So the coefficient of Δ in this term is

$$\alpha_1 \sum_{j \geq 2} \frac{1}{\alpha_1 - \alpha_j} V(\alpha_1, ..., \alpha_m)$$

It follows that the coefficient of Δ in g is

$$\sum_{i<j} (\frac{\alpha_i}{\alpha_i - \alpha_j} + \frac{\alpha_j}{\alpha_j - \alpha_i})$$

$$= \binom{m}{2} V(\alpha_1, ..., \alpha_m)$$

■

Theorem 47 *The number of Standard Young Tableau of shape* λ

$$f(\lambda_1, ..., \lambda_m) = \frac{n!}{(\lambda_1 + m - 1)!(\lambda_2 + m - 2)! \cdots \lambda_m!} \cdot V(\lambda_1 + m - 1, \lambda_2 + m - 2, \cdots, \lambda_m$$

Proof. If $\lambda_i + m - i = \lambda_i + m - (i+1)$ then $LHS = RHS = 0$ So $\lambda_i + m - i > \lambda_{i+1} + m - (i+1)$. Want to show RHS h satisfies i, ii, iii, iv. The base cases are easy to check. So consider (iv). Set

$$\alpha_i = \lambda_i + m - i$$

and $\Delta = -1$ in claim. Then $\frac{(n-1)!}{(\lambda_1+m-1)! \cdots \lambda_m!}(\lambda_1 + m - 1)V(\lambda_1 + m - 2, \lambda_2 + m - 2) + (\lambda_2 + m - 2)V(\lambda_1 + m - 1, \lambda_2 + m - 3) + ... + \lambda_m V(\lambda_1 + m - 1, ..., \lambda_{m-1})$
$= \frac{(n-1)!}{(\lambda_1+m-1)! \cdots \lambda_m!}(\sum \lambda_i + m^2 - (1 + 2 + ... + m) - 1 \cdot \binom{m}{2})V(\lambda_1 + m - 1, ..., \lambda_m)$
$= \frac{(n-1)! \sum \lambda_i}{1!} V = \frac{n!}{\lambda_1 + m - 1 ... \lambda_m} V(..)$ ■

This is not very illuminating(!) We can rewrite it in a nice way. Let $h_{ij} = hooklength$ of box $ij = 1 + \#$ of boxes below + number of boxes to the right.

Theorem 48 $f^\lambda = \frac{n!}{\prod_{ij \in \lambda} h_{ij}}$

let h_{ij} = hooklength of box ij = 1+ number of boxes below + number of boxes to right.
Proof. Consider the products of hook lengths in row i. e.g. $i = row1$. We can check

$$\frac{(\lambda_1 + m - 1)!}{\prod_{2 \leq j \leq m} [(\lambda_1 + m - 1) - (\lambda_j + m - j)]}$$

= product of row 1 hooks.

So $\prod f_{ij} \in \lambda h_{ij} = \frac{\prod_{i=1}^m (\lambda_i + m - i)!}{\prod_i \prod_{i<j \leq m} (\alpha_i - \alpha_j)} = \frac{\prod (\lambda_i + m - i)!}{V(\lambda_1 + m - 1, ..., \lambda_m)}$ Therefore $f^\lambda = \frac{n!}{\prod h_{ij}}$ ■

27 The Transfer Matrix Method

Take a directed graph D with arc weights w_a. Let A be the corresponding adjacency matrix. Let $\mathbf{P}_{ij}^n$ be the set of paths (walks) from i to j containing exactly n arcs. The weight of a path $P := \{a_1, a_2, ..., a_k\}$ is

$$w(P) = \prod_{a \in P} w_a$$

Theorem 49

$$a_{ij}^n = \sum_{P \in \boldsymbol{P}_{ij}^n} w(P)$$

Proof. True for $n = 1$. Follows by induction using the definition of matrix multiplication

$$\begin{aligned} a_{ij}^n = \sum_k a_{ik}^{n-1} a_{kj} &= \sum_k a_{ik}^{n-1} w_{k_j} \\ &= \sum_k w_{k_j} \sum_{P \in \mathbf{P}_k^{n-1}} w(P) \\ &= \sum_{P \in \mathbf{P}_{ij}^n} w(P) \end{aligned}$$

■

This observation is useful.

Theorem 50 $F_{ij}(x) = \frac{Cof_{ij}(I-xA)}{Det(I-xA)}$ *(cofactor , det after removing the ith row, jth col, signed by* $(-1)^{i+j}$ *may be slightly incorrect...)*

Proof. $F_{ij}(x)$ is just ijth entry at

$$\sum_{n \geq 0} x^n A^n = (I - xA)^{-1}$$

■

If we just consider circuits

Corollary 17 *If* $d(x) = det(I - Ax)$ *then*

$$\sum_i \sum_{n \geq 1} a_{ii}^n x^n = \frac{-xd'(x)}{d(x)}$$

Proof. $\sum_i a_{ii}^n = tr(A^n) = \lambda_1^n + \lambda_2^n + ... + \lambda_q^n$ where the λ_i are eigen values of A. So

$$= \sum_{n \geq 1} tr(A^n) x^n = \frac{\lambda_1 x}{1 - \lambda_1 x} + \frac{\lambda_2 x}{1 - \lambda_2 x} + ... + \frac{\lambda_q x}{1 - \lambda_q x}$$

Now

$$d(x) = \prod_{i=1}^q (1 - \lambda_i x)$$

and the result follows. ■

For example

27.1 Restricted Walks

Let $f(n)$ be the number of grid walks using only N, E, W such that NN and EW cannot be consecutive. We set this up as

$$\begin{array}{cccc} & N & E & W \\ N & 0 & 1 & 1 \\ E & 1 & 1 & 0 \\ W & 1 & 1 & 1 \end{array}$$

Observe that a walk of k arcs is $k+1$ nodes.

$$\sum_{n\geq 0} f(n+1)x^n = \sum_{ij} F_{ij}(x)$$

$$= \frac{3+x-x^2}{1-2x-x^2+x^3}$$

(find $-det(I-xA)$, and then find coefficients in numerator).

$$\sum_{n\geq 0} f(n)x^n = 1 + \frac{x(3+x-x^2)}{1-2x-x^2+x^3} = \frac{1+x}{1-2x-x^2+x^3}$$

Note to prohibit subsequence $NSNN$ say , label the nodes by sequences of size 3.

28 Free Monoids

- Let **A** be an alphabet(finite set).
- A^* is the set of all words.
- A_n^* is words with n letters.

Then $(A^*, \cdot)$ is a **Free Monoid** on **A** where $\cdot$ is concatenation. If $B \subseteq A^*$ then B^* consists of all words that can be obtained by concatenating words in B. We say that B^* is *freely generated* if $b \in B^*$ has a unique factorization in terms of words in B. Give each letter a a weight $w(a)$, and let $w(a_1, ..., a_k) = w(a_1)\cdots w(a_k)$.

For any subset $H \subseteq A^*$ let

$$H(x) = \sum_{v\in H} w(v)x^{L(v)}$$

where $L(v)$ is the number of letters of v. So the coefficient $h(n)$ of $H(x)$ is

$$\sum_{v\in H_n} w(v)$$

Theorem 51 *Let B freely generate B^*. Then*

$$B^*(x) = \frac{1}{1-B(x)}$$

Proof.

$$b^*(n) = \sum_{n=n_1+n_2+\ldots+n_k} \prod_{j=1}^{k} \sum_{v\in B_{n_j}} w(v)$$

by uniqueness. ■

For example: **Dominoes** How many ways can we fill a $2\times n$ grid of dominoes of size 1×1 and 1×2 (rotations not allowed) ?

Here factorising breaks the grid into smaller $2\times k$ grids. What grids can not be factored further?

$$\begin{pmatrix}[.]\\ [.]\end{pmatrix}, \begin{pmatrix}[..]\\ [..]\end{pmatrix}, \begin{pmatrix}[..][.]\\ [.][..]\end{pmatrix} \rightarrow, \begin{pmatrix}[..][..]...[..][.]\\ [.][..]...[..]\end{pmatrix}, \begin{pmatrix}[.][..]...[..]\\ [..][..]...[..][.]\end{pmatrix}$$

Words in B.

$$B(x) = x + x^2 + 2\sum_{n\geq 2} x^n = x + x^2 + \frac{2x^2}{1-x}$$

So

$$B^*(x) = \frac{1}{1-(x^2+x^2+\frac{2x^2}{1-x})} = \frac{1-x}{1-2x-2x^2+x^3}$$

Another example: $f(n) =$ the number of permutations such that $\pi_i - i \in \{0, \pm 1, \pm 2\}$

$$B(x) = x + x^2 + x^3 + x^4 + 2\sum_{n\geq 3} x^n$$

$$\begin{aligned} B^*(x) &= (1 - x - x^2 - x^4 - \frac{2x^3}{1-x})^{-1} \\ &= \frac{1-x}{1-2x-2x^3+x^5} \end{aligned}$$

29 Statistics

Generating functions can easily be used to find moments of distributions etc, so they can be useful in statistics. For example, let Ω be a finite set of objects. Let each $w \in \Omega$ possess a collection of properties. Let $f(k)$ be the number of objects with exactly k properties. What is the average number of properties that an object has?

$$\mu = \frac{1}{|\Omega|}\sum_{w\in\Omega} p(w) = \frac{1}{|\Omega|}\sum_k kf(k)$$

where $p(w)$ is the number of properties of w.

$$\mu = \frac{\sum_k k\cdot f(k)}{\sum_k f(k)} = \frac{xDF(x)}{F(x)}|_{x=1}$$

$$D\log(F(x))|_{x=1}$$

How about variance?

$$Var = \frac{1}{|\Omega|} \sum_{w \in \Omega} (p(w) - \mu)^2$$

$$= \frac{1}{|\Omega|} \sum_k (k-\mu)^2 f(k)$$

$$= \frac{1}{|\Omega|} \sum_k k^2 f(k) - 2\mu k f(k) + \mu^2 f(k)$$

$$= \frac{(xD)^2 F - 2\mu(xD)F + \mu^2 F}{F}|_{x=1}$$

therefore

$$\mu = \frac{F'}{F}|_{x=1}$$

$$(xD)^2 F = xD(xF') = x^2 F'' + xF'$$

$$= \frac{x^2 F''}{F} + \frac{xF'}{F} - \frac{2\mu F'}{F} + \mu^2|_{x=1}$$

$$= \frac{x^2 F''}{F} + \frac{xF'}{F} - (\frac{F'}{F})^2|_{x=1}$$

Now

$$D\log(F) = \frac{F'}{F}$$

$$D^2 \log F = D(\frac{F'}{F}) = \frac{FF'' - F'F'}{F^2}$$

So

$$D\log(F) + D^2\log(F)|_{x=1} = \frac{F''}{F} - (\frac{F'}{F})^2 + \frac{F'}{F}|_{x=1} = Var$$

29.1 example: signless Stirling numbers

Let $f(k) = s(n,k)$ be the number of permutations with k cycles. So

$$F(x) = S(x) = \sum_k s(n,k)x^k = x(x+1)\cdots(x+n-1)$$

$$\mu = D\log(F)|_{x=1}$$

$$\log(F) = \log(x) + \log(x+1) + ... + \log(x+n-1)$$

$$D\log(F) = \frac{1}{x} + \frac{1}{x+1} + ... + \frac{1}{x+n-1}$$

$$D\log(F)|_{x=1} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

$$= H_n$$

Now

$$D^2 \log(F) = \frac{-1}{x^2} - \frac{1}{(x+1)^2} - ... - \frac{1}{(x+n-1)^2}$$

$$D^2 \log(F)|_{x=1} = -(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}) = \frac{-\pi^2}{6} + o(1)$$

so

$$Var = \sigma^2 = D\log(F) + D^2\log(F)|_{x=1} = H_n - \frac{\pi^2}{6} + o(1)$$

$$\sigma \approx \sqrt{\log(n)}$$

i.e. the number of cycles is concentrated around $\log(n)$ if we take a random permutation. What about other functions that can be viewed in terms of the exponential formula?

Let $h(n,k)$ be the number of objects of size n with k blocks. Then

$$\mu(n) = \frac{\sum_k kh(n,k)}{h(n)}$$

But

$$H(x,y) = \sum_n \sum_k h(n,k) \frac{x^n}{n!} y^k = \exp(yF(x))$$

$$\frac{d}{dy} H(x,y)|_{y=1} = \sum_n \frac{x^n}{n!} \sum_k h(n,k)k = F(x)\exp(F(x))$$
$$= F(x)H(x)$$

$$\sum_n \frac{x^n}{n!} h(n)\mu(n) = H(x)F(x)$$

Lemma 28 $\mu(n) = \frac{1}{h(n)} \sum_i \binom{n}{i} f(i)h(n-i)$

Proof.

$$\mu(n) = [\frac{h(n)}{n!}x^n]H(x)F(x)$$
$$= [x^n]\frac{n!}{h(n)}H(x)F(x)$$

$$= \frac{1}{h(n)} \sum_i \binom{n}{i} f(i)g(n-i)$$

■

e.g. consider $s(n,k)$ again.

$$\mu(n) = \frac{1}{h(n)} \sum_i \binom{n}{i} f(i)h(n-i)$$

$$= \frac{1}{n!} \sum_i \binom{n}{i} (i-1)!(n-i)!$$

$$= \frac{1}{n!} \sum_i \frac{n!}{i!(n-i)!} (i-1)!(n-i)!$$

$$= \sum_i \frac{1}{i} = H_n$$

30 Inclusion Exclusion

What do generating functions have to do with IE? We have a set Ω of objects, and a collection $\mathbf{P}$ of properties which objects may or may not posses. For example a property is just $q \in \mathbf{P}$. Basically , IE is useful when

- It is hard to see how many objects have exactly k properties.
- But it is easy to see how many have at least k properties: Given a set $P \subseteq \mathbf{P}$ let
  - N_p^+ be the number of objects with at least the properties in P
  - $P(w) \subseteq \mathbf{P}$ be the properties of $w \in \Omega$.

Set $l_n := \sum_{|P|=n} N_P^+$

$$= \sum_{|P|=n} \sum_{w:P \subseteq P(w)} 1 = \sum_{w \in \Omega} \sum_{|P|=n, P \subseteq P(w)} 1$$

Consider the ordinary generating function for L

$$L(x) = \sum_{n \geq 0} L_n x^n = \sum_{n \geq 0} (\sum_{t \geq 0} \binom{t}{n} e_t) x^n$$

$$= \sum_{t \geq 0} e_t \sum_{n \geq 0} \binom{t}{n} x^n$$

$$= \sum_{t \geq 0} e_t (x+1)^t = E(x+1)$$

So

$$E(x) = L(x-1)$$

e.g. To find a formula for e_n we equate coefficients of x^n ($[x^n] \sum L_t (x-1)^t$)

$$e_n = \sum_t (-1)^{t-n} \binom{t}{n} L_t$$

e.g.(objects with no properties)

$$e_0 = \sum_t (-1)^t L_t$$

So IE method is:

1. Given Ω and $\mathbf{P}$
2. Find N_P^+'s
3. Find L_n's
4. Find e_n's by $E(x) = L(x-1)$

30.1 Non-Attacking Rooks

How many ways can we place k rooks on a chess board of size $C \subseteq [n] \times [n]$ such that no rooks can take each other?

Look at something similar. Let e_k be the number of permutations that hit C in exactly k squares. We have a property for each square $s = (i,j) \in C$. $P(s) =$ set of π that hit s. $(\pi(i) = j)$ .So a set $P \subseteq \mathbf{P}$ of properties is just a set of squares in C.

$$L_k = \sum_{|P|=k} N_P^+$$

$$= r_k \cdot (n-k)!$$

Given k hit squares then we can complete π in $(n-k)!$ ways.

$$L(x) = \sum_k r_k (n-k)! x^k$$

So

$$e_j = [x^j]C(x-1) = [x^j] \sum_k r_k (n-k)!(x-1)^k$$

In particular e_0 is the number of permutations that miss C.

$$= \sum_k (-1)^k r_k (n-k)!$$

e.g. if $C :=$ *diagonal* : $\{(1,1),(2,2),...,(n,n)\}$ then e_0 is the number of derangements. $r_k = \binom{n}{k} =$ the number of ways to put k non attacking rooks on diagonal.

$$e_0 = \sum_k (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum_k (-1)^k \frac{n!}{k!}$$

Next let $C := \{(1,1),(2,2),...,(n,n),(1,2),(2,3),...,(n,1)\}$
r_k is the number of ways to put non attacking rooks on C or the number of ways to pick k non adjacent points in a $2n$ cycle.

Lemma 29 *The number of ways to pick k non adjacent points in an m cycle is*

$$\frac{m}{m-k}\binom{m-k}{k}$$

Proof. Call this $f(m,k)$.

- Color the chosen points red.
- Color one of the other points blue

We can do this in $(m-k)f(m,k) = g(m,k)$ ways. We can also count $g(m,k)$ as follows:

1. Color a point blue in m ways
2. Arrange $m-k-1$ clear points in a line and put k red points in the spaces between them(can use ends). There are $\binom{m-k}{k}$ ways to do this.

So $g(m,k) = m\binom{m-k}{k}$. Therefore $f(m,k) = \frac{m}{m-k}\binom{m-k}{k}$ ■(note this sort of trick changes ring problems to line problems).

So

$$r_k = \frac{2n}{2n-k}\binom{2n-k}{k}$$

Thus

$$L(x-1) = \sum_k \binom{2n-k}{k}(n-k)!(x-1)^k$$

$$e_0 = \sum_k (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

This is the number of permutations missing C. This is *Problem des Menage:* The number of ways to seat n couples around a table such that couples do not sit next to each other.

30.1.1 another example

Let Ω be n sets in a $2n$ set. Let $S \subseteq [2n]$ have property $i \in [n]$ if $i \notin S$. So how many subsets have exactly k properties.

$$e_k = \binom{n}{n-k}\binom{n}{k} = \binom{n}{k}^2$$

(have exactly $n-k$ elements from $[n]$ and the rest come from $[n, ..., 2n]$)

What does IE tell us? Let $P \subseteq [n]$ be a set of properties. Then

$$N_P^+ = \binom{2n-|P|}{n}$$

$$L_k = \sum_{|P|=k} N_P^+ = \binom{n}{k}\binom{2n-k}{n}$$

So

$$\sum_k \binom{n}{k}^2 x^k = E(x) = L(x-1) = \sum_k \binom{n}{k}\binom{2n-k}{k}(x-1)^k$$

So

$$\sum_k \binom{n}{k}^2 x^k = \sum_k \binom{n}{k}\binom{2n-k}{n}(x-1)^k$$

30.2 Increasing Subsequence

Given a sequence

$$S := 6\ 3\ 3\ 8\ 7\ 2\ 4\ 1\ 1\ 5\ 2\ 4\ 9\ 3$$

What is the length of the longest non-decreasing subsequence?

We can answer this with Schensted's algorithm:

$$6 \begin{pmatrix} 3 \\ 6 \end{pmatrix} \begin{pmatrix} 33 \\ 6 \end{pmatrix} \begin{pmatrix} 338 \\ 6 \end{pmatrix} \begin{pmatrix} 337 \\ 68 \end{pmatrix} \ldots$$

$$(237, 38, 6)\ (234, 37, 68)\ (134, 27, 38, 6)\ (114, 23, 37, 68)\ (1145, 23, 37, 68)\ (1125, 234, 37, 68)$$

$$(1125, 234, 37, 68)\ (1125, 234, 37, 68)\ (1124, 2345, 37, 68)\ (11245, 2345, 37, 68)$$

$$(11239, 2344, 35, 67, 8)$$

- Largest non decreasing subsequence is the number of columns
- Longest decreasing subsequence is equal to the number of rows.

Corollary 18 (Erdos-Szekeres) *A sequence of length n has either a non decreasing subsequence of length $\sqrt{n}$ or a decreasing subsequence of length $\sqrt{n}$.*

We get stronger results, for example

Lemma 30 *If $\lambda(S) = (\lambda_1, \lambda_2, \lambda_3, ...)$ then S contains disjoint non decreasing subsequences of length $\lambda_1, \lambda_2, \lambda_3, ...$*

Exam: 10 : 00 – 12 : 00 room 1205