CS 764 Algorithmic Game Theory	**Scribe: Matt Drescher**
Instructor: Adrian Vetta	**Winter, 2006**

LECTURE 1: See Tardos lecture 1.

1 Load balancing

Social objective

$$max_{s \in S} r_s(L_s)$$

We showed last time that the best NE optimizes this. What about the worst NE? In general this can be really bad socially. EXERSIZE(find an Example). What about for special response functions? e.g. $r_s(L_s) = L_s$.

Theorem 1 *Any NE has max response time at most twice the optimal response time(i.e. min max response time)*

Proof. Let server 1 have max load in a NE M. Let job j be assigned to server 1.

$$r_1(L_1) \leq r_s(L_s + w_j) \forall s \in S$$

by definition of NE. since the response functions are linear we have $r_i(L_i) = L_i$ so we can just write

$$L_1 \leq L_s + w_j \forall s \in S$$

$$mL_1 \leq \sum_{s \in S} (L_s + w_j)$$

$$L_1 \leq (\frac{1}{m} \sum_{s \in S} L_s) + w_j$$

$$\leq_{W.T.S} 2opt$$

Take optimal allocation m^*. Job j is assigned to some server so $opt \geq w_j$. Now

$$\sum_{s \in S} L_s = \sum_{j \in J} w_j$$

max loaded server has load at least the average

$$= \frac{1}{m} \sum_{j \in J} w_j = \frac{1}{m} \sum_{s \in S} L_s$$

i.e.

$$opt \leq c$$

■

What about the other social objective function ? Total Time(average time per job)

$$= \sum_{s \in S} L_s \cdot r_s(L_s)$$

Note here average response time can increase when a player makes a best response move.

2 Potential Games

A game is a potential game if there exists a potential function ϕ such that if player i changes strategy from s to s' then the change in $\phi =$ change in i's payoff.

Why are potential functions useful? #1. if players make best response moves then ϕ always decreases. If it stops we have a (pure) NE.

Theorem 2 *A potential game with finite strategy sets has a pure NE.*

Proof. Process must terminate. No cycles since potential function is always decreasing. Finite strategy set. ■

#2. We can use potential functions to give bounds on the social quality of NE. Consider the case $w_j = 1 \forall j \in J$. Let

$$\phi := \sum_{s \in S} \sum_{k=1}^{L_s} r_s(k) = \sum_{s \in S} [r_s(1) + r_s(2) + ... + r_s(L_s)]$$

How does this relate to the social objective ?

$$\gamma(m) := \sum_{s \in S} r_s(L_s) L_s = \sum_{s \in S} [r_s(L_s) + r_s(L_s) + .. + r_s(L_s)]$$

What happens if job i moves from server s to server t? payoff i $r_s(L_s) \to r_t(L_t + 1)$ changes by $r_t(L_t + 1) - r_s(L_s)$

This is exactly the change in ϕ.

corollary 1 *There exists pure NE*

Lets consider the social value of NE. Suppose $\phi(m) \leq \gamma(m) \leq \beta \cdot \phi(m) \forall m$

Theorem 3 *Then there exists a pure NE m with $\gamma(m) \leq \beta \cdot \gamma(m^*)$ where m^* is optimal social solution.*

Proof. Let m minimize ϕ(i.e. m is NE)

$$\gamma(m) \leq \beta \cdot \phi(m) \leq \beta \phi(m^*) \leq \beta \gamma(m^*)$$

by minimality and assumptions. ■

2.1 Load balancing game

we have $r_s(L_s) = L_s$

$$2\phi(m) = \sum_{s \in S} 2(r_s(1) + r_s(2) + ... + r_s(L_s))$$

$$= \sum_{s \in S} 2 \frac{1}{2} L_s(L_s + 1) \geq \sum_{s \in S} L_s^2$$

$$\gamma(m) = \sum_{s \in S} L_s - r_s(L_s)$$

so set $\beta := 2\phi(m) \geq \gamma(m) \geq \phi(m)$.

corollary 2 *Best NE within factor 2 of optimal.*

2.2 Broadcasting game

Directed network G with arc costs $c_a \geq 0$. A broadcaster makes a broadcast from a source s to customers $t_1, ..., t_n$. Each customer shares the cost of any link it needs for its broadcasts. Look at best NE.

$$\gamma(T) = \sum_{a \in T} c_a$$

$$\phi(T) = \sum_{a \in T} c_a(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n_a})$$

where $n_a := \#$ *customers using arc a in* T. Its easy to show ϕ is a potential function. In this case(example on chalk board) $\gamma(T) \leq \phi(T) \leq log(n)\gamma(T)$. So there exists pure NE and best NE costs atmost $logn \cdot opt$ we have seen $log(n)$ is tight by example.

2.3 Open Problems

1. Observe that upper bound also applies to undirected networks. But the examples lower bound does not. is gap $o(logn)$? is gap $O(1)$?

2. What happens if there is more then 1 source? i.e. source - destinations $s_1 - t_1, s_2 - t_2, ..., s_n - t_n$ problem there may be no NE! (paid edges are not shared).

possible approach use sink equilibria

3 Selfish Routing

Given a graph $G = (V, A)$ with source sink pairs $(s_1, t_1), (s_2, t_2), ..., (s_k, t_k)$. We want to route F_i units of flow from s_i to t_i. The r_i is made up of tiny units eg cars, data. Associated with arc is a cost function $c_a()$ that is load dependent and non-negative, non-decreasing, convex. Goal is to find a flow(traffic routing) that minimizes cost. This is the *social goal. Private Objective:* Minimize your own travel costs. i.e. players pick shortest paths. Thus we have a game. A reasonable solution therefore may be a NE.

3.1 Example

one source since pair (s, t) everyone wants to go from s to t. $r = 1$ we have two edges from s to t, call them roads x_1, x_2. Suppose $c_1(x_1) = 1$ and $c_2(x_2) = x_2$ per unit cost of travel. We have 1 unti of traffic. What happens if we send half through each road. we have cost $1/2 \cdot c_1(1/2) + 1/2 \cdot c_2(1/2) = 1/2 + 1/4 = 3/4$. This is not stable since traffic taking road one would rather take road 2 and pay less. Therefore in out NE the total cost is $c_2(1) = 1$. We see that the NE is worse then the optimal solution. Here NE is $\frac{4}{3}$ worse then optimal. Surprisingly this is the worst ratio(if cost functions are linear).

$$\max_{NE} \frac{cost(NE)}{cost(opt)} = price\ of\ anarchy$$

Cost of a lack of coordination.

Theorem 4 *R,T Price of anarchy $\leq \frac{4}{3}$ for all networks with linear cost functions*

Proof. **observations:** At a NE $x^n := (x^n_{a_1}, x^n_{a_2}, ...)$. Players use the shortest paths. So for $p_1, p_2 \in S_i$, $S_i :=$ *the set of* $s_i \rightarrow t_i$ *paths. assume* $x^n_p > 0$

$$c(p_1) = \sum_{a \in p_1} c_a(x^n_a) \leq c(p_2) = \sum_{a \in p_2} c_a(x^n_a)$$

therefore all $p \in S_i$ with $x_p > 0$ have the same cost. *look at cost with respect to paths:*

$$c(x^n) = \sum_{i=1}^{k} \sum_{p \in S_i, x^n_p > 0} c(p) \cdot x^n_p$$

or look at cost with respect to arcs: Let x^* be the optimal flow.

$$= \sum_{a \in A} c_a(x^n_a) \cdot x^n_a$$

so

$$c(x^n) = \sum_{a \in A} c_a(x^n_a) x_a = \sum_{i=1}^{k} \sum_{p \in S_i, x_p > 0} x^n_p$$

$$\leq \sum_{i=1}^{k} \sum_{p \in S_i} c(p) \cdot x^*_p$$

$$= \sum_{a \in A} c_a(x^n_a) \cdot x^*_a$$

using opt flow with respect to Nash costs is worse

$$c(x^n) \leq \sum_{a \in A} (c_a(x^*_a) x^*_a - c_a(x^*_a) x^*_a + c_a(x^n_a) x^*_a)$$

$$= opt + \sum_{a \in A} x^*_a (c_a(x^n_a) - c_A(x^*_a))$$

$$\leq opt + \sum_{a \in A, x^n_a > x^*_a} x^*_a (c_a(x^*_a) - c_a(x^*_a))$$

the summand is a rectangle with length $c_a(x^n_a) - c_a(x^*_a)$ and width c^*_a . This rectangle is at most $\frac{1}{4}$ of the region which is Nash. so $nash \leq \frac{4}{3} opt$ ■. The important thing here is to realize that we can thin of cost in terms of arcs *or* in terms of paths.

This result is not true for cost functions that are not linear. If $c_a(x)$ can be polynomial of degree d then nash can be $\Omega(\frac{d}{log(d)}) \cdot opt$. What can we say in this case? -Bicriteria result.

Theorem 5 *Cost of a NE x^n is at most the cost of the optimal solution that routes twice as much traffic i.e. $r'_i = 2r_i$ $\forall i$*

Proof. Replace c by cost function c'.

$$c'_a(x_a) = \{c_a(x^n_a)\ if\ x_a < x^n_a, = c_a(x_a)\ if\ x_a \geq x^n_a$$

$$c'(x^n) = c(x^n)$$

$$c'(x^*) \leq c(x^*) + c(x^n)$$

cost can increase by at most Nash cost on every arc.

Consider any path $pinS_i$

$$c'(p_i) \geq \sum_{a \in p_i} c_a(x^n_a) = c^n(p_i)$$

i.e. the cost per unit of any path is at least the nash cost. But we route twice the traffic in x^* $so_{\#2}$ $c'(x^*) \geq 2c'(x^n) \geq 2c(x^n)$. therefore

$$c(x^*) + c(x^n) \geq 2c(x^n) \Rightarrow c(x^*) \geq c(x^n)$$

■

4 Braess Paradox

four nodes(like a square) and we want to route flow from s to t with the same cost functions we saw previously $c_a(x_1) = 1, c_2(x_2) = x_2$. Consider a different network obtained by adding directed arc from the top corner to the bottom with cost 0. Now its best to go up and then take the new edge down and then go to t. In this case adding an extra road makes things worse!

4.1 Extensions and Open Problems

Add capacities , Changing objective functions(fairness, etc..), Larger players, priorities,Braess Paradox

5 The existence of NE

Given $k-player-game$. Player i has strategy set S_i. Payoff to i is $\pi_i(s_1, s_2, ..., s_k)$, where $s_i \in S_i$. If players use mixed strategies $p_1, p_2, ..., p_k$ then the expected payoff to i is $\pi_i(p_1, p_2, p_k)$

$$= \sum_{s_1 \in S_1, s_2 \in S_2, ..., s_k \in S_k} \prod_{j=1}^{k} p_j(s_j)\pi_i(s_1, s_2, ..., s_k)$$

So $p_1, ..., p_k$ form a (mixed) NE if $\forall i$ p_i maximizes expected payoff to i given $p_1, ..., p_{i-1}, p_{i+1}, .., p_k$. Nash's result says that NE always exist in finite games. We will give a combinatorial proof of this.

Brouwer's fixed point 1 *Any CTS function $f : S_d \to S_d$ has a fixed point $f(x) = x$*

We give the proof when $d = 2$(its easy to generalize to higher dimensions). **Proof.** Idea: Take a "'big"' triangle, triangulate it into smaller triangles. 3 color the vertices such that $col(v_i) = i$. All vertices on edge (v_i, v_j) are colored i or j.

Sperner's Lemma 1 *Any such coloring gives a "small". $3-colored$ triangle.*

Proof. We show there exists an odd number of 3 colored small triangles. Use a dual graph-we have a node for each small triangle and a node for the big outside triangle. $\triangle_A$ and $\triangle_B$ have an edge between them if they share an edge with endpoints colored 1 and 2. Observations: The inside triangles have degree: 2 if $\triangle$ has only the colors 1 and 2, 1 if $\triangle$ has all 3 colors, 0 if $\triangle$ misses color 1 or color 2. Degree of outer $\triangle$ is odd as $\exists$ odd number of color changes from v_1 to v_2. There is an even number of odd degree vertices, therefore there exists at least 2 odd degree triangles as outer triangle is odd degree. Therefore there exists an inner $\triangle$ of degree 1 . therefore there exists a $3-colored$ inner $\triangle$ ■

Let the simplex be the triangle in R^3 with vertices $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$. Take any infinite sequence $T_1, T_2, ...$ or triangulations. Where $\delta(T_n) \to 0$ as $n \to \infty$ and $\delta(T_n)$ is max edge length in T_n. We 3 color T_n as follows: let

$$col(v) = min_{1 \leq i \leq 3}(i : f(v)_i < v_i)$$

-$col(v_i) = i,\ f(1,0,0) = (<1,,),\ f(0,1,0) = (\geq 0, <1,,)$
Take vertex x on edge opposite v_i has $x_i = 0$ as its a convex combination, *i.e.* not colored i. So T_n has a 3 colored triangle $\triangle_n = \{x_n^1, x_n^2, x_n^3\}$ Focus on $x_1, x_2, x_3, ..., x_n...$ is an infinite sequence. It need not converge but it has a convergent subsequence. (sequence is compact-Bolzano Weisstras thm). Let this subsequence converge to x^*. Look at the traingles corresponding to this subsequence. So there exists a subsequence of this $(x_1^2, x_2^2, ...)$ which tends to x^*. There exists a subsequence of $x_1^3, x_2^3, ...$ which tends to x^*. By continuity we have

$$f(x^*) \leq x_1^*$$

$$f(x^*)_2 \leq x_2^* \Rightarrow f(x^*) = x^*$$

$$f(x^*)_3 \leq x_3^*$$

This easily generalizes to higher dimensions(eg use tetrahedrons in 3d). Also applies to spaces homeomorphic to the simplex. How does this apply to NE? What is the space -each player picks a probability distribution p_i on $s \in S_i$ let $f(f_1, f_2, ..., f_h) = (p'_1, p'_2, ..., p'_k)$. Where p'_i is best response of i to $p_1, ..., p_{i-1}, p_{i+1}, ..., p_h$ So a fixed point $f(p_1, ..., p_h) = p_1, ..., p_h$ is NE. Can we do this in a CTS way? Consider p_i and p'_i. Why not just increase the prob of any strategy that is an improvement.

Aside: Given $p_2, ..., p_k$ what can i do? Any pure strategy gives some payoff $\pi_1(s, p_2, ..., p_k)$. i.e. a mixed strategy p_1 is only a best response is if all the pure strategies give > 0 probability to all best responses.

$$p'_i(s) = p_i(s) + max[0, \pi_i(p_1, ..., s, p_{i+1}, ..., p_h) = \pi_1(p_1, ..., p_i, ..., p_h)$$

We now scale p'_i to make it a prob distribution.

$$p'_i(s) = \frac{p_i(s) + max(0, \epsilon_{i,s})}{1 + \sum_s max(0, \epsilon_{i,s})}$$

p'_i improves on p_i i.e. if $p'_i = p_i\ \forall i$ we are done. This is CTS so by Brouwers FPT we have a NE.

■

6 Auctions

Start simple:
we have 1 good to sell
k bidders, player i values good at v_i

assumptions

No collusion auction.
Quasilinear utilities: $u_i = v_i - p_i$ if pays p_i
players know their valuations from the start and they dont change.

6.1 English Auction

Bidders successively increase their bids b_i.
winner is player with the highest valuation.
Pays the second highest valuation(i.e. all the other players have dropped out).

If this is the outcome what if we try to incorporate it into the auction?

6.2 Vickrey Auction

Players make sealed bids b_i
Highest bidder wins
pays the second highest bid

Clearly if $b_i = v_i \ \forall i$ then the outcome is the same as the english.
Vickrey auctions have several nice properties:
(1) **Incentive Compatibility**
(i) Truthfulness: bidding truthfully is a dominant strategy.

Lemma 1 *Player i maximizes utility by bidding $b_i = v_i$ no matter what the other bids $\{b_j\}_{j \neq i}$ are.*

Proof. Let $B = max_{j \neq i} b_j$. If $v_i \geq B$, if you win you get $u_i = v_i - B \geq 0$. Win if $b_i \geq B$ " lose you get 0. i.e. $b_i = v_i$ is best response. If $v_i < B$ then any bid greater then B wins and gives $u_i = v_i - B < 0$, bid $< B$ lose so get $u_i = 0$, i.e. $b_i = v_i$ is best response. ∎

(ii) Strongly truthful: For any other bid $b_i \neq v_i$ there are cases where you do worse than $b_i = v_i$. e.g. $v_i > B > b_i$ lose but could of got $u_i > 0$. $b_i > B > v_i$ win and get negative payoff.

(iii) Individual Rationality: Truthful bids always get ≥ 0 utility.
Losers get 0 utility.
Truthful bids only win when $v_i > B$ i.e. get ≥ 0 utility.
Caveat: In practice players do not always bid truthfully in tests!

So incentive constraints "'Should"' get players to bid truthfully. Why is this useful?
-Auctioneer: has some objective to optimize(e.g. revenue, efficiency) to max this you have to know the *valuations* not the bids. (auctioneer bribes players to get this info: pay second price)
-Bidders: You don't care *what* the other players do- you just bid truthfully. [dont spend money

figuring out beliefs of other players. Don't investigate how auction works.]
-simplicity

(2) Efficiency. An efficient outcome maximizes total utility. $\sum_{i \in bidder} u_i + u_{auctionear}$ (Auctioneer has utility = price). Prices transfer money i.e. just transfer utility so have no effect on total utility.

So Vickrey auction is efficient if everyone bids truthfully.(Bad for revenue max).

(3) General valuations. This works no matter what the structure/ type of valuations.

(4) Polytime: The auction is polytime (in number of players and number of items). (Go through k bids)

However it is not possible to get all these 4 properties in more complicated auctions.

There are other useful properties too: "fairness", "transparency".

7 Combinatorial Auctions

What if we have a set I of n items to sell. We could sell them off one at a time. But this may not be very effective.

-compliments a set of items might be worth more then the sum of the parts. e.g. Art collection.

-substitutes a set of items might be worth less then the sum of their parts.

Combinatorial auction allows players to bid on subsets $S \subseteq I$. These are most useful if we have compliments.

7.1 VCG(Vickery Clark Groves) Mechanism

Players submit bids $b_i(S)$ on each $S \subseteq I$
Auctioneer chooses an allocation(disjoint perturbation of I) $(S_1^*, S_2^*, ..., S_k^*)$ that maximizes

$$\sum_{i=1}^{n} b_i(S_i)$$

over all feasible allocations.

Player i pays price P_i.

Properties
(3) General Valuations
(2) Efficient: If truthful biddes then we max efficiency.
But will bidders bid truthfully ??

First we will find some "'silly"' prices that waork.

Lemma 2 *Using VCG mechanism with prices* $p_i = -\sum_{j\neq i} b_j(S_j^*)$ *induces truth telling.*

Proof. these prices mean auctioneer pays the bidders. Utility of player i is :

$$v_i(S_i^*) + \sum_{i\neq j} b_j(S_j^*)$$

Should player i tell the truth? The auction max $\sum_i b_i(S_i)$. So i can influence the allocation by its bids, but i has no influence on $b_j(S)$ $\forall S,\ \forall j$

So setting $b_i(S) = v_i(S)$ $\forall S$ mean the auctioneer is also maximizing its utility. ■

Note setting

$$p_i = -\sum_{j\neq i} b_j(S_j^*) + c$$

is also truth telling. Moreover so is

$$p_i = -\sum_{j\neq i} b_j(S_j^*) + f_i(\{b_j\}_{j\neq i})$$

Each player has same optimization problem as before.

Auctioneer allocates according to bids not prices.

$$f_i(\{b_j\}_{j\neq i}) = max_{\{S_j\}_{j\neq i}} \sum_{j\neq i} b_j(S_i)$$

(best allocation where $S_i = 0$. So

$$p_i = max_{S_i=0} \sum_{j\neq i} b_j(S_j) - \sum_{j\neq i} b_j(S_j^*) \geq 0$$

as $\{S_1^*, ..., S_k^*\} - S_i^*$ is feasible for max problem.

So the prices are reasonable. Is the auction individually rational? i.e. $u_i \geq 0$. if truth telling So

$$p_i = b_i(S_i^*) - \sum_{j=1}^{k} b_j(S_j^*) + max_{\{S_j\}_{j\neq i}} \sum_{j\neq i} b_j(S_j)$$

(*)

$$= b_i - (\sum_{j=1}^{k} b_j(S_j^*) - max_{\{S_j\}_{j\neq i}} \sum_{j\neq i} b_j(S_j)$$

$$\geq 0$$

$$u_i = v_i - p_i$$

$$= v_i(S_i^*) - (b_i(S_i^*) - (*)) = (*) \geq 0$$

(since $b_i(S_i^*) = v_i(S_i^*)$)

the following two lines are the key to understanding the VCG auction:

$p_i =$ damage your participation does to other players.
$u_i =$ what you contribute to extra efficiency by taking part.

$$-\sum_{j\neq i} b_j(S_j^*) - - - \ \textit{what the others get}$$

$$max_{\{S_j\}_j} \sum_{j\neq i} b_j(S_j) - - - - - - \ \textit{what they would get without you}$$

$$u_i = \sum_{j=1}^{k} b_j(S_j^*) - max_{\{S_j\}_{j\neq i}} \sum_{j\neq i} b_j(S_j)$$

(4) Polytime ? No chance. Run time is exponential in number of items.
a: each player submits exponential number of bids.
b: the allocation problem is NP hard. Even if the bids could be submitted succinctly in polytime. Its also a disaster to approximate.

7.2 Bidding for contracts

Another type of auction = bidding for contracts. Contractors submit bids("'quotes"') for company/gov work. e.g. Suppose we want to build a road from s to t. Contractors can build/rent out links. Each link a incurs a cost of c_a for the contractor $a \in A$.

Social goal: min cost $s \to t$ path.
Contractors goal: Max Profit $\pi_a = p_a(\textit{payment it receives}) - c_a$
Contractors will not reveal there true costs so will bid higher. Lets use VCG mechanism to get truth telling (i.e. bribe contractors to tell truth)

VCG Utility = added efficiency by having agent a. Want $u_a = 0\ if\ a \notin P^*\ for\ any\ min\ cost\ path\ P^*$, $c(P^{-a}) - c(P^*)\ if\ a \in P^*$

where P^* is min cost path, P^{-1} optimal path not using a, $c(P) = \sum_{a\in P} c_a$
remember our bids are c' not c
Payments should then be :

$$p_a = c'_a(\textit{to cover their costs}) + (c'(P'^{-a}) - c'(P'^*))(\textit{bribe})\ if\ a \in P'^*, c'(P'^{-a}) - c'(P'^*) = 0\ if\ a \notin P'^*$$

P'^{-a} = cheapest path not using a with respect to cost c'

Payment depend on c'.

Lemma 3 *Truth telling is dominant strategy.*

Proof. Utility of a is $p_a - c_a = c'_a - c_a + (c'(P'^{-a} - c'(P'^{*}))$ if a wins

$$= [c'(P' - a) - c_a]_{does\ not\ depend\ on\ c'_a} + [c'_a - c'(P'^{*})]$$

So if bidding $c'_a = c_a$ wins this is a best response. (if change too much you could lose). If bidding $c'_a = c_a$ loses then $u_a = 0$. So

$$c'(P'^{-a}) < c'(P'^{+a})$$

(P'^{+a}–best path that uses a.)

To win you must cut your bid by at least $c'(P'^{+a}) - c(P'^{-a})$ and this is all loss. never get > 0 payoff ■

This is polytime. Run a min cost $s-t$ path alg, run a min cost $s-t$ path algorithm on $G-a$ $\forall a$.

8 Bidding for Contracts

VCG mechanism.
$\leq m$ min s-t path algorithms.

8.1 Open Problems

Can you do this faster, i.e. use less min s-t path algorithms in directed case. eg $O(1)$ Efficient but transfer payments are huge!

9 Single minded Bidders

For general valuation functions we can't satisfy all the properties (1)-(4) at once. Lets keep (4) polytime but relax (3) general value functions.

Single minded bidders: Player i only cares about one subset $S_i \subseteq I$.

$$V_i(T_i) = v_i \ if \ S_i \subseteq T_i$$

$$v_i(T_i) = 0 \ otherwise$$

Communication complexity is now clearly polytime i.e. 1 bid/player . Before we had 2^n bids/player.

Allocation problem: instead of $2^n \cdot k$ bids we have k bids.

But even if the players bid truthfully we still have a really hard allocation problem

9.1 Weighted Stable Set problem

Stable set = set of pairwise non adjacent vertices. Find a max weight stable set

$$\sum_{v \in S} w_v$$

We can formulate this as an allocation problem:
Players are vertices , edges are items, S_v for vertex v is $\delta(v)$ = edges incident to v .
$v(S_v) = w_v$. Feasible allocation. disjoint partition of items to players. Stableset $S \Rightarrow$ all of S can win their bids. Not stable set $\Rightarrow (x, y) \in E \Rightarrow x, y$ can not both win their bids.

9.2 Weighted Stable Set Problem

Stable set is not approximable to within a factor $O(|V|^{1-\epsilon}$ or $O(|E|^{\frac{1}{2}-\epsilon})$ (unless $NP \subseteq ZPP$)

corollary 3 *Allocation problem can not be approximated to within $O(R^{1-\epsilon})$–players or $O(n^{\frac{1}{2}-\epsilon})$–$|I|$.*

So even assuming true bids we still do awfully. We try this anyway! Lets look for approx algorithms for allocation problem with singleminded bidders.

9.3 Simple Algs

Sort $v_1 \geq v_2 \geq ... \geq v_k$. Allocate sets greedily in this order(i.e. whilst they are disjoint). could be a factor of n off.

–Sort

$$\frac{v_1}{|S_1|} \geq \frac{v_2}{|S_2|} \geq ... \geq \frac{v_k}{|S_k|}$$

Allocate greedily $n - \epsilon$

Sort

$$\frac{v_1}{\sqrt{|S_1|}} \geq \frac{v_2}{\sqrt{|S_2|}} \geq ... \geq \frac{v_k}{\sqrt{|S_k|}}$$

Allocate greedily

This is an $O(\sqrt{n})$ approximation algorithm **Proof.** Given $(S_1, v_1)...(S_k, v_k)$. Let $J^* \subseteq \{1, 2, ..., k\}$ be the optimal allocation. $J \subseteq \{1, 2, .., k\}$ be the greedy allocation. Want to show

$$\sum_{j \in J^*} v_j \leq \sqrt{n} \sum_{j \in J} v_j$$

How do we prove this? Observe when $j \in J$ is chosen then S_j may "‘block"’ some sets in J^*. For any $j \in J$ let $B_j = \{i \in J^* : S_i \; is \; first \; blocked \; by S_j\}$

Clearly $\cup_{j \in J} B_j = J^*$ if we say $i \in J^*$, J clocks i. Since $i \in B_j$ was available when we chose S_j we have

$$\sum_{i \in J^*} v_i = \sum_{j \in J} \sum_{i \in B_j} v_i \leq \sum_{j \in J} \frac{v_j}{|\sqrt{S_j}|} \cdot \sqrt{n} \sqrt{|B_j|}$$

$$\sqrt{n} \sum_{j \in J} v_j \sqrt{\frac{|B_j|}{|S_j|}}$$

$$\leq \sqrt{n} \sum_{j \in J} v_j = \sqrt{n}$$

value greedy allocation. $|S_j| \geq |B_j|$ as optimal sets are disjoint so S_j blocks at most $|S_j|$ of them ■

So we have a $\sqrt{n}$ approximation algorithm which is "‘the best possible"’. tight example with one guy getting $\sqrt{n} + \epsilon$ and the other getting 1.

But is this truthful? -VCG utility of $i = i's$ effect on efficiency. Cant find this. We can find i's effect on efficiency using our mechanism. But value may increase without a player. e.g. i's utility is $\sqrt{n} + \epsilon - n < 0$ not irrational.

Single minded bidders i: S_i value v_i rank by $\frac{b_i}{\sqrt{|S_i|}}$ VLG:doesnt work. Is there a truthful payment scheme? –Let i pay p_i where $p_i = \bar{v}_i$ is the lowest bid i could have made to win the auction (under this mechanism) given other bids.

Why is this truthtelling? $v_i \geq \bar{v}_i$, Any bid $\geq \bar{v}_i$ gives utility $v_i - \bar{v}_i \geq 0$

$v_i < \bar{v}_i$ any winning bid gives utility $v_i - \bar{v}_i < 0$ bid $b_i = v_i$ gives 0 utility.

So we have

(1) Truthfulness

(2)dont have Efficiency –approximate

(3) dont have general calculations –single minded bidders.

(4) we have polytime.

10 Auctions with Large Supply

Same format as before:

Single minded bidders. $j : (S_j, v_j)$ $\S_j \subseteq I$

But

Large supply: m_i copies of item i.

e.g

club membership

Highway tolls

Partitioning bandwidth

(in fact sometimes we have ∞ supply and we chose to restrict it.)

10.1 Unlimited Supply

in other words $M_i \geq k$

Assume bidders are truthful. How do we price the items? We assume that the pricing mechanism must be *envy - free* i.e. all players get the same deal. e.g. cant charge v_j for S_i and v_i for $S_i = S_j$. Envy free may be a legal requirement, price discrimination is common.

An envy-free mechanism is:

charge p_i for item i, j gets S_j if $v_j \geq \sum_{i \in S_j} p_i$

Lets set $p_i = p$ $\forall i \in I$

So we can choose

$$p = \frac{v_1}{|S_1|} \ or \ \frac{v_2}{|S_2|} \ or ... \ \frac{v_k}{|S_k|}$$

set $B_i := \frac{v_1}{|S_1|}$

So try all k and pick the best one.

Theorem 6 *Alg is an $O(logk + logn)$ approximation algorithm.*

Proof. Let R_j be the revenue at price B_j. $R^* = max_j R_j$. Then $R_j = \sum_{1 \le t \le j} B_j |S_t| = \frac{v_j}{|S_j|} \sum_{1 \le t \le j} |S_j|$
$Opt \le \sum_{1 \le j \le k} v_j = \sum_{1 \le j \le k} \frac{|S_j| R_j}{\sum_{1 \le t \le j} |S_t|}$

$$\le R^* \sum_{1 \le j \le k} \frac{|S_j|}{\sum_{1 \le t \le j} |S_t|}$$

$$= R^* \sum_{1 \le j \le k} \sum_{s=1}^{|S_j|} \frac{1}{\sum_{1 \le t \le j} |S_t|}$$

$$\le R^* \sum_{1 \le j \le k} \sum_{s=1}^{|S_j|} \frac{1}{s + \sum_{1 \le t < j} |S_t|}$$

$$= R^* (1 + \frac{1}{2} + ... + \frac{1}{\sum_{1 \le j \le k} |S_j|}$$

$$\le R^* log(\sum_{1 \le j \le k} |S_j|)$$

$$\le R^* log(nk)$$

■

This analysis is tight. Let $k = n$. Player i wants item i at value $\frac{1}{i}$. Clearly $p_i = \frac{1}{i} \Rightarrow$ reveue $= \sum_{1 \le i \le n} \frac{1}{i} \approx log(n)$
We have prices p. i.e. $p = 1$ or $\frac{1}{2}$ or $\frac{1}{3}$ $or\therefore$ or $\frac{1}{n}$
They all give revenue = 1.

10.2 Open Problems

There are lots of open problems even for special cases of this problem.

-Beat $O(lognk)$
Highway Problem- I is a road: $1 - -2 - -3 - -4 - -5 - -6 - -7$. Player j wants interval(subpath) $S_j \subseteq I$. Open to beat $O(log(n))$.

Vertex Pricing Open truthful mechanisms. Given $G = (V, E)$ players are edges with values v_e. Items are vertices. Price vertices p_i to maximize $\sum_{(i,j), v_e \ge p_i + p_j} p_i + p_j$. Give better then 4 approximation algorithm.

What if we get rid of envy-free pricing? Can we do better? (1) Obviously for infinite supply sell S_j for v_j

(2) Large but limited supply: $m_i \geq c \cdot log(n)$

$$max \sum_j b_j x_j$$

$$s.t. \sum_{j: i \in S_j} x_j \leq m_i \; \forall i \in I$$

$$x_i \in \{0,1\} \; \forall j$$

Relax integrality constraints with $0 \leq x_j \leq 1 \; \forall j$.

What does it mean to fractionally accept a bid? We use x_j as a probability to round to $0-1$. Accept bid b_j with probability x_j

$$Exp[\sum_{j's \; bid \; accept} b_j] = \sum_j b_j x_j = LPsol \geq opt$$

We expect to do better then opt! The problem is we expect to sell at least m_i of items i but we may oversell it. To get around this use probability $\frac{x_j}{1+\delta}$. So expected revenue $= \frac{1}{1+\delta} \cdot opt$ Expected number of copies of i sold is $\leq \frac{m_i}{1+\delta}$. Probability that i is oversold? Suppose $Z_1, ..., Z_t$ are $0 - -1$ random variables with $E(\sum_s Z_s) = \mu$ then Chernoffs bound says

$$P(\sum_s Z_s > (1+\delta) - \mu) \leq (\frac{e}{(1+\delta)^{1+\delta}})^{\mu}$$

$$P(\# \; items \; sold \geq m_j) \leq ($$

Choose c large enough with respect to $\delta \leq \epsilon/n$

10.3 Auctions with large Supply

So with failure probability ϵwe get within δ of optimal. If we over sell any item we can just run it again until we get a good solution.

Can we implement this as a truthful auction? What are the prices?

Observe that if the other bids are fixed then if b_j increases this causes an increase in x_j. We maximize an LP so the LP solution increases as b_j increases. So the probability that j wins the auction is monotonically increasing in b_j (if this was not the case then clearly there would be some incentive to lie).

Caveat: we have to be careful as an increasing b_j changes the other x values and this could change failure probability. Maybe this could give incentive to lie? To get around this just cancel the auction with a total probability ϵ whether or not the allocation is good.

Theorem 7 *Using prices $p_j = b_j x_j(b_j) - \int_0^{b_j} x_j(t_j) dt_j$ is truthful.*

Proof.

Truthtelling gives $v_j x_j(v_j - [v_j x_j(v_j) - \int_0^{v_j} x_j(t_j) dt_j] = \int_0^{v_j} x_j(t_j) dt_j \geq 0$

Case 1: $b_j > v_j$. Expected utility

$$= v_j x_j(b_j) - (b_j x_j(b_j) - [b_j x_j(b_j) - \int_0^{b_j} x_j(t_j)dt_j]$$

$$= \int_0^{b_j} x_j(t_j)dt_j - (b_j - v_j)x_j(b_j)$$

$$\leq \int_0^{v_j} x_j(t_j)dt_j$$

Case 2: $b_j < v_j$ similar proof by picture.

$\blacksquare$

11 Zero-Sum Games

A 2-player game is zero-sum if payoff to player (1) + playoff to player (2) = 0. Can we find NE in these games? Suppose x, y are mixed strategies of (1) and (2) respectively. What are the players objectives? The outcome of the game is: Expected pay off to $I = x^T Ay = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij} = \sum_i \sum_j P(i, j\ occurs)a_{ij}$

Given x , player (2) wants to

$$max_y x^T(-A)y = min_y x^T Ay$$

$$= min_y \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i y_j$$

$$= min_y \sum_{j=1}^{n} y_j (\sum_{i=1}^{n} a_{ij} x_i)$$

$$= min_{1\leq j} \sum_{i=1}^{n} a_{ij} x_i$$

your best response to a mixed strategy is always a pure strategy!(in all games) But any convex combination of pure best responses(if $\exists > 1$) is a best response. So NE may only be mixed.

Fine but player (1) knows this so wants to

$$max_x min_y x^T Ay$$

$$= max_x min_{1\leq j\leq n} \sum_{j=1}^{m} a_{ij} x_i$$

$$max\ min_{1\leq j\leq n} \sum_{i=1}^{n} x_i a_{ij}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0 \forall i$$

we can write this as

$$max\ z$$

$$s.t. \sum_{i=1}^{m} x_i a_{ij} \geq z\ \forall j = 1...n$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0$$

So player (1) can guarantee a payoff of $z^* = max\ z$ Symmetric argument for player (2)

$$min\ max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} y_j$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \geq 0$$

$$minw$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq w \forall i = 1...n$$

$$\sum y_j = 1$$

$$y_j \geq 0$$

Player (2) can guarantee that (1) gets at most $w^* = min\ w$ if $w^* = z^*$ this is a solution to the game. i.e. a NE.

Recall LP duality:

$$max(cx|Ax \leq b,\ x \geq 0) = min(by|yA \geq c, y \geq 0)$$

(assume some $a_{ij} \geq 0$ by adding constant to all payoffs). can change $' =' $ to $' \leq'$ since solution must still give $\sum x_i = 1$ or we don't have a max.

$$max\ z$$

$$s.t. z - \sum_{i=1}^{m} x_i a_{ij} \geq 0\ \forall j = 1...n$$

$$\sum_{i=1}^{n} x_i \leq 1$$
$$x_i \geq 0$$

$$max(1,0,0,...,0)^T(z,x_1,x_2,...,x_m)$$
$$A'x \leq (1,0,...,0)$$
$$x_i \geq 0, z\ unrestricted$$

where $A' =$

$$(,1,1,...,1)$$
$$-A^T$$

$x := (z, x_1, ..., x_n)^T$

dual is

$$min(1,0,...,0)^T(w,y_1,...,y_n)^j$$
$$-A'y \geq (1,0,...,0)$$
$$y_i \geq 0, w\ unrestricted$$

`min max theorem:`

$$min_y x^* Ay = max_x x^T Ay^*$$

What about 3 player zero-sum games ?
Ex: Show that multiplayer zero-sum games are as hard as multiplayer games(general games)

An LP approach cant work here exactly for 3+ player games.

12 Correlated Equilibria

Consider the game of chicken:

$$\begin{pmatrix} & G & S \\ G & (0,0) & (5,1) \\ S & (1,5) & (4,4) \end{pmatrix}$$

NE at SS,GG.

If we put any probability distribution on NE (included mixed NE) and we are told to play this way according to these probabilities then it is an equlibria for all players to obey these instructions. So we can get any expected payoff within the convex hull of the payoffs of NE. Can we do better? No: if all players receive the same signal(i.e.get same info). restriction $\Rightarrow$ can only put weight on NE solutions. Yes: if we give players different signals.

Observations: NE are correlated equlibria including mixed NE.
Allows a wider range of payoffs.
Can sometimes do even better.

Questions: How much better can we do? Can we quickly find CE ? What about more players? We have k players, $S = (S_1, ..., S_k)$ pure $\Rightarrow \pi(S) = (\pi_1(S), ..., \pi_k(S))$ For a CE: (1) it has a probability distribution $p(S) \geq 0 \forall S$ and $\sum_S p(S) = 1$.
(2) There is no incentive to deviate.

e.g. If I is told to play T_1 I knows the signal is $(T_1, *, *, ...)$. So her payoff is

$$\sum_{S:S_1=T_1} \frac{p(S)}{\sum_{S:S_1=T_1} p(S)} \pi_1(S)$$

If she plays $\bar{T}_1$ she gets

$$\sum_{S:S_1=T_1} \frac{p(S)}{\sum_{S:S_1=T_1} p(S)} \pi_1(\bar{T}_1, S_2, ..., s_k)$$

So obey the signal if

$$\sum_{S:S_1=T_1} p(S)\pi_1(S) \geq \sum_{S:S_1=\pi} p(S)\pi_1(\bar{T}_1, S_2, ..., S_k)$$

This must be true for every signal she receives and true for all the other players.
This is just an LP

$$max \sum_S p(S)(\sum_i \pi_i(S))$$

$$s.t. \forall players i, T_i, \bar{T}_i \quad \sum_{S:S_i=T_i} p(S)\pi_i(S) \geq \sum_{S:S_i=T_i} p(S)\pi_i(S_1, ..., \bar{T}_i, ..., S_k)$$

$$\sum_S p(S) = 1$$

$$p(S) \geq 0$$

This LP has $n_1, .., n_k$ variables as player i has n_i pure strategies. $\sum_i n_i^2$ constraints
So for constant # players we have polytime algorithm. Remark: can implement CE without outside signals by using prenegotiated contracts.

13 Evolutionary Stable Equilibria

Idea: Suppose the payoffs in a game are related to reproductive fitness. Then over time the players with the highest payoff reproduce more and their characteristics/behaviors become dominant.

13.1 Simple Model

-Large number of players.
-Each player competes for a resource against another player(≥ 1 player)
-Players reproduce asexually at rate dependent upon payoff.
    -children are identical to parent.
-game repeats for children.

A strategy S^* is evolutionarily stable if $\delta > 0$ such that if at most a δ fraction of the population change strategy from S^* to S then those players do worse then the rest of the population.

$$\begin{bmatrix} & X & Y \\ X & (2,2) & (0,1) \\ Y & (0,0) & (1,1) \end{bmatrix}$$

Pure strategy X is ESS. suppose δ fraction change to Y.

$$Mutants : (1-\delta) - 0 + \delta \cdot 1 = \delta$$

$$Original : (1-\delta)2 + \delta \cdot 0 = 2 - 2\delta$$

If players switch to a mixed strategy they still do worse. Pure strategy Y is also an ESS.

$$mutants : (1-\delta) \cdot 0 + \delta \cdot 2 = 2\delta$$

$$original : (1-\delta) \cdot 1 + \delta 0 = 1 - \delta$$

Theorem 8 *An ESS must be Nash eq.*

Proof. If S^* is not NE then any single individual can do better by switching strategy-we dont need a $\delta - group$ to switch. ■

So an ESS is stable against small coalitions.

corollary 4 *S^* is an ESS iff*
(i) (S^*, S^*) is a NE
(ii) $\forall$ best responses S to S^*, $S \neq S^*$ $u(S,S) < u(S,S^*)$

Proof.

$$Mutants : \pi(S) : \delta\pi_1(S,S) + (1-\delta)\pi(S,S^*)$$

$$non-mutants : \pi(S^*) : \delta\pi_1(S^*,S) + (1-\delta)\pi_1(S^*,S^*)$$

Since (S^*, S^*) is NE we have $\pi(S^*,S^*) \geq \pi(S,S^*)$
if $\pi(S^*,S^*) > \pi(S,S^*)$ then $\exists\delta > 0$ such that $\delta - groups$ lose out.
if $\pi(S^*,S^*) = \pi(S,S^*)$ then any $\delta - group$ still loses as $\pi(S,S) < \pi(S^*,S)$.

if either (i) or (ii) does not hold then some $\delta - group$ can do better. ■

13.2 Examples

(1) Hawk -Dove

$$\begin{array}{ccc} & H & D \\ H & (\frac{v-c}{2}, \frac{v-c}{2}) & (v,0) \\ D & (0,v) & (\frac{1}{2}v, \frac{1}{2}v) \end{array} \;]$$

-resourse worth $v := 4$ fighting costs loser $c := 10$. Doves share resource

$$\{(\frac{2}{5}, \frac{3}{5}), (\frac{2}{3}, \frac{3}{5})\}$$

is NE

$$\begin{array}{ccc} & H & D \\ H & (-3,-3) & (4,0) \\ D & (0,4) & (2,2) \end{array} \;](i)$$

P$fraction of population are Hawks.$
$(ii) All players can be Hawks/Doves they randomize their behavior (common as well e.g. wasps).$

Is S^* an ESS? Exersize.

Another example Side Blotched Lizards

$$\begin{array}{cccc} & Y & O & B \\ Y & (2,2) & (3,0) & (0,3) \\ O & (0,3) & (2,2) & (3,0) \\ B & (3,0) & (0,3) & (2,2) \end{array} \;]$$

Orange throat: Aggressive Large territories
Blue throat: Less Aggressive small territories
Yellow throat: No territory, sneak onto other territories to mate.

Yellows invade orange, orange invade blue, blue beats yellow. Unique Nash is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = S^*$. Is this an ESS?
-Is there a best response S to S^* with $\pi(S,S) > \pi(S^*,S)$ e.g. $\pi(Y,Y) = 2$ $v(S^*,Y) = \frac{2+0+3}{3} = \frac{5}{3}$ nY is a best response to S^* i.e. there exist games with no ESS. In practice relative numbers of colored lizards varies a lot.

We can deal with non-symmetric payoffs by assuming that if 2 players are paired then they are 1 with prob 1/2.

e.g. Asymmetric H-D

$$\begin{array}{ccc} & H & D \\ H & (\frac{V-c}{2}, \frac{v-c}{2}) & (V,0) \\ D & (0,v) & (\frac{1}{2}v, \frac{1}{2}v) \end{array} \;]$$

Player I has value V, player II has value v, and $c > V > v$. rows: $Owner$, columns:$Intruder$. Then e.g. baboons, moths, etc

$$Bourgovise\ Strategy : \{(H^{if\ owner}, D^{if\ intruder}), (H^{if\ owner}, D^{int})\}$$

is ESS . So is

$$Paradoxical\ Strategy\{(D^0, H^I), (D^0, H^I)\}$$

e.g. Oecibus Cuitas Spider–abandons hideout if intruder enters.

13.3 Repeated Games

These kind of arguments also apply to repeated games. Suppose we want to max expected payoff over t games.

$$\begin{array}{ccc} & C & D \\ C & (1,1) & (S,0) \\ D & (0,S),(4,4) & \end{array}]$$

Always playing C still dominates always playing D.

Tit-for-Tat

-cooperate at first (D)

-if opponent plays C then play C in next game. Nice, Not a pushover, Forgiving.

Observe that tit-for-tat is an ESS

$$\begin{array}{cccc} & C & D & TFT \\ C & (1,1) & (S,0) & (1+\epsilon_1, 1-\epsilon_1) \\ D & (0,S) & (4,4) & (4,4) \\ TFT & (1-\epsilon_2, 1+\epsilon_2) & (4,4) & (4,4) \end{array}]$$

(C is also just about an ESS! for small δ, t). Of course there are lots of other strategies.

14 Hardness of NE

Here we show that it is $NP-Hard$ to find to find the (socially) best NE in a 2-player(symmetric) game. Social payoff: $\sum$ *individual payoffs*. We use a reduction from $3-SAT$: n variables $x_1, ..., x_n$

Each variable has two literals $\pm$, $x_i, \bar{x_i}$

There are m clauses $C_1, ..., C_m$ of the form $C_j = (x_1 \vee \bar{x_3} \vee x_7)$ (union of 3 literals)

Is there an assignment $\pm$ of the literals such that all the clauses are satisfied? i.e. $\emptyset = C_1 \wedge ... \wedge C_m$ *is true (Formula in conjunctive normal form) .*

What does this have to do with games? Coitzer and Sanholm gave a reduction from $3-SAT$ to a game in which if we can find the best NE we can test whether or not $\emptyset$ is satisfiable.

2 player symmetric game

$$S_1 = V \cup L \cup C \cup \{S\} = S_2$$

for a total of $n + 2n + m + 1 = 3n + m + 1$ strategies.

Payoffs are:

$$\begin{bmatrix} & l' \in L & v' \in V & c' \in C & S \\ l \in L & l' = \bar{l} : -2, l' \neq \bar{l} : 1 & -2 & -2 & -2 \\ v \in V & l' = v\ or\ \bar{v} : 2-n\ l' \neq v, \bar{v} : 2 & -2 & -2 & -2 \\ c \in C & c' \in C : 2-n,\ c' \notin c : 2 & -2 & -2 & -2 \\ S & 1 & 1 & 1 & 0 \end{bmatrix}$$

<u>intuition</u>
Use of L strategies give assignment which is good NE.
Use of S gives bad NE.
$\pi_1(l, \bar{l}) = -2$ is to enforce consistency ($-V$ gives the corollaries).

Lemma 4 *The pure strategy (S, S) is a NE.*

Proof. Get 0 get -2 if switch from S. ■

i.e. $\exists NE$ with social value $0 + 0 = 0$.

Lemma 5 *if $L_1, ..., L_n$ satsifies $\emptyset$ then $\exists NE$ in which each player plays l_i with probability $\frac{1}{n}$.*

Proof.
Assume player 2 plays $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ on $L_1, ..., L_n$. If $L_1, ..., L_n$ are all best responses then any mixed strategy on them is also. if player 1 plays L_1 he gets 1(same for $L_2, .., L_n$).
Show get ≤ 1 for any other pure strategy.
player 1 plays $\bar{l}_i$: $\frac{1}{n} - 2 + (1 - \frac{1}{n}) - 1 < 1$
player 1 plays $v \in V : \frac{1}{n}(2 - n) + (1 - \frac{1}{n})2 = 1$.

player 1 plays $c \in C' :\leq \frac{1}{n}(2 - n) + (1 - \frac{1}{n})2 = 1$(at least 1 literal is in C as $\emptyset$ is satisfied).
player 1 plays S: gets 1 ■

Value of this NE is $1 + 1 = 2$.
If these are the only NE then we are done.

Lemma 6 *There are no other NE*

$$\left[\begin{array}{cccc}
 & l' \in L & v' \in V & \\
l \in L & l' = \bar{l} : (-2, l') \neq \bar{l} : 1 & l = v \ or \bar{v} : (-2, 2 - n) \ l \neq v', \bar{v}' : (-2, 2) & l \in C' : (-2, 2 \\
v \in V & l' = v \ or \ \bar{v} : (2 - n, -2) \ l' \neq v, \bar{v} : (2, -2) & (-2, -2) & (- \\
c \in C & c' \in C : (2 - n, -2) \ c' \notin c : (2, -2) & (-2, -2) & (- \\
S & (1, -2) & (1, -2) & (
\end{array}\right.$$

Proof. If player 2 plays S with probability 1 then player 1 must play S with probability 1.
Suppose player 2 plays S with prob p (inc $p = 0$).
I can guarantee $1 - p$ by playing S.
S player 1 gets $\geq 1 - p$ simply play with prop q gets...erased board... $Sum \geq 2 - p - q$
But sum of payoffs

$$(1 - p)(1 - q) + -1(1 - p)q + -1(1 - q)p + pq \cdot 0$$

< 2 if players use V or C.

$$= 2 - 3p - 3q + 4pq \leq (1 - p) + (1 - q)$$

i.e. in NE V and C are not used. So players use only $L \cup \{S\}$ at NE.

$$\begin{array}{c|cc} & l' & S \\ \hline l & \leq 1 & -2 \\ S & 1 & 0 \end{array}$$

S weakly dominates l
S strictly dominates l if player 2 plays S with prob $p > 0$. Expected payoff for S is $>$ then with l. for any other NE both players only use L. Suppose player 2 plays $\bar{l_i}$ and l_i with combined prob $< \frac{1}{n}$. If player 1 plays v_i he gets $> 2(1 - \frac{1}{n}) + \frac{1}{n}(2 - n) = 1$. i.e. player 1 has no best response in L. $\Rightarrow$ player 2 use $l_i, \bar{l_i}$ with probability $\frac{1}{[}n \forall i$ (also for $\bar{H}$ by symmetry). If player 2 plays L_i and player 1 plays $\bar{L_i}$ sometime then payoffs are < 1 and we can do better playing S. So player 1 and player 2 both play l_i (or both play $\bar{l_i}$) with prob $\frac{1}{n}$. So both play $l_1, l_2, ..., l_n$ with prob $(\frac{1}{n}, ..., \frac{1}{n})$. Suppose there is an unsatisfied clause C. Playing C gives payoff of 2. So this is not NE. ■

So if we find in polytime the best NE then if social value $> 0\ \emptyset satisfiable$, $= 0\ \emptyset not\ satisfiable$.

Theorem 9 *Finding best NE is NP-Hard*

■

corollary 5 *Counting the number of NE is $\sharp P$ hard*

\# Nash = 1+# of sat assignments. ■

15 Graphical Games

Get notes for first lecture

Open Problems
Find NE exactly in polytime? Paper on this which was recently shown to be flase!
What about on trees with bounded pathwidth/treewidth? What about other graphs ? e.g. find NE in other sorts of graphs.

Payoff matrices for players I and II:

$$A := \begin{bmatrix} 1 & 7 & 6 & 3 & 4 & 1 \\ 0 & 5 & 6 & 2 & 0 & 1 \\ 2 & 0 & 3 & 2 & 4 & 1 \\ 4 & 5 & 1 & 3 & 5 & 0 \\ 3 & 1 & 7 & 4 & 6 & 2 \\ 5 & 2 & 0 & 5 & 1 & 4 \end{bmatrix}$$

$$B := \begin{bmatrix} 3 & 4 & 0 & 0 & 1 & 2 \\ 0 & 3 & 1 & 4 & 5 & 0 \\ 0 & 6 & 4 & 3 & 2 & 2 \\ 2 & 2 & 5 & 0 & 1 & 6 \\ 7 & 1 & 0 & 3 & 4 & 2 \\ 1 & 5 & 6 & 1 & 4 & 1 \end{bmatrix}$$

with $columns : c_1, ..., c_6,\ rows : r_1, ..., r_6$

Suppose I knows that II will play c_1 Obviously I will play r_6, as this gives the highest payoff.

Geometrically

I plots points on line and picks the one furthest to the right i.e. an "'extreme point"'.

Suppose I knows II will play only c_1 or c_2 with probability $p, 1-p$ say. Plot the points in two dimensional space.

The extreme points are on the dominant of the convex hull. Instead of rightmost point which we wanted in one dimension, we want points on Convex Hull with positive norms. Points on faces(facets) are also best responses if p corresponds to the normal to the face.

e.g. $p = \frac{3}{5}$, then r_4 gives I

$$\frac{3}{5} \cdot 4 + \frac{2}{5} \cdot 5 = \frac{22}{5}$$

and r_6 gives

$$\frac{3}{5} \cdot 6 + \frac{2}{5} \cdot 2 = \frac{22}{5}$$

i.e. any convex combination of (r_4, r_6) is b.r. to $p = \frac{3}{5}$. (r_4, r_6) are brs to some probability distribution on (c_1, c_2) iff $r_4 - r_6$ is a facet of $P(c_1, c_2)$.

similarly (c_1, c_2) are b.r.s to some prob distribution on (r_4, r_6), iff $c_1 - c_2$ is facet of $P(r_4, r_6)$. This generalizes to higher dimensions. b.r.s correspond to facets $d{-}dim\ polytopes$. This observation helps if we , say , want to count the expected number of NE in random games. Payoff entries are iid from some distribution e.g. $U[0, 1]$.

 # pure NE r_i is a best response to c_j iff it is highest number in c_j probability of this is $\frac{1}{n}$. c_j is best response to r_i with prob $\frac{1}{n}$. since the matrices are independent , the probability of the event is $P(r_i \leftrightarrow c_j) = \frac{1}{n^2}$ So the expected number of NE is $n^2 \cdot \frac{1}{n^2}$.

Note: As points in general position therefore only need to search for 1×1, $2 \times 2, ...$

2×2 **NE**:

What is the probability that (r_i, r_j) is a best response to (c_r, c_s) for some p ? it is $P(r_i r_j\ is\ face) = \frac{E(\#faces)}{\binom{n}{2}}$.

$$P(r_i r_j \leftrightarrow c_s c_r) = (\frac{E(\#faces)}{\binom{n}{2}})^2 = q$$

$$E(\#NE) = \binom{n}{2}^2 q = E(\#faces)^2 = E(\#CH\ points)^2$$

So lets count # CH points. Fix columns 1 and 2. We have n points (1 per row). For a point to be onCH we need there to be some hyperplane H such that $x \in H$ and all other pts lie bellow H. Take (x, y) and the hyperplane shown all points below with probability $(1-2xy)^{n-1}$

$$E(\#CH) \geq n \cdot E[(1-2xy)^{n-1}]$$

$$= n \int_0^{1/2} (1-2z)^{n-1} f(z) dz$$

where z is product of two independent $U[0,1]$ distributions. We have $f(z) = log(\frac{1}{z})$ $0 < z \leq 1$. So

$$E(\#CH) \geq \int_0^{\epsilon} n(1-2z)^{n-1} f(z) dz$$

$$\geq f(\epsilon) \int_0^{\epsilon} (1-2z)^{n-1} dz$$

$$= f(\epsilon)[-\frac{1}{2}(1-2z)^n]_0^{\epsilon}$$

Setting $\epsilon = \frac{1}{n}$ gives

$$E(\#CH) \geq log(n)[-\frac{1}{2}(1-\frac{2}{n})^n + \frac{1}{2}]$$

$$\geq \frac{log(n)}{3}$$

Theorem 10 *$\exists \Omega(log^2(n))$ 2×2 NE in expectation*

$E = exp$ $f(x) = e^{-x}$ $G_d = Gamma(d)$, $f(x) = \frac{e^{-x}x^{d-1}}{(d-1)!}$ $U = U[0,1]$ $E \equiv -log(U)$, $P(E \leq z) = \int_0^x e^{-y} dy$

$$= [-e^{-y}]_0^x = 1 - e^{-x}$$

$$= P(U \geq e^{-x}) = P(-logU \leq x)$$

....

For a good algorithm you want "'concentration bounds"' on # NE. Therefore with high probability $\exists 2 \times 2$ NE.

Open Problem

Expected Polytime algorithm? Other distributions? Other games .e.g. 0-1 games

ϵ approximate NE.

16 Approximate NE

Recall an approximate NE is a set of probability distributions $p_1, ..., p_k$ such that $\forall i$ p_i gives player i an expected payoff within ϵ of optimal of her best response to $p_1, ..., p_{i-1} p_{i+1} ... p_k$ i.e.

$$\prod_i (p_1 \cdot p_i' p_k) \leq \prod (p_1 \cdots p_k) + \epsilon \;\forall i \;\forall p_i'$$

Conceptually this is a problem: What does it mean to be almost an equilibrium?

But there is very little incentive to change so it is reasonable to study these approximate NE. It is easier to find approximate NE . We show this for 2-players but it generalizes easily to a fixed number of players. The key structural result is that $\exists$ approximate NE in which both players only use $O(lgn)$ strategies. i.e. small supports. This gives Quasi polynomial time algorithm...try all possibilities.

So player (1) pick prob dist p andplayer (2) pick prob dist q

The payoffs are

$$(1) : p^T Aq$$

$$(2) : p^T Bq$$

where A is the payoff matrix for (1), B for (2). Assume entries of A, B are in range $[0, 1]$.

Theorem 11 *(Lipton et al)*
For any NE p^*, q^* and for any $\epsilon > 0$ there exists an approximate NE $\bar{p}, \bar{q}$ with supports of size $O(\frac{logn}{\epsilon^2})$.

Proof.
Let $p^{*T} Aq^* = \pi_1,\ p^{*T} Bq^* = \pi_2$ Since p^*, q^* is NE we have $\forall$ strategies $s \in S_1$ for which $p^*_s > 0$

$$e_s^T Aq^* = \pi_1 \ \ i.e. \ \ (Aq^*)_s = \pi_1 \ \forall s \ played\ by (1) \ in \ p^*$$

similarly $(p^{*T}B)_t = \pi_2\ \forall t\ played\ by\ (2)\ in\ q^*$.

Define $\bar{p}$ as: sample k rows according to the distribution p^*(allow repeats). Let $\bar{p}$ give weight $\frac{1}{k}$ to each sample. (Similarly for $\bar{q}$). We will show $\bar{p}\bar{q}$ satisfy conditions of the theorem with positive probability. So by probabilistic method there exists small support approximate NE.

WTS

$$e_s A\bar{q} \leq \bar{p}A\bar{q} + \epsilon \ \forall s \in S_1$$

$$\bar{p}Be_t \leq \bar{p}B\bar{q} + \epsilon \ \forall t \in S_2$$

Claim 1: $e_s A\bar{q} \leq e_s Aq^* + \frac{\epsilon}{2}$.
Claim 2: $p^* Aq^* \leq \bar{p}A\bar{q} + \frac{\epsilon}{2}$.
Proof of claim 2: by previous thing $\bar{p}(Aq^*) = p^*(Aq^*) = \pi_1$ So

$$\bar{p}(Aq^*) = (\bar{p}A)q^* = \pi_1$$

As $q^{*]}$ is a probability distribution, the entries of $\bar{p}A$ have expectation π_1 according to q^*. Since $\bar{q}$ is chosen with respect to q^* we see each sample of $\bar{q}$ has expectation π_1 when multiplied by $\bar{p}A$. Now $\bar{p}A\bar{q}$ is sum of r.v.s. with expectation $\frac{\pi_1}{R}$ SO the expectation is π_1.

Hoeffding's Tail inequality: Let $X_1, X_2, ..., X_N$ be independant r.v.s such that X_i is in range $[a_i, b_i]$. Then for any $t > 0$

$$P[|\sum_{i=1}^{N} X_i - E(\sum_{i=1}^{N} X_i)|] \leq exp(\frac{-2t^2}{\sum_{i=1}^{N}(b_i - a_i)^2}$$

Proof. Application of Chernoffs Bound. ∎

So

$$P[|\bar{p}A\bar{q} - p^*Aq^*| \geq \frac{\epsilon}{2}] = P[|_i - E(\sum X_i)| \geq \frac{k\epsilon}{2}]$$

$$\leq exp(\frac{-2\frac{k\epsilon}{2}^2}{k})$$

$$= e^{-\frac{k\epsilon^2}{2}} = \frac{1}{n^2}$$

if $k = 4logn/\epsilon^2$ ■

Proof of claim 1:
Similar argument shows

$$(e_sA)_{\bar{q}} \leq (e_sA)q^* + \frac{\epsilon}{2} \; with \; prob \geq 1 - \frac{1}{n^2}$$

As we have n rows the probability one of these is 'bad' is $\leq n\frac{1}{n^2} = \frac{1}{n}$

So

$$e_sA\bar{q} \leq \bar{p}A\bar{q} + \epsilon$$

is not true with probability $\frac{1}{n^2} + \frac{1}{n}$

Similarly with respect to player (2) and B things are good with probability $1 - \frac{C}{n} > 0$.

So there exists approximate NE with support s $O(\frac{logn}{s^2}$.

corollary 6 *there exists quasi poly time algorithm*

Proof. Try all pairs of support of size ....erased board... ■

17 Sink Equilibrium

So far our equilibria concepts have mainly been mixed strategy equilibrium. But many games involve pure strategies only. We have the concept of PSNE. Often these do not exist ! Even they do exist, why should PSNE be an outcome of a game? What if players iteratively make moves. Maybe we converge to a PSNE eg potential games.

Lets suppose this is how our game evolves. What happens? We model this using a strategy profile graph.

18 Sink Eq

Why are they of interest? consider the following game :

n-players

player i has one 'nice; strategy y_i

i has n 'naughty‘ strategies $X_i^1 X_i^2 ... X_i^n$

player i gets payoff 1 if she plays y_i

PLayer i gets 2 if she plays $X_i^{r_i}$ for some r_i and

$$i \equiv [\sum_{j:j \ plays \ anaughty \ strategy \ X_j^{r_i}} r_j] \ mod \ n$$

$$0 \ otherwise$$

Social utility = sum of private payoffs.

At any point player i has a strategy that gives 2.

So players will only ever use X strategies.

So there exists a sink in which all states only have X strategies. Each state sink has value 2.

Optimal value is $n+1$:

$n-1$ play y_i

one player plays X_j^j

Price of sinking = worst case ratio of value of sink to opt

$$= \frac{n+1}{2}$$

$Price\ of\ anarchy \leq \frac{n+1}{n}$ each player always gets 1 in any NE.

i.e. factor n difference between performance you expect if people use mixed vs pure strategies. why naughty strategies dont work against mixed strategies. conclusion price of anarchy is not a good measure of the cost of the lack of coordination in games where players use pure strategies.

What is the valie of sink? max value of a state in Q.

min value of a state in Q .

expected value of a state in W.

Expected value of random walk over Q(value of steady state dist). So we are interested in whether a random walk has good social value. speed of convergence. (e.g. after polytime walk what is expected value?)

Open: Many games without PSME have have not been analyzed. They can be analyzed wrt sink equilibria and random walks.

Use RWS to give speed of convergence.

Other types of underlying state graph: simultaneous moves

-best response moves

-forward thinking.

19 Market Equilibria

Suppose we have k players and n commodities.

PLayer i has a vector$v_i \in R_+^n$ of communities she owns

PLayer i has a linear utility function u_i.
i.e. limited utility is

$$u_i \cdot v_i = \sum_{i \in C} u_{ij} v_{ij}$$

How can people trade to improve their utility.
Pareto Allocation: An allocation A of the commodities such that there does not exist an allocation A such that no player is worse off under A' and at least one player is strictly better off.
(i.e. no-one can be made better of without making someone worse off). We want to find a Pareto allocation(there may be more then one). We will show one exists use commodities prices.

Given a set p of prices what will i do?
she can sell her goods for $p \cdot v_i$
She can spend this...
choose set of items v_i^* that maximizes $u_i \cdot v_i^*$ such that

$$p \cdot v_i^* \leq p \cdot v_i$$

For this to be a realistic solution we need demand at most

$$\sum_i v_i^*(p) \leq \sum_i v_i$$

If so we say the market clears.

Theorem 12 *(Arrow-Debrew) There exists a market clearing set of prices (a general equilibrium).*

Proof.
Assume

$$\sum_{j \in C} p_j = 1 \ (a \ simplex)$$

Suppose demand is bigger then supply for some item(otherwise we are done). Use idea from before. Raise prices

$$\phi(p) = \frac{p_1 + max(0, d_1)}{1 + \sum_{j \in C} max(0, d_j)}, \ldots,$$
$$, \ldots, \frac{p_n + max(0, d_n)}{1 + \sum_{j \in C} max(0, d_j)}$$

By Browers FPT there is some p^* such that

$$\phi(p^*) = p^*$$

Let $d'_j = max(0, d_j)$(excess demand). want $d'_j = 0 \forall j \in C$ Observe

$$\phi(p) = \frac{1}{\beta}(p + d')$$

Since

$$p^* \cdot v_i^* \leq p \cdot v_i \ \forall i$$

$$p^*(v_i^* - v_i) \leq 0 \ \forall i$$

$$p^*(\sum_i (v_i^* - v_i)) \leq 0$$

$$p^* \cdot d \leq 0$$

Now $\phi(p^*) = p^* = \frac{1}{\beta}(p^* + d')$. So $p^* \cdot d = \frac{1}{\beta}(p^* + d') \cdot d$

$$p^* d = \frac{1}{\beta}(p^* d + d' d)$$

$$(\beta - 1)_{\geq 0}(p^* d)_{\leq 0} = d' d$$

therefore $d'd \leq 0$.

but $d' \cdot d \geq 0$ if $d_j < 0$ then $d'_j = 0$ therefore $d = 0$ i.e. market clears. ∎

20 Coalition Games

20.1 The Core

What happens if players work together in groups? eg if players together buy/build/rent a network, how are costs divided between them? One possible solution concept is *the core*

A set I of players.

A subset $S \subseteq I$ can guarantee a payoff $v(S)$.

How should $v(I)$ be shared so that no subset is unhappy ?

e.g. Treasure: n people find treasure on an island. 2 people are needed to carry each piece of treasure.

$$v(S) = \lfloor \frac{|S|}{2} \rfloor$$

if $n = 2$ $(\epsilon, 1 - \epsilon)$ is a stable sharing i.e. in core.

If $n \geq 3$ $(1 - \epsilon, \epsilon, 0)$, $(1/3, 1/3, 1/3)$ the core is empty.

e.g. Majority Voting

$$v(S) = 1 \ if \ |S| > \frac{I}{2} \ 0 \ otherwise$$

again the core is empty if $|I| > 3$. When is the core nonempty? We need to assign x_i to player $i \in I$ such that

$$\sum_{i \in S} x_i \geq v(S)$$

(this implies transferable payoffs)

$$\sum_{i \in I} x_i = v(I)$$

We can answer this using LP we need 1 technical lemma:

Lemma 7 $\exists x \geq 0$ *such that* $Ax \leq b$ *iff* $\forall y \geq 0 \ yA \geq 0 \Rightarrow yb \geq 0$

Proof.
take $y \geq 0, A^T y \geq 0$ and $yb < 0$ $\exists x \geq 0, Ax \leq b$ So $0 > y^T b \geq y^T(Ax) \geq 0x \geq 0$

next suppose $Ax \leq 1, x \geq 0$ has a solution. Then the LP

$$max \ \ -1z$$

$$Ax - Iz \leq b$$

$$x, z \geq 0$$

has value 0.
The dual is

$$min \ b^T y$$

$$[A^T, -I]^T y \geq [0, -1]^T$$

$$y \geq 0$$

$$min y^T b$$

$$A^T y \geq 0$$

$$1 \geq y \geq 0$$

also has value 0
$\exists x \geq 0, Ax \leq b \ iff \ \forall y \geq 0, yA \geq 0 \Rightarrow yb \geq 0.$

■

Theorem 13 *A coalition game with (trans payoffs) has non empty core iff* $\forall y \geq 0 \sum_{S:i\in S} y_S = 1 \forall i \Rightarrow \sum_S y_S v(S) \leq v(I)$

Proof.

$$x_i \geq 0 \sum_{i\in I} x_i \leq v(I), \sum_{i\in S} x_i \geq v(S) \forall S \subseteq I$$

$$Ax \leq b$$

$$x \geq 0$$

$$[I, S]^T x \leq [v(I), -v(S)]^T \Leftrightarrow \forall y \geq, yA \geq 0 \Rightarrow yb > 0$$

$[I, S]y \geq 0, \ y \geq 0 \Rightarrow (y_I \ y_S)[V(I), v(S)]^T \geq 0.$

$$y_I - \sum_{S:i\in S} y_S \geq 0 \ \forall i, y \geq 0$$

$$\Rightarrow v(I)y_I - \sum_S v(S)y_S \geq 0$$

So

$$\frac{y_I}{y_I} \geq \sum_{S:i\in S} \frac{y_S}{y_I}, y \geq 0 \Rightarrow v(I) \geq \sum_S v(S)\frac{y_S}{y_I}$$

$$\forall \hat{y} > 0, 1 = \sum_{S:i\in S} \hat{y}_S \Rightarrow v(I) \geq \sum_S \hat{y}_S v(S)$$

■

20.2 The Core + Stable Matchings

Set of boys and set of girls, have a list of preferred partners. e.g.

$$g_j : b_3 \geq b_7 \geq b_{26} \geq ...$$

A 'Solution' is a matching of boys to girls H is a complete(perfect) matching if everyone prefers being married to single.

What does it mean for M to be in the core?

Suppose a subset $S \subseteq B \cup G$ prefers to deviate to give a matching M^*. So each player in S gets at least as good a partner as in M. So any edge $e \in M^* - M$ is between members in S, otherwise the other end vertices are needed in the coalition.

corollary 7 *M is in the core iff i) Each i prefers $M(i)$ to being single.*
ii) No pair i, j prefer each other to their partners. i.e. not $i : j \geq m(i), j : i \geq M(j)$

Proof.
if S deviates to give M^* then either some $i \in S$ becomes single or $\exists i, j \in S$ such that $(i, j) \in M^* - M$.

Clearly if (i) or (ii) is violated then we have $|S| \leq 2$ showing M is not in core. ■

questions: Is the core always non-empty? Can we efficiently find an M in the core? We want an M that satisfies (i) and (ii) such an M is a stable matching(marriage). We do this algorithmically.

20.3 Boy Proposal Algorithm

```
Pick any single man i
   the man proposes to next girl j on his list.
   j accepts if she prefers i to her current partner M(j).  If so Set M(j) = i
Repeat until no single men or we have gone through all the lists of each single man.
```

First see that this is efficient. i.e. runtime is $O(n^2)$ Each list is of size $\leq n$ Property 1
the quality of a girls partner only gets better as the algorithm runs.
On the other hand the quality of the boys partner only gets worse.

Theorem 14 *We have a stable marriage at the end*

Proof. At the end of the algorithm he has been rejected by every girl on his list. They wont change their minds... Similarly take i and $i, j \geq M(i)$ then j has rejected i and wont change her mind ■

extensions Girls accept at least 1 boy. i.e. Girls = Hospitals. Same algorithm works. Room mate problem- non bipartite graph. here the core can be empty.

21 Stable Matchings: Fairness and Truth telling

The game may have many stable matchings. Let $j \in S(i)$ if $(j, i) \in M$ for some stable matching M. We say j is a 'stable partner of i'. The boy-proposal algorithm has several interesting properties.

(0) The order in which boys are chosen to make proposals does not matter.
(1) The alg is truthful for boys.
(2) Every boy is matched to his 'best' stable partner.
(2′) Every girl is matched to her 'worst' stable partner.
Obviously girl proposal algorithm has symmetric properties.

But the mechanism is not truthful for girls. What if g_2 lies and rejects b_1 and stays with b_2 when she has $b_3 > b_1 > b_2$? We end up with a different matching...with g_2 getting b_3 instead of b_1 i.e. she does better by lying. So there can be incentive to lie for girls. Remarks: Hospitals used the hospital proposal algorithm but changed it
biased against students (didn't want to get sued)
incentive for students to lie
In fact things are not so bad.

Suppose boys rank the top k girls. Girls have a complete ranking implicitly(i.e. only compare 2 boys at a time). How many girls have an incentive to lie? Assume each boys list is drawn at random. (Girls can be anything.)

Theorem 15 *The expected number of girls who may benefit from lying is $O(e^k)$.*

Proof. A girl can only have an incentive to lie if she has at least 2 stable partners. So lets calculate how many stable partners a girl has. Find the truthful boy-proposal stable matching M. Let girl j discard her current partner $M(j)$. i.e. delete from list. Run the algorithm from here-this is OK as order of proposals doesn't matter. If j gets a partner then she has another stable partner. As girls partners get better it suffices to check if anyone proposes to j after she discards $M(j)$. If not she has one stable partner. So boy $M(j)$ makes proposal. If this is accepted by a matched girl then another boy becomes single and we repeat with that boy.

This ends if
(i) the boy proposes to an unmatched girl(includes j)
(ii) we get to the end of the boys list.

So we only need to calculate the probability that the single woman chosen in (i) is j. But the lists are random so each single girl is equally likely. i.e. j has at least 2 stable partners with propbability $\leq \frac{1}{1+\#single\ girls}$.

$$E(\#single\ girls) = nP(g\ single)$$

$$\geq nP(girl\ is\ on\ no\ lists)$$

$$= n(1-\frac{k}{n})^n \approx ne^{-k} \ \ (this\ is\ concentrated)$$

i.e. $E(\#girls\ with\ \geq 2\ stable\ partners) \leq n\frac{1}{1+o(\frac{n}{e^k})} \leq O(e^k)$
i.e. if k is small then truth telling is good strategy.

Open: What happens in roommate problem?

21.1 Coalition Games: Shapley Value

This a different solution concept for coalition games. Unlike the core, the shapely value always exists. Recall we have $v(S)\ \forall S \subseteq I, v(\emptyset) = 0$. We want to assign x_i values such that $\sum_{i\in I} x_i = v(I)$. (we get rid of the constraint $\sum_{i\in S} x_i \geq v(S)$) What axiomatic properties should x have?

Axiom I: Symmetry: The label of a player should not affect its payoff. In particular, if players i and j have the same effect on any coalition i.e.

$$\forall S\ v(S \cup i) - v(S) = v(S \cup j) - v(S)$$

then $x_i = x_j$.

Axiom II Dummy Players that contribute nothing receive nothing. Specifically, we say R is a carrier iff

$$\forall S \subseteq I\ v(S \cap R) = v(S)$$

The set $I - R$ are dummies. Only players in R receive anything. Given 2 coalition games v, v', let w be the coalition game that has value v a fraction p of the time, value v' $1-p$ of the time. i.e.

$$w(S) = pv(S) + (1-p)v'(S)$$

Axiom III Linearity: if $w(S) = pv(S) + (1-p)v'(S) \forall S$ then $x_i(w) = px_i(v) + (1-p)x_i(v')$.

These are all pretty natural. Surprisingly there is only *one* solution concept that satisfies all 3 axioms.

Shapely Value:

$$x_i = E_\pi[v(\{\pi^{-1}(1), \pi^{-1}(2), ..., i = \pi^{-1}(\pi_i)\}) - v(\{\pi^{-1}(1), ..., \pi^{-1}(\pi_i - 1)\})$$

632145 , $v(1236) - v(632)$ what is this? Randomly select the players and see the average marginal increase due to player i.

Theorem 16 *Shapely value satisfies axioms* I, II, III.

Proof.

- Look at all permutations.
- Dummies never increase value of subset so always get 0.
- $$x_i = p \cdot x_i(v) + (1-p)x_i(v')$$

  By construction.

■

Theorem 17 *Only the Shapley value satisfies axioms I, II, III.*

Proof. Let O be the coalition game with $v(S) = 0 \forall S \subseteq I$. Then $x_i(0) = 0 \forall i \in I$ by axiom II. Linearity axiom implies $x : R^{2^{|I|}} \to R^{|I|}$ is a linear mapping. So the set of coaliion games on I is a $2^{|I|} - 1$ dimensional vector space. Lets find a basis. Given $\emptyset \neq R \subseteq I$ let the coalition game V_R be $V_R = 1\ if\ R \subseteq S,\ 0\ otherwise$ i.e. R is a carrier in game V_R, $I - R$ are dummies.

So for x we have Dummy Axiom $\Rightarrow x_i(v_R) = 0\ i \notin R$, $\sum_{i \in R} x_i(v_R) = 1$. Symmetry: $x_i(v_R) = \frac{1}{|R|} \forall i \in R$. These are $2^{|I|} - 1$ such games. Are they linearly independent? If so we are done as then these games give all game payoffs (as they are linear combinations of the basis). By III these are values Shapely gives too.

Suppose $\exists \lambda_S$ such that $\sum_S \lambda_S v_S = 0$. Take S^* the smallest set with $\lambda_S \neq 0$ then $\sum_S \lambda_S v_S(S^*) = 0 + \sum_{S:S^*S} \lambda_S v_S(S^*) = \lambda_{S^*} v_{S^*}(S^*) = \lambda_{S^*}$ contradiction. ■

So Shapely value exists and has some nice properties. But it has some problems

- Complexity: there are $n!$ permutations so how do we calculate it?
- it may not be fair. e.g. election example.
- Shapely value may not be in the core even if core is non-empty.