From self-interpreters to normalization by evaluation

Mathieu Boespflug

École Polytechnique

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$$E \ulcorner M \urcorner =_\beta M$$

$$
\begin{aligned}
\ulcorner x \urcorner &= \mathsf{Var}\ \ x \\
\ulcorner \lambda x.t \urcorner &= \mathsf{Lam}\ (\lambda x.\ulcorner t \urcorner) \\
\ulcorner t_0\ t_1 \urcorner &= \mathsf{App}\ \ulcorner t_0 \urcorner\ \ulcorner t_1 \urcorner
\end{aligned}
$$

An example self-interpreter

$$
\begin{aligned}
E_\alpha\ (\textsf{Var}\ x) &= x \\
E_\alpha\ (\textsf{Lam}\ t) &= t \\
E_\alpha\ (\textsf{App}\ t_0\ t_1) &= (E_\alpha\ t_0)\ (E_\alpha\ t_1)
\end{aligned}
$$

Self-reducers

$$R \ulcorner M \urcorner =_{\beta} \ulcorner NF_M \urcorner$$

$$E_{NF} \ulcorner M \urcorner \triangleq E_{\alpha}(R \ulcorner M \urcorner) \longrightarrow_{\text{whnf}} NF_M$$

An evaluator

$$
\begin{aligned}
eval\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
eval\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ t \\
eval\ (\mathsf{App}\ t_0\ t_1) &= \mathsf{case}\ eval\ t_0\ \mathsf{of} \\
&\qquad \mathsf{Lam}\ t \rightarrow eval\ (t\ t_1) \\
&\qquad t_0' \rightarrow \mathsf{App}\ t_0'\ (eval\ t_1)
\end{aligned}
$$

A normalizer (call-by-name)

$$
\begin{aligned}
\mathit{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
\mathit{norm}\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ (\lambda x.\mathit{norm}\ (t\ x)) \\
\mathit{norm}\ (\mathsf{App}\ t_0\ t_1) &= \mathsf{case}\ \mathit{norm}\ t_0\ \mathsf{of} \\
&\quad \mathsf{Lam}\ t \to t\ t_1 \\
&\quad t_0' \to \mathsf{App}\ t_0'\ (\mathit{norm}\ t_1)
\end{aligned}
$$

A normalizer (call-by-value)

$$
\begin{aligned}
\mathrm{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
\mathrm{norm}\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ (\lambda x.\mathrm{norm}\ (t\ x)) \\
\mathrm{norm}\ (\mathsf{App}\ t_0\ t_1) &= \mathsf{case}\ \mathrm{norm}\ t_0\ \mathsf{of} \\
\mathsf{Lam}\ t &\to t\ (\mathrm{norm}\ t_1) \\
t_0' &\to \mathsf{App}\ t_0'\ (\mathrm{norm}\ t_1)
\end{aligned}
$$

The target (untyped normalization by evaluation)

$$\begin{aligned}
\mathrm{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
\mathrm{norm}\ (\mathsf{Lam}\ \mathrm{t}) &= \mathsf{Lam}\ (\lambda x.\mathrm{norm}\ (\mathrm{t}\ x)) \\
\mathrm{norm}\ (\mathsf{App}\ \mathrm{t}_0\ \mathrm{t}_1) &= \mathsf{App}\ \mathrm{t}_0\ (\mathrm{norm}\ \mathrm{t}_1)
\end{aligned}$$

CPS transformation to the rescue...

$$\text{Term} \supset \text{Term}_\text{V} \ni v ::= x \mid \lambda x.t_c$$
$$\text{Term} \supset \text{Term}_\text{CPS} \ni t_c ::= v \mid v\, v$$

$$\begin{aligned}
\mathrm{norm}\ (\text{Var}\ x) &= \text{Var}\ x \\
\mathrm{norm}\ (\text{Lam}\ t) &= \text{Lam}\ (\lambda x.\mathrm{norm}\ (t\ x)) \\
\mathrm{norm}\ (\text{App}\ (\text{Lam}\ t_0)\ t_1) &= \mathrm{norm}\ (t_0\ t_1) \\
\mathrm{norm}\ (\text{App}\ (\text{Var}\ x)\ t_1) &= \text{App}\ (\text{Var}\ x)\ (\mathrm{norm}\ t_1)
\end{aligned}$$

CPS transformation to the rescue...

$$\begin{array}{llll} \mathsf{Term} \supset \mathsf{Term_V} & \ni & v & ::= x \mid \lambda x.t_c \\ \mathsf{Term} \supset \mathsf{Term_{CPS}} & \ni & t_c & ::= v \mid v\ v \end{array}$$

$$\begin{aligned} \mathrm{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\ \mathrm{norm}\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ (\lambda x.\mathrm{norm}\ (t\ x)) \\ \mathrm{norm}\ (\mathsf{App}\ (\mathsf{Lam}\ t_0)\ t_1) &= \mathrm{norm}\ (t_0\ t_1) \\ \mathrm{norm}\ (\mathsf{App}\ (\mathsf{Var}\ x)\ t_1) &= \mathrm{norm}(\mathsf{App}\ (\mathsf{Var}\ x) t_1) \end{aligned}$$

since $\mathsf{App}\ (\mathsf{Var}\ x)\ (\mathrm{norm}\ t_1) \equiv \mathrm{norm}\ (\mathsf{App}\ (\mathsf{Var}\ x)\ t_1)$ for all x, t_1.

$$
\begin{aligned}
\mathrm{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
\mathrm{norm}\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ (\lambda x.\mathrm{norm}\ (t\ x)) \\
\mathrm{norm}\ (\mathsf{App}\ t_0\ t_1) &= \mathrm{norm}\ (\mathrm{app}\ t_0\ t_1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{app}\ (\mathsf{Lam}\ t_0)\ t_1 &= t_0\ t_1 \\
\mathrm{app}\ (\mathsf{Var}\ x)\ t_1 &= \mathsf{App}\ (\mathsf{Var}\ x)\ t_1
\end{aligned}
$$

A new representation scheme (aka intepretation)

$$
\begin{aligned}
[\![x]\!] &= \mathsf{Var}\ x \\
[\![\lambda x.t]\!] &= \mathsf{Lam}\ (\lambda x.[\![t]\!]) \\
[\![t_0\ t_1]\!] &= app\ [\![t_0]\!]\ [\![t_1]\!]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{norm}\ (\mathsf{Var}\ x) &= \mathsf{Var}\ x \\
\mathrm{norm}\ (\mathsf{Lam}\ t) &= \mathsf{Lam}\ (\lambda x.\mathrm{norm}\ (t\ x)) \\
\mathrm{norm}\ (\mathsf{App}\ t_0\ t_1) &= \mathsf{App}\ t_0\ (\mathrm{norm}\ t_1)
\end{aligned}
$$

$$\mathrm{nbe}\ t = E_\alpha\ (\mathrm{norm}\ [\![t]\!])$$

Bottom line

- Obtained a normalization by evaluation algorithm for terms in CPS.
- This algorithm also works for terms in direct style.
- Correctness of the initial self-reducer implies correctness of the algorithm, since all transformations involved are meaning preserving.