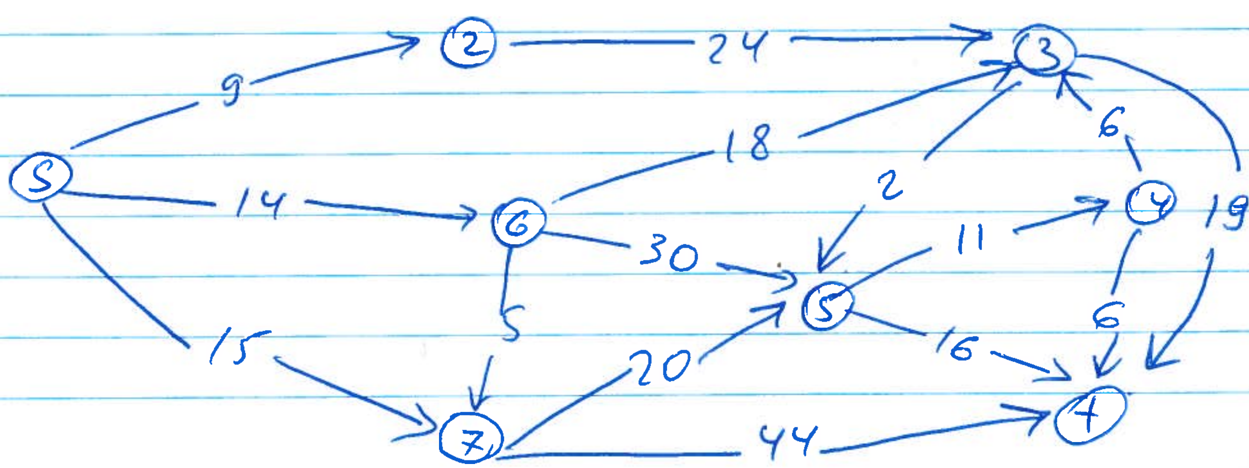
# Finding Shortest Path Tree

We start with the following directed graph and want to find its shortest path tree. For that, we use the local improved algorithm (LIA).

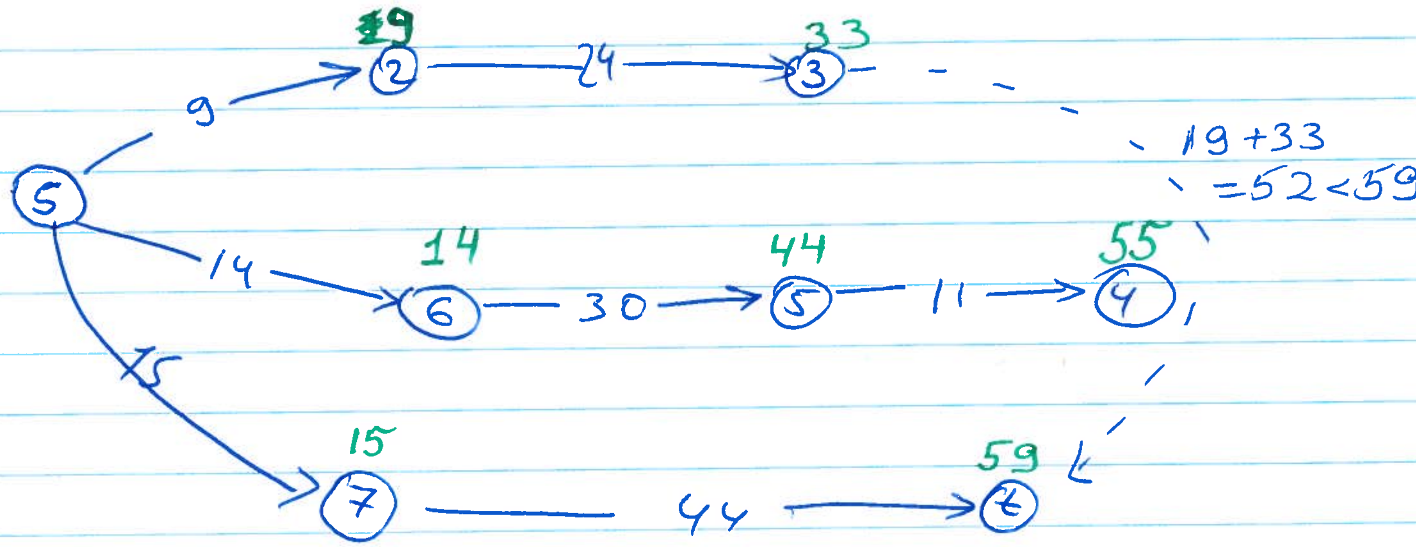
1. Take any directed spanning tree T rooted at s. (Use BFS for that)

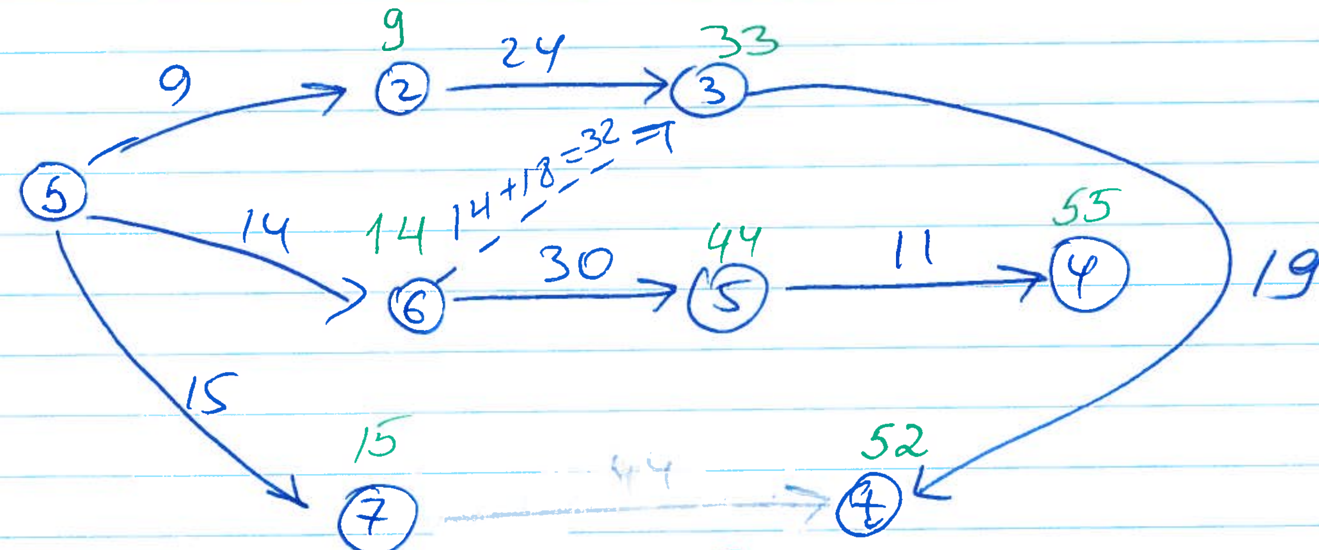
2. $\forall v \in V(G)$ compute $d(v)$, the distance from s to v in T.
$\forall e \in G \setminus T$, say $e=(v,w)$ if
$d(v) + c(v,w) < d(w)$, then
delete the edge that enters w and add $(v,w)$ to T.

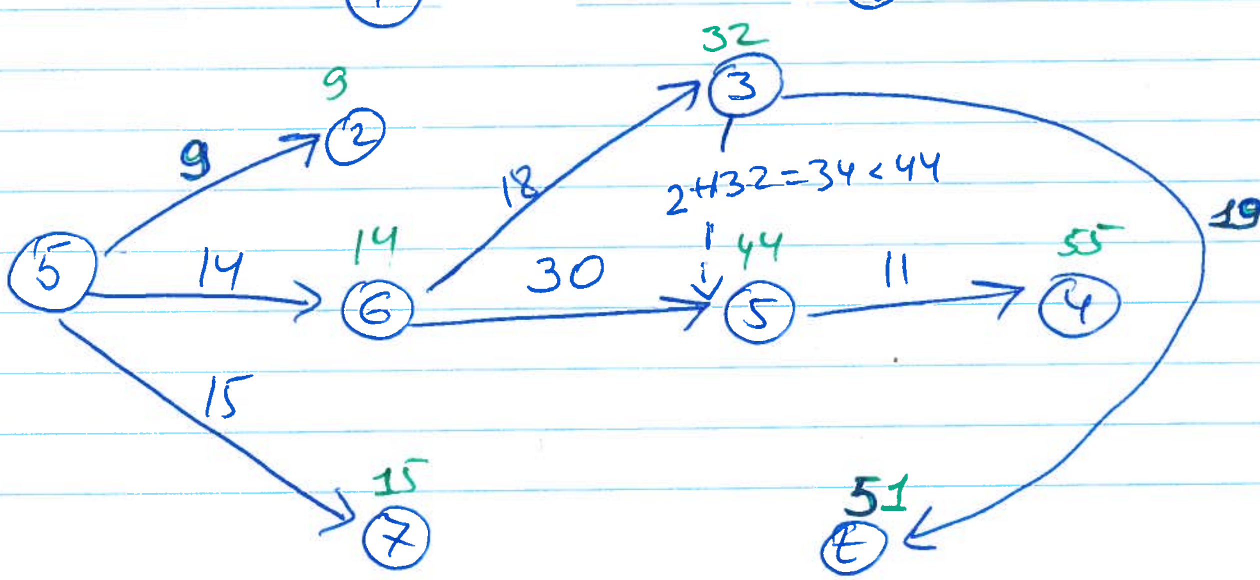
If no such edge edge exists, stop! (you found SPT).

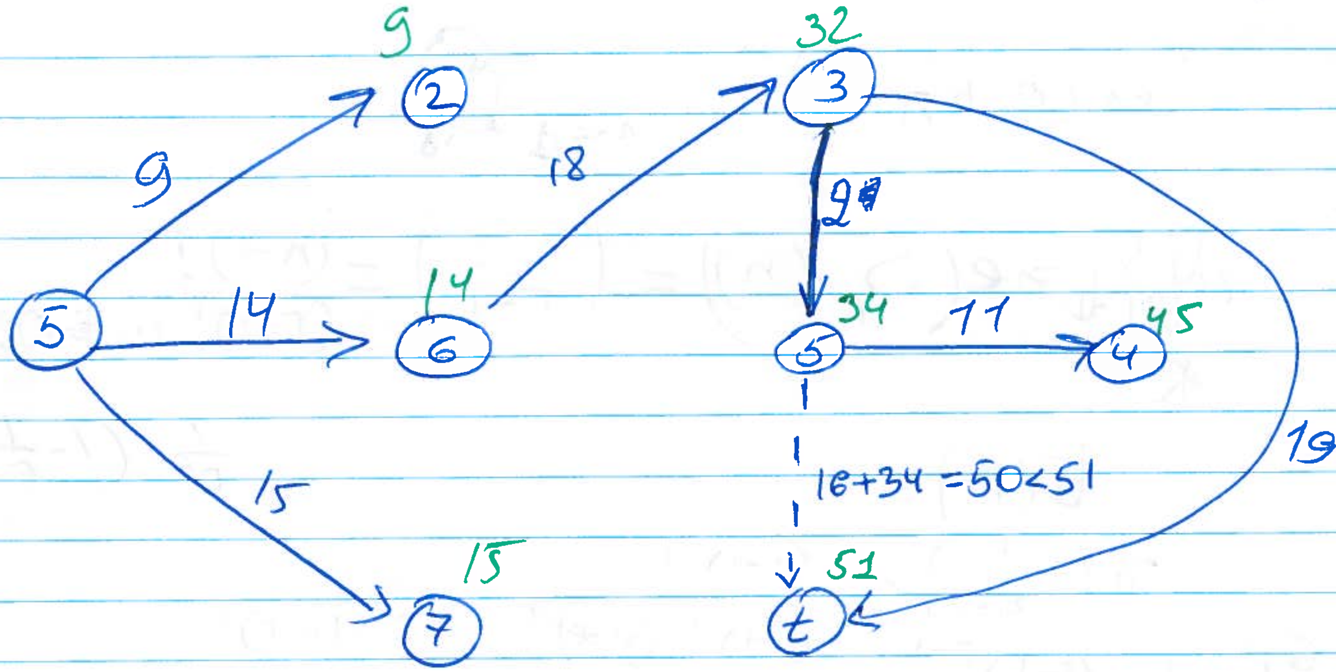


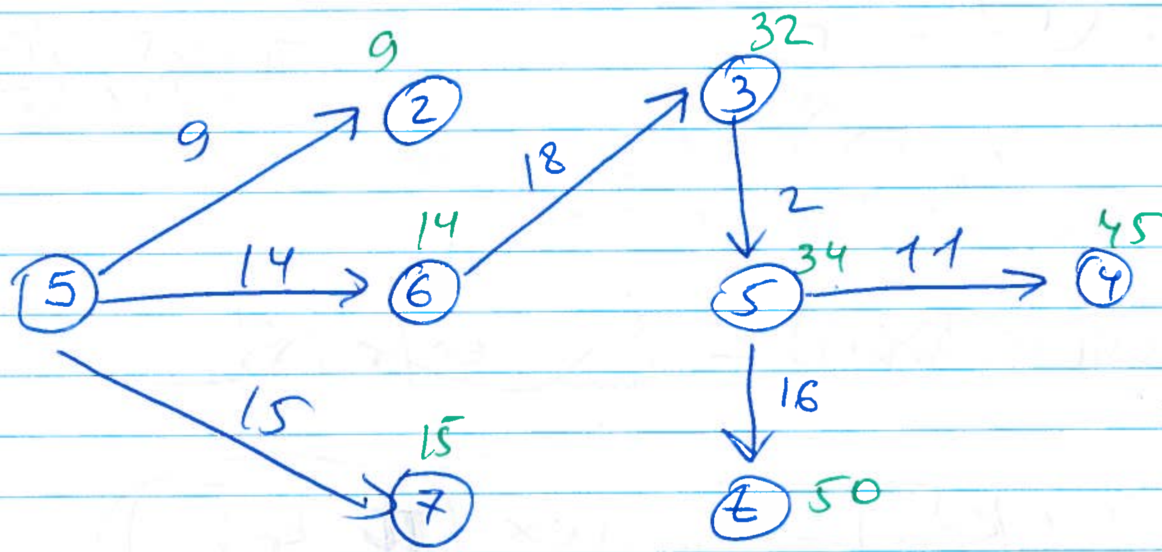
Start with some spanning tree, say this one.
You can use BFS for that.






The algorithm stops here,

since $\nexists\, e \in G \setminus T$ s.t.

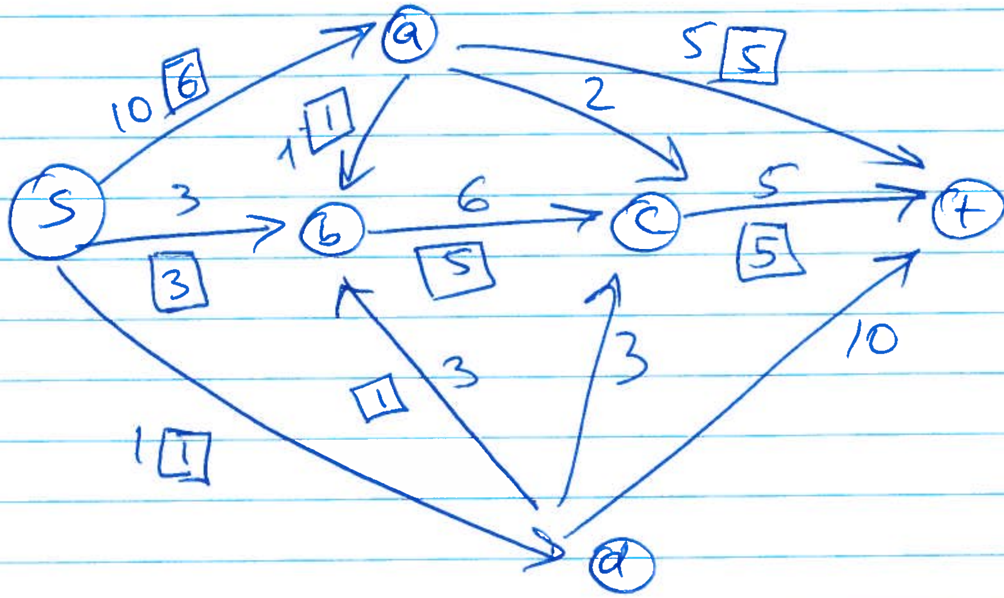
if $e = (v, w)$, then

$d(v) + c(v,w) < d(w)$, that is

$\forall e \in G \setminus T;\ d(v) + c(v,w) \geq d(w)$ !

# Exercise 7.3






The value of this flow is 10, since

$\delta^-(t) = 5+5 = 10$ (the flow that goes into $t$).

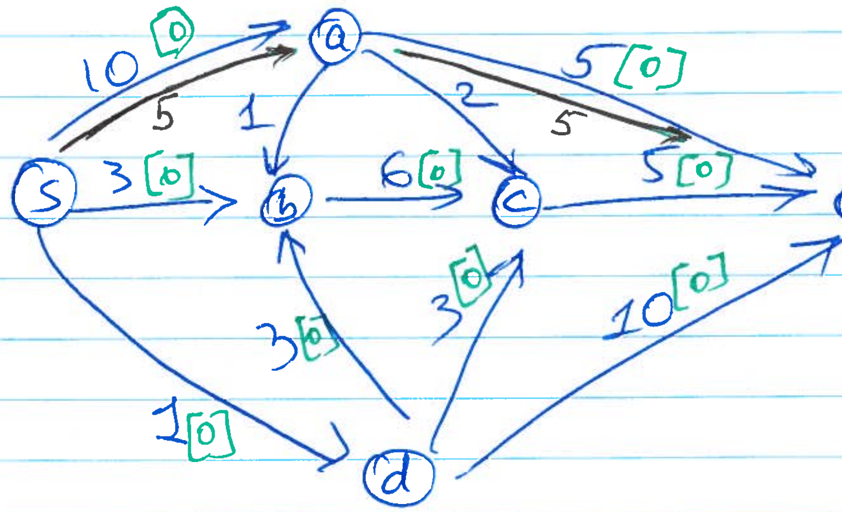
It is not maximal, since one can find a better one. You can see that $d$ doesn't send anything to $t$, but he could, if the flow from $s$ to $a$ is bigger than 6, say

$f(s,a) = 7$,
$f(a,c) = 2$
$f(s,b) = f(b,c) = 3$
$f(c,t) = 5$
$f(a,t) = 5$
$f(s,d) = 1$, $f(d,t) = 1$.
otherwise $f(e) = 0$.

In this case the vaule of the flow will be 11.

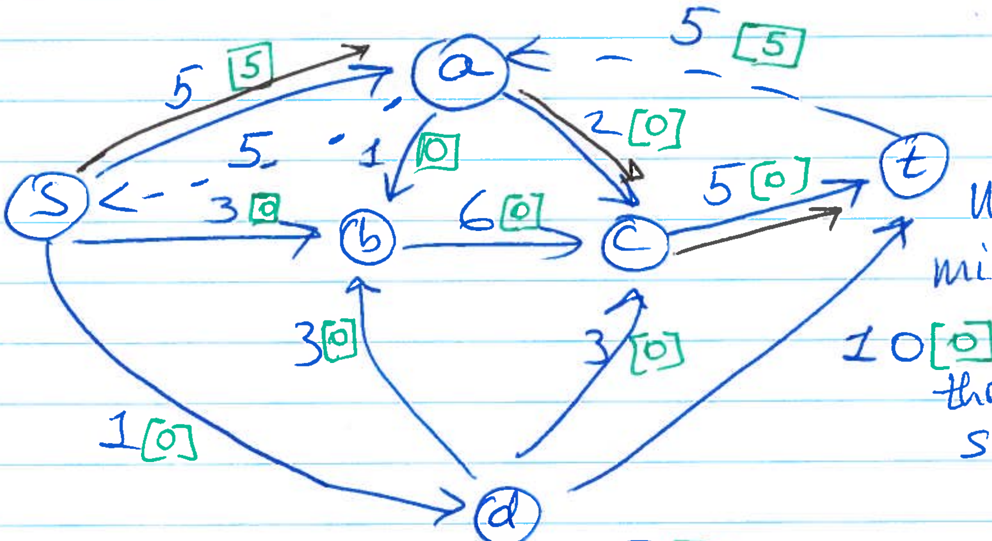
Actually, this is maximal. You can use Ford–Fulkerson algorithm to see that

Start with 0 flow.



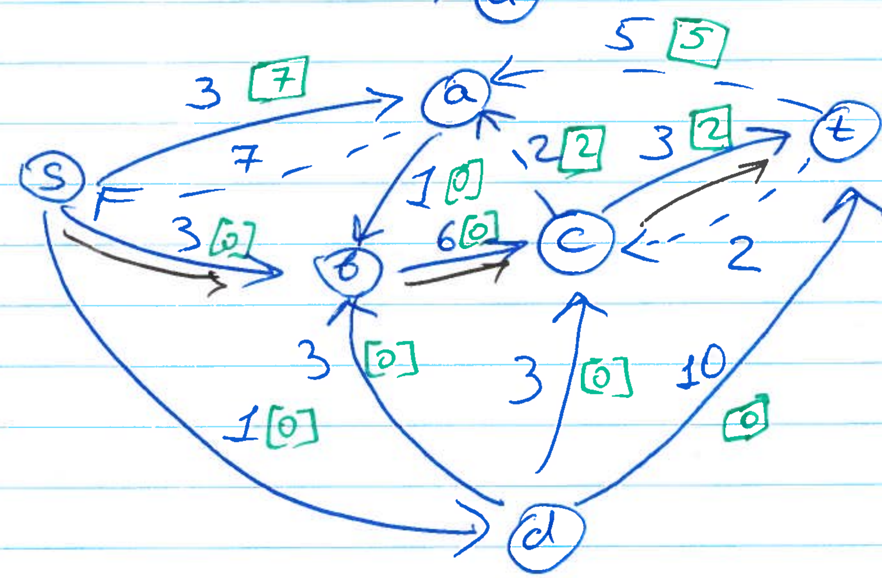
We can push $\min(10,5)=5$ units of flow through $s \to a \to t$ path

$G_{f_1}$

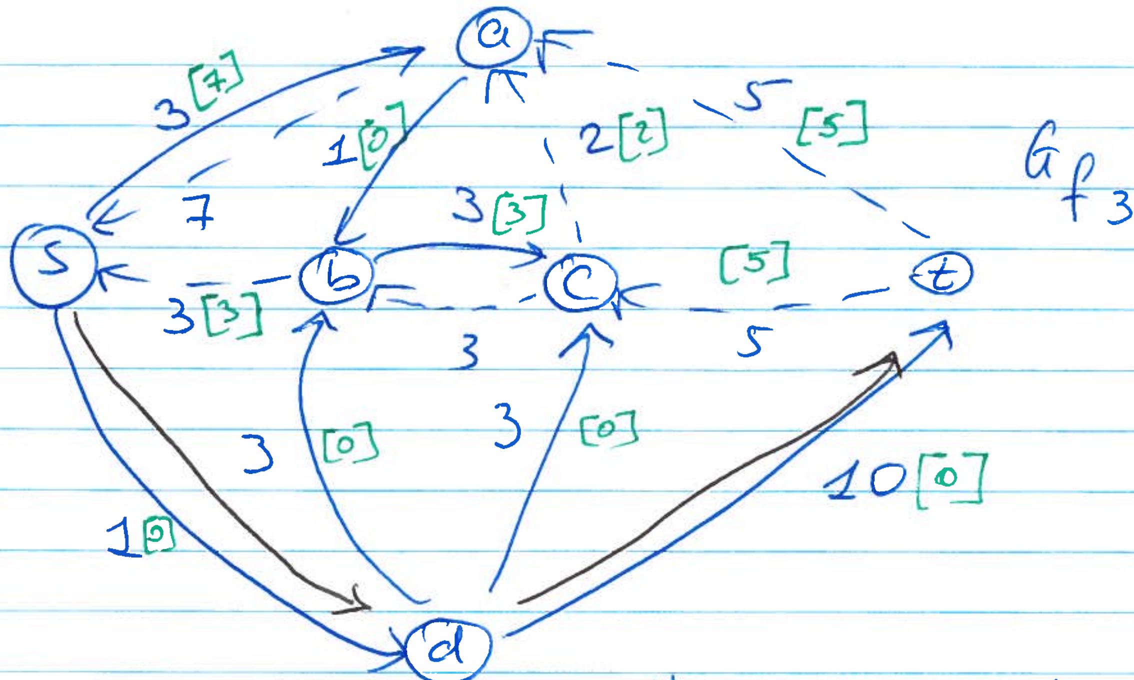


We can push $\min(5,2,5)=2$ units of flow through path $s \to a \to c \to t$

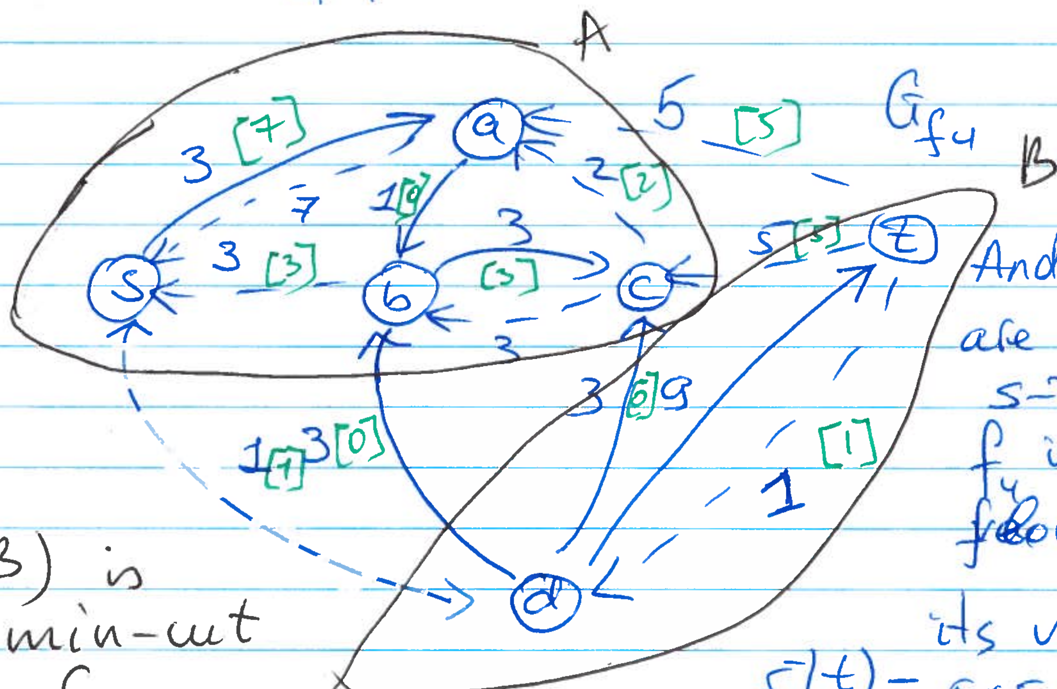
$G_{f_2}$



We can push $\min(3,6,3)=3$ units of low through the path $s \to b \to c \to t$




We can push trough the path $s \to d \to t$ one more unit of flow.




$(A, B)$ is the min-cut in $G$.

And since there are no more $s$-$t$ paths $f_4$ is the max flow and its value is $\delta(t) = 5+5+1 = 11$.

To find the min-cut look which vertices are reachable from $s$? They are $\{a, b, c\}$, hence the min-cut is $A = \{s, a, b, c\}$, $B = \{d, t\}$

And indeed

$c(A, B) = \underline{11}$. (you can check).