Solutions to Assignment2

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1 Exercise 3.9

(a) We want to solve the following system.

$$\begin{aligned} \text{maximize } & z = 3x_1 + x_2 \\ \text{subject to } & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We introduce new slack variable x_0 and get a new system.

$$\begin{aligned} & x_1 - x_2 - x_0 \leq -1 \\ & -x_1 - x_2 - x_0 \leq -3 \\ & 2x_1 + x_2 - x_0 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Our new optimization problem is the following.

$$\begin{aligned} \text{maximize } & z' = -x_0 \\ \text{subject to } & x_1 - x_2 - x_0 = -1 \\ & -x_1 - x_2 - x_0 = -3 \\ & 2x_1 + x_2 - x_0 = 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

At this point we know that $x_1 = x_2 = 0$ and $x_0 = 3$ is feasible solution. We also get new slack variables.

$$
\begin{aligned}
x_3 &= -x_1 + x_2 + x_0 - 1 \\
x_4 &= x_1 + x_2 + x_0 - 3 \\
x_5 &= -2x_1 - x_2 + x_0 + 4 \\
z' &= -x_0
\end{aligned}
$$

We pivot on x_0 and get

$$
\begin{aligned}
x_3 &= -x_1 + x_2 + (x_4 - x_1 - x_2 + 3) - 1 \\
x_0 &= x_4 - x_1 - x_2 + 3 \\
x_5 &= -2x_1 - x_2 + (x_4 - x_1 - x_2 + 3) + 4 \\
z' &= -x_4 + x_1 + x_2 - 3
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= -2x_1 + x_4 + 2 \\
x_0 &= x_4 - x_1 - x_2 + 3 \\
x_5 &= -3x_1 - 2x_2 + x_4 + 7 \\
z' &= -x_4 - x_1 - x_2 + 3
\end{aligned}
$$

A feasible solution is
$x_1 = x_2 = x_4 = 0, x_0 = 3, x_3 = 2, x_5 = 7.$
Now we pivot on x_1 and get the following.

$$
\begin{aligned}
x_1 &= -\frac{x_3}{2} + \frac{x_4}{2} + 1 \\
x_0 &= x_4 - (-\frac{x_3}{2} + \frac{x_4}{2} + 1) - x_2 + 3 \\
x_5 &= -3(-\frac{x_3}{2} + \frac{x_4}{2} + 1) - 2x_2 + x_4 + 7 \\
z' &= -x_4 + (-\frac{x_3}{2} + \frac{x_4}{2} + 1) + x_2 - 3
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= -\frac{x_3}{2} + \frac{x_4}{2} + 1 \\
x_0 &= \frac{x_4}{2} + \frac{x_3}{2} + 2 - x_2 \\
x_5 &= \frac{3x_3}{2} - \frac{x_4}{2} - 2x_2 + 4 \\
z' &= -\frac{x_4}{2} - \frac{x_3}{2} + x_2 - 2
\end{aligned}
$$

Now we pivot on x_2 and get the next system.

$$
\begin{aligned}
x_1 &= -\frac{x_3}{2} + \frac{x_4}{2} + 1 \\
x_0 &= \frac{x_4}{2} + \frac{x_3}{2} - (\frac{3x_3}{4} - \frac{x_4}{4} - \frac{x_5}{2} + 2) + 2 \\
x_2 &= \frac{3x_3}{4} - \frac{x_4}{4} - \frac{x_5}{2} + 2 \\
z' &= -\frac{x_4}{2} - \frac{x_3}{2} + (\frac{3x_3}{4} - \frac{x_4}{4} - \frac{x_5}{2} + 2) - 2
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= -\frac{x_3}{2} + \frac{x_4}{2} + 1 \\
x_0 &= -\frac{x_3}{4} + \frac{3x_4}{4} + \frac{x_5}{2} \\
x_2 &= \frac{3x_3}{4} - \frac{x_4}{4} - \frac{x_5}{2} + 2 \\
z' &= \frac{x_3}{4} - \frac{3x_4}{4} - \frac{x_5}{2}
\end{aligned}
$$

And at this point we can see that $x_1 = 1, x_2 = 2, x_0 = x_3 = x_4 = x_5 = 0$ is a feasible solution. Now we can pivot on x_3 and remove x_0.

$$
\begin{aligned}
x_1 &= -\frac{1}{2}(3x_4 + 2x_5) + \frac{x_4}{2} + 1 \\
x_3 &= 3x_4 + 2x_5 \\
x_2 &= \frac{3}{4}(3x_4 + 2x_5) - \frac{x_4}{4} - \frac{x_5}{2} + 2 \\
z &= 3x_1 + x_2
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= -x_4 - x_5 + 1 \\
x_3 &= 3x_4 + 2x_5 \\
x_2 &= 2x_4 + x_5 + 2 \\
z &= 3x_1 + x_2
\end{aligned}
$$

So we get the optimal solution to be $z = 5$ when $x_1 = 1, x_2 = 2$.

(b)

$$\begin{aligned} \text{maximize } & z = 3x_1 + x_2 \\ \text{subject to } & x_1 - x_2 \le -1 \\ & -x_1 - x_2 \le -3 \\ & 2x_1 + x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We introduce new variable x_0.

$$\begin{aligned} & x_1 - x_2 - x_0 \le -1 \\ & -x_1 - x_2 - x_0 \le -3 \\ & 2x_1 + x_2 - x_0 \le 2 \end{aligned}$$

A feasible solution is $x_1 = 0, x_2 = 0, x_0 = 3$. Our new optimization problem is the following.

$$\begin{aligned} \text{maximize } & z' = -x_0 \\ \text{subject to } & x_1 - x_2 - x_0 \le -1 \\ & -x_1 - x_2 - x_0 \le -3 \\ & 2x_1 + x_2 - x_0 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We introduce new slack variables.

$$\begin{aligned} x_3 &= x_0 - x_1 + x_2 - 1 \\ x_4 &= x_1 + x_2 + x_0 - 3 \\ x_5 &= -2x_1 - x_2 + x_0 + 2 \\ z' &= -x_0 \end{aligned}$$

We pivot on x_0 and get

$$\begin{aligned} x_3 &= (x_4 - x_1 - x_2 + 3) - x_1 + x_2 - 1 \\ x_0 &= x_4 - x_1 - x_2 + 3 \\ x_5 &= -2x_1 - x_2 + (x_4 - x_1 - x_2 + 3) + 2 \\ z' &= -(x_4 - x_1 - x_2 + 3) \end{aligned}$$

$$
\begin{aligned}
x_3 &= x_4 - 2x_1 + 2 \\
x_0 &= x_4 - x_1 - x_2 + 3 \\
x_5 &= -3x_1 - 2x_2 + x_4 + 5 \\
z' &= -x_4 + x_1 + x_2 - 3
\end{aligned}
$$

Now we pivot on x_1.

$$
\begin{aligned}
x_1 &= \frac{x_4}{2} - \frac{x_3}{2} + 1 \\
x_0 &= x_4 - (\frac{x_4}{2} - \frac{x_3}{2} + 1) - x_2 + 3 \\
x_5 &= -3(\frac{x_4}{2} - \frac{x_3}{2} + 1) - 2x_2 + x_4 + 5 \\
z' &= -x_4 + (\frac{x_4}{2} - \frac{x_3}{2} + 1) + x_2 - 3
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{x_4}{2} - \frac{x_3}{2} + 1 \\
x_0 &= \frac{x_4}{2} + \frac{x_3}{2} - x_2 + 2 \\
x_5 &= -\frac{x_4}{2} + \frac{3x_3}{2} - 2x_2 + 2 \\
z' &= -\frac{3x_4}{2} - \frac{x_3}{2} + x_2 - 2
\end{aligned}
$$

Now we pivot on x_2.

$$
\begin{aligned}
x_1 &= \frac{x_4}{2} - \frac{x_3}{2} + 1 \\
x_0 &= \frac{x_4}{2} + \frac{x_3}{2} - (-\frac{x_4}{4} + \frac{3x_3}{4} - \frac{x_5}{2} + 1) + 2 \\
x_2 &= -\frac{x_4}{4} + \frac{3x_3}{4} - \frac{x_5}{2} + 1 \\
z' &= -\frac{3x_4}{2} - \frac{x_3}{2} + (-\frac{x_4}{4} + \frac{3x_3}{4} - \frac{x_5}{2} + 1) - 2
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{x_4}{2} - \frac{x_3}{2} + 1 \\
x_0 &= \frac{3x_4}{4} - \frac{x_3}{4} + \frac{x_5}{2} + 1 \\
x_2 &= -\frac{x_4}{4} + \frac{3x_3}{4} - \frac{x_5}{2} + 1 \\
z' &= -\frac{7x_4}{4} + \frac{x_3}{4} - \frac{x_5}{2} - 1
\end{aligned}
$$

Now we pivot on x_3.

$$
\begin{aligned}
x_3 &= x_4 - 2x_1 + 2 \\
x_0 &= \frac{3x_4}{4} - \frac{1}{4}(x_4 - 2x_1 + 2) + \frac{x_5}{2} + 1 \\
x_2 &= -\frac{x_4}{4} + \frac{3}{4}(x_4 - 2x_1 + 2) - \frac{x_5}{2} + 1 \\
z' &= -\frac{7x_4}{4} + \frac{1}{4}(x_4 - 2x_1 + 2) - \frac{x_5}{2} - 1
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_4 - 2x_1 + 2 \\
x_0 &= \frac{x_4}{2} + \frac{x_1}{2} + \frac{x_5}{2} + \frac{1}{2} \\
x_2 &= \frac{x_4}{2} - \frac{3x_1}{2} - \frac{x_5}{2} + \frac{5}{2} \\
z' &= -\frac{3x_4}{2} - \frac{x_1}{2} - \frac{x_5}{2} - \frac{1}{2}
\end{aligned}
$$

We get that $x_0 \neq 0$, therefore the system is infeasible.