Point-based approximations for fast POMDP solving

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Abstract

The Partially Observable Markov Decision Process has long been recognized as a rich framework for real-world planning and control problems, especially in robotics. However exact solutions in this framework are typically computationally intractable for all but the smallest problems. Furthermore, until recently, the efficient approximations that were available offered few theoretical guarantees regarding their performance. This paper describes a new class of approximate POMDP algorithms, called *Point-Based Value Iteration* (PBVI), which feature both good empirical performance (in simulation and robotic tasks), as well as solid theoretical guarantees.

1. Introduction

The concept of planning has a long tradition in the AI literature (Fikes & Nilsson, 1971; Chapman, 1987; McAllester & Roseblitt, 1991; Penberthy & Weld, 1992; Blum & Furst, 1997). Classical planning is generally concerned with agents which operate in environments that are fully observable, deterministic, finite, static, and discrete. While these techniques are able to solve increasingly large state-space problems, the basic assumptions of classical planning—full observability, static environment, deterministic actions—make these unsuitable for most robotic applications.

Planning under uncertainty aims to improve robustness by explicitly reasoning about the type of uncertainty that can arise. The Partially Observable Markov Decision Process (POMDP) (Ästrom, 1965; Sondik, 1971; Monahan, 1982; White, 1991; Lovejoy, 1991b; Kaelbling, Littman, & Cassandra, 1998; Boutilier, Dean, & Hanks, 1999) has emerged as possibly the most general representation for (single-agent) planning under uncertainty. The POMDP supersedes other frameworks in terms of representational power simply because it combines the most essential features for planning under uncertainty.

First, POMDPs handle uncertainty in both *action effects* and *state observability*, whereas many other frameworks handle neither of these, and some handle only stochastic action effects. To handle partial state observability, plans are expressed over *information states*, instead of world states, since the latter ones are not directly observable. The space of information states is the space of all beliefs a system might have regarding the world state. Information states are easily calculated from the measurements of noisy and imperfect sensors. In POMDPs, information states are typically represented by probability distributions over world states.

Second, many POMDP algorithms form plans by optimizing a *value function*. This is a powerful approach to plan optimization, since it allows one to numerically trade-off between alternative ways to satisfy a goal, compare actions with different costs/rewards, as well as plan for multiple interacting goals. While value function optimization is used in other planning approaches—for example Markov Decision Processes (MDPs) (Bellman, 1957)—POMDPs are unique in expressing the value function over information states, rather than world states.

Finally, whereas classical and conditional planners produce a sequence of actions, POMDPs produce a full *policy* for action selection, which prescribes the choice of action for any possible information state. By producing a universal plan, POMDPs alleviate the need for re-planning, and allow fast execution. Naturally, the main drawback of optimizing a universal plan is the computational complexity of doing so. This is precisely what we seek to alleviate with the work described in this paper

POMDPs offer a rich framework to optimize a control strategy. However, computational considerations limit the usefulness of POMDPs in large domains. Known algorithms for exact planning in POMDPs operate by optimizing the value function over all possible information states (also known as *beliefs*). These algorithms can run into the well-known curse of dimensionality, where the dimensionality of planning problem is directly related to the number of states. But they can also suffer from the lesser known curse of history, where the number of belief-contingent plans increases exponentially with the planning horizon. In fact, exact POMDP planning is known to be PSPACE-complete, whereas propositional planning is only NP-complete (Littman, 1996). As a result, many POMDP domains with only a few states, actions and sensor observations are computationally intractable.

This paper describes a new class of POMDP approximations known as *Point-Based Value Iteration* (PBVI) algorithms which specifically target the curse of history, in an attempt to overcome complexity issues in POMDP planning. Whereas exact POMDP algorithms attempt to optimize a value function for *all* information states, PBVI-type algorithms select a small set of representative belief points, and iteratively apply value updates to those points only. The point-based update is significantly more efficient than an exact update (quadratic vs. exponential). And while the number of points can grow quickly, our empirical results show good performance with very few belief points (sometimes less than the number of states).

From a high-level stand-point, PBVI-class solutions share many similarities with earlier grid-based methods (Brafman, 1997; Zhou & Hansen, 2001). However there are a few key differences. When performing value backups, PBVI algorithms update both the *value* and *value gradient*; this choice provides better generalization to unexplored beliefs. Another important aspect is the strategy employed to select belief points. Rather than picking points randomly, or according to a fixed grid, PBVI algorithms generally use exploratory stochastic trajectories to focus on reachable regions of the belief, thus reducing the number of belief points necessary to find a good solution compared to earlier approaches.

This paper discusses many instances of PBVI-class algorithms. These range from a naive version that combines point-based value updates with random belief point selection, to a sophisticated algorithm that combines the standard point-based value update with an estimate of the error bound between the approximate and exact solutions to select belief points.

The PBVI class of algorithms has a number of important properties, which are discussed at greater length in the paper:

- **Theoretical guarantees**. There is a known bound on the error of our value function approximation, with respect to the exact solution. To the best of our knowledge, no previous approximations exist for which similar bounds have been proven.
- **Scalability**. We are able to handle problems on the order of 10^3 states, which is an order of magnitude larger than problems typically used to test scalability of POMDP algorithms. The empirical performance is evaluated extensively in realistic robot tasks, including a search-for-missing-person scenario.
- **Wide applicability**. The approach makes few assumptions about the nature or structure of the domain. The PBVI framework does assume known discrete state/ action/observation spaces and a known model (i. e. state-to-state transitions, observation probabilities, costs/rewards), but no additional specific structure (e. g. constrained policy class).
- **Anytime performance**. An anytime solution can be achieved by gradually alternating phases of belief point selection and phases of point-based value updates. This allows for an effective trade-off between planning time and solution quality.

This paper expands on these ideas. Section 2 begins by exposing the basic concepts in POMDP solving, including representation, inference, and exact planning. Section 3 presents the general anytime PBVI algorithm and its theoretical properties, as well as discussing a range of strategies to select good belief points. Section 4 presents an empirical evaluation of PBVI-type algorithms using standard simulation problems. Section 5 pursues the empirical evaluation by tackling complex robot domains. Finally, Section 6 surveys a number of existing POMDP approaches that are closely related to PBVI.

2. Review of POMDPs

Partially Observable Markov Decision Processes provide a general planning and decision-making framework for acting optimally in partially observable domains. They are well-suited to a great number of real-world problems where decision-making is required despite prevalent uncertainty. They generally assume a complete and correct world model, with stochastic state transitions, imperfect state tracking, and a reward structure. Given this information, the goal is to find an action strategy which maximizes expected reward gains. This section first establishes the basic terminology and essential concepts pertaining to POMDPs, and then reviews optimal techniques for POMDP planning.

2.1 Basic POMDP Terminology

Formally, a POMDP is defined by seven distinct quantities, denoted $\{S, A, Z, b_0, T, O, R\}$. The first three of these are:

- *States*. The state of the world is denoted s, with the finite set of all states denoted by $S = \{s_0, s_1, \ldots\}$. The state at time t is denoted s_t, where t is a discrete time index.

The state is not directly observable in POMDPs, where an agent can only compute a belief over the state space S.

- *Observations.* To infer a belief regarding the world's state s, the agent can take sensor measurements. The set of all measurements, or observations, is denoted $Z = \{z_0, z_1, \ldots\}$. The observation at time t is denoted z_t. Observation z_t is usually an incomplete projection of the world state s_t, contaminated by sensor noise.

- *Actions.* To act in the world, the agent is given a finite set of actions, denoted $A = \{a_0, a_1, \ldots\}$. Actions stochastically affect the state of the world. Choosing the right action as a function of history is the core problem in POMDPs.

Throughout this paper, we assume that states, actions and observations are discrete and finite. It is commonly assumed that actions and observations are alternated over time. This assumption does not restrict the general expressiveness of the approach, but is adopted throughout for notational convenience.

To fully define a POMDP, we have to specify the probabilistic laws that describe state transitions and observations. These laws are given by the following distributions:

- The *initial state probability distribution*,

$$b_0(s) \quad := \quad Pr(s_0 = s), \tag{1}$$

  is the probability that the domain is in state s at time $t = 0$. This distribution is defined over all states in S.

- The *state transition probability distribution*,

$$T(s, a, s') \quad := \quad Pr(s_t = s' \mid s_{t-1} = s, a_{t-1} = a) \ \ \forall t, \tag{2}$$

  is the probability of transitioning to state s', given that the agent is in state s and selects action a, for any (s, a, s'). Since T is a conditional probability distribution, we have $\sum_{s' \in S} T(s, a, s') = 1, \forall (s, a)$. As our notation suggests, T is time-invariant, that is, the stochastic matrix T does not change over time. For time-variant state transition probabilities, the state s must include a time-related variable.

- The *observation probability distribution*,

$$O(s, a, z) \quad := \quad Pr(z_t = z \mid s_{t-1} = s, a_{t-1} = a) \ \ \forall t, \tag{3}$$

  is the probability that the agent will perceive observation z upon executing action a in state s. This conditional probability is defined for all (s, a, z) triplets, for which $\sum_{z \in Z} O(s, a, z) = 1, \forall (s, a)$.

Finally, the objective of POMDP planning is to optimize action selection, so the agent is given a reward function describing its performance:

- The *reward function.* $R(s, a) : S \times A \longrightarrow \Re$, assigns a numerical value quantifying the utility of performing action a when in state s. We assume the reward is bounded,

$R_{min} < R < R_{max}$. The goal of the agent is to maximize the sum of its reward over time. Mathematically, this is commonly defined by a sum of the form:

$$E[\sum_{t=t_0}^{T} \gamma^{t-t_0} r_t], \tag{4}$$

where r_t is the reward at time t, $E[\ ]$ is the mathematical expectation, and γ where $0 \leq \gamma < 1$ is a *discount factor*, which ensures that the sum in Equation 4 is finite.

These items together, the states S, actions A, observations Z, reward R, and the three probability distributions T, O, and b_0, define the probabilistic world model that underlies each POMDP.

2.2 Belief Computation

POMDPs are instances of Markov processes, which implies that the current world state, s_t, is sufficient to predict the future, independent of the past $\{s_0, s_1, ..., s_{t-1}\}$. The key characteristic that sets POMDPs apart from many other probabilistic models (like MDPs) is the fact that the state s_t is not directly observable. Instead, the agent can only perceive observations $\{z_1, \ldots, z_t\}$, which convey incomplete information about the world's state.

Given that the state is not directly observable, the agent can instead maintain a complete trace of all observations and all actions it ever executed, and use this to select its actions. The action/observation trace is known as a *history*. We formally define

$$h_t \quad := \quad \{a_0, z_1, \ldots, z_{t-1}, a_{t-1}, z_t\} \tag{5}$$

to be the history at time t.

This history trace can get very long as time goes on. A well-known fact is that this history does not need to be represented explicitly, but can instead be summarized via a *belief distribution* (Ästrom, 1965), which is the following posterior probability distribution:

$$b_t(s) \quad := \quad Pr(s_t = s \mid z_t, a_{t-1}, z_{t-1}, \ldots, a_0). \tag{6}$$

Because the belief distribution b_t is a sufficient statistic for the history, it suffices to condition the selection of actions on b_t, instead of on the ever-growing sequence of past observations and actions. Furthermore, the belief b_t at time t is calculated *recursively*, using only the belief one time step earlier, b_{t-1}, along with the most recent action a_{t-1} and observation z_t:

$$\begin{aligned} b_t(s) &= \tau(b_{t-1}, a_{t-1}, z_t) \\ &= \frac{\sum_{s'} O(s', a_{t-1}, z_t)\ T(s, a_{t-1}, s')\ b_{t-1}(s')}{Pr(z_t | b_{t-1}, a_{t-1})} \end{aligned} \tag{7}$$

where the denominator is a normalizing constant.

This equation is equivalent to the decades-old Bayes filter (Jazwinski, 1970), and is commonly applied in the context of hidden Markov models (Rabiner, 1989), where it is known as the forward algorithm. Its continuous generalization forms the basis of Kalman filters (Kalman, 1960).

It is interesting to consider the nature of belief distributions. For finite state spaces, which will be assumed throughout this paper, the belief is a continuous quantity. It is defined over a simplex describing the space of all distributions over the state space S. For very large state spaces, calculating the belief update (Eqn 7) can be computationally challenging. Recent research has led to efficient techniques for belief state computation that exploit structure of the domain (Dean & Kanazawa, 1988; Boyen & Koller, 1998; Poupart & Boutilier, 2000; Thrun, Fox, Burgard, & Dellaert, 2000). However, by far the most complex aspect of POMDP planning is the generation of a policy for action selection, which is described next. For example in robotics, calculating beliefs over state spaces with 10^6 states is easily done in real-time (Burgard et al., 1999). In contrast, calculating optimal action selection policies exactly appears to be infeasible for environments with more than a few dozen states (Kaelbling et al., 1998), not directly because of the size of the state space, but because of the complexity of the optimal policies. Hence we assume throughout this paper that the belief can be computed accurately, and instead focus on the problem of finding good approximations to the optimal policy.

2.3 Optimal Policy Computation

The central objective of the POMDP perspective is to compute a *policy* for selecting actions. A policy is of the form:

$$\pi(b) \quad \longrightarrow \quad a, \tag{8}$$

where b is a belief distribution and a is the action chosen by the policy π.

Of particular interest is the notion of *optimal policy*, which is a policy that maximizes the expected future discounted cumulative reward:

$$\pi^*(b_t) \quad = \quad \underset{\pi}{\operatorname{argmax}}\, E_\pi \left[\sum_{t=t_0}^{T} \gamma^{t-t_0} r_t \mid b_t \right]. \tag{9}$$

There are two distinct but interdependent reasons why computing an optimal policy is challenging.

The more widely-known reason is the so-called curse of dimensionality: in a problem with n physical states, π is defined over all belief states in an $(n-1)$-dimensional continuous space.

The less-well-known reason is the curse of history: POMDP solving is in many ways like a search through the space of possible POMDP histories. It starts by searching over short histories (through which it can select the best short policies), and gradually considers increasingly long histories. Unfortunately the number of distinct possible action-observation histories grows exponentially with the planning horizon.

The two curses—dimensionality and history—are related: the higher the dimension of a belief space, the more room it has for distinct histories. But, they often act independently: planning complexity can grow exponentially with horizon even in problems with only a few states, and problems with a large number of physical states may still only have a small number of relevant histories.

To see how these curses work, we consider the most straightforward approach to finding optimal policies, as described by Sondik (1971). The overall idea is to apply multiple iterations of dynamic programming, to compute increasingly more accurate values for each belief state b. Let V be a value function that maps belief states to values in $\Re$. Beginning with the initial value function:

$$V_0(b) \;=\; \max_a \sum_{s\in S} R(s,a)b(s), \tag{10}$$

then the t-th value function is constructed from the $(t-1)$-th by the following recursive equation:

$$V_t(b) \;=\; \max_a \left[\sum_{s\in S} R(s,a)b(s) + \gamma \sum_{z\in Z} Pr(z \mid a,b) V_{t-1}(\tau(b,a,z)) \right], \tag{11}$$

where $\tau(b,a,z)$ is the belief updating function defined in Equation 7. This value function update maximizes the expected sum of all (possibly discounted) future pay-offs the agent receives in the next t time steps, for any belief state b. Thus, it produces a policy that is optimal under the planning horizon t. The optimal policy can also be directly extracted from the previous-step value function:

$$\pi_t^*(b) \;=\; \operatorname*{argmax}_a \left[\sum_{s\in S} R(s,a)b(s) + \gamma \sum_{z\in Z} Pr(z \mid a,b) V_{t-1}(\tau(b,a,z)) \right]. \tag{12}$$

Sondik (1971) showed that the value function at any finite horizon t can be expressed by a set of vectors: $\Gamma_t = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$. Each α-vector represents an $|S|$-dimensional hyper-plane, and defines the value function over a bounded region of the belief:

$$V_t(b) \;=\; \max_{\alpha\in\Gamma_t} \sum_{s\in S} \alpha(s)b(s). \tag{13}$$

In addition, each α-vector is associated with an action, defining the best immediate policy assuming optimal behavior for the following $(t-1)$ steps (as defined respectively by the sets $\{V_{t-1}, ..., V_0\}$).

The t-horizon solution set, Γ_t, can be computed as follows. First, we rewrite Equation 11 as:

$$V_t(b) \;=\; \max_{a\in A} \left[\sum_{s\in S} R(s,a)b(s) + \gamma \sum_{z\in Z} \max_{\alpha\in\Gamma_{t-1}} \sum_{s\in S} \sum_{s'\in S} T(s,a,s')O(s',a,z)\alpha(s')b(s) \right]. \tag{14}$$

The value $V_t(b)$ cannot be computed directly for each belief $b \in B$ (since there are infinitely many beliefs), but the corresponding set Γ_t can be generated through a sequence of operations on the set Γ_{t-1}.

The first operation is to generate intermediate sets $\Gamma_t^{a,*}$ and $\Gamma_t^{a,z}, \forall a \in A, \forall z \in Z$ *(Step 1)*:

$$\begin{aligned} \Gamma_t^{a,*} &\leftarrow \alpha^{a,*}(s) = R(s,a) \\ \Gamma_t^{a,z} &\leftarrow \alpha_i^{a,z}(s) = \gamma \sum_{s'\in S} T(s,a,s')O(s',a,z)\alpha_i(s'), \forall \alpha_i \in \Gamma_{t-1} \end{aligned} \tag{15}$$

where each $\alpha^{a,*}$ and $\alpha_i^{a,z}$ is once again an $|S|$-dimensional hyper-plane.

Next we create Γ_t^a ($\forall a \in A$), the cross-sum[1] over observations, which includes one $\alpha^{a,z}$ from each $\Gamma_t^{a,z}$ *(Step 2)*:

$$\Gamma_t^a \quad = \quad \Gamma_t^{a,*} + \Gamma_t^{a,z_1} \oplus \Gamma_t^{a,z_2} \oplus \dots \tag{16}$$

Finally we take the union of Γ_t^a sets *(Step 3)*:

$$\Gamma_t \quad = \quad \cup_{a \in A}\, \Gamma_t^a. \tag{17}$$

This forms the pieces of the backup solution at horizon t. The actual value function V_t is extracted from the set Γ_t as described in Equation 13.

Using this approach, bounded-time POMDP problems with finite state, action, and observation spaces can be solved exactly given a choice of the horizon T. If the environment is such that the agent might not be able to bound the planning horizon in advance, the policy $\pi_t^*(b)$ is an approximation to the optimal one whose quality improves in expectation with the planning horizon t (assuming $0 \leq \gamma < 1$).

As mentioned above, the value function V_t can be extracted directly from the set Γ_t. An important aspect of this algorithm (and of all optimal finite-horizon POMDP solutions) is that the value function is guaranteed to be a piecewise linear, convex, and continuous function of the belief (Sondik, 1971). The piecewise-linearity and continuous properties are a direct result of the fact that V_t is composed of finitely many linear α-vectors. The convexity property is a result of the maximization operator (Eqn 13). It is worth pointing out that the intermediate sets $\Gamma_t^{a,z}$, $\Gamma_t^{a,*}$ and Γ_t^a are also composed entirely of segments that are linear in the belief. This property holds for the intermediate representations because they incorporate the expectation over observation probabilities (Eqn 15).

In the worst case, the exact value update procedure described could require time doubly exponential in the planning horizon T (Kaelbling et al., 1998). To better understand the complexity of the exact update, let $|S|$ be the number of states, $|A|$ the number of actions, $|Z|$ the number of observations, and $|\Gamma_{t-1}|$ the number of α-vectors in the previous solution set. Then Step 1 creates $|A|\,|Z|\,|\Gamma_{t-1}|$ projections and Step 2 generates $|A|\,|\Gamma_{t-1}|^{|Z|}$ cross-sums. So, in the worst case, the new solution requires:

$$|\Gamma_t| = O(|A||\Gamma_{t-1}|^{|Z|}) \tag{18}$$

α-vectors to represent the value function at horizon t; these can be computed in time $O(|S|^2|A|\,|\Gamma_{t-1}|^{|Z|})$.

It is often the case that a vector in Γ_t will be completely dominated by another vector over the entire belief simplex:

$$\alpha_i \cdot b < \alpha_j \cdot b, \quad \forall b. \tag{19}$$

Similarly, a vector may be fully dominated by a set of other vectors (e. g. α_2 in Fig. 1). This vector can then be pruned away without affecting the solution. Finding dominated vectors can be expensive. Checking whether a single vector is dominated requires solving a linear

1. The symbol $\oplus$ denotes the cross-sum operator. A cross-sum operation is defined over two sets, $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$, and produces a third set, $C = \{a_1 + b_1, a_1 + b_2, \dots, a_1 + b_n, a_2 + b_1, a_2 + b_2, \dots, \dots, a_m + b_n\}$.

program with $|S|$ variables and $|\Gamma_t|$ constraints. Nonetheless it can be time-effective to apply pruning after each iteration to prevent an explosion of the solution size. In practice, $|\Gamma_t|$ often appears to grow singly exponentially in t, even given clever mechanisms for pruning unnecessary linear functions. This enormous computational complexity has long been a key impediment toward applying POMDPs to practical problems.

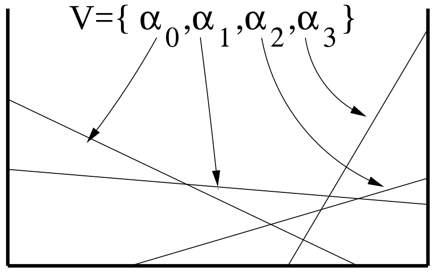


Figure 1: POMDP value function representation

2.4 Point-Based Value Backup

Exact POMDP solving, as outlined above, optimizes the value function over all beliefs. Many approximate POMDP solutions, including the PBVI approach proposed in this paper, gain computational advantage by applying value updates at specific (and few) belief points, rather than over all beliefs (Cheng, 1988; Zhang & Zhang, 2001; Poon, 2001). These approaches differ significantly (and to great consequence) in how they select the belief points, but once a set of points is selected, the procedure for updating their value is standard. We now describe the procedure for updating the value function at a set of known belief points.

As in Section 2.3, the value function update is implemented as a sequence of operations on a set of α-vectors. If we assume that we are only interested in updating the value function at a fixed set of belief points, $B = \{b_0, b_1, ..., b_q\}$, then it follows that the value function will contain at most one α-vector for each belief point. The point-based value function is therefore represented by the corresponding set $\{\alpha_0, \alpha_1, \dots, \alpha_q\}$

Given a solution set Γ_{t-1}, we simply modify the exact backup operator (Eqn 14) such that only one α-vector per belief point is maintained. This is the key idea behind all algorithms presented in this paper, and the reason for the large computational savings associated with this class of algorithms.

To obtain solution set Γ_t from the previous set Γ_{t-1}, we begin once again by generating intermediate sets $\Gamma_t^{a,*}$ and $\Gamma_t^{a,z}, \forall a \in A, \forall z \in Z$ (exactly as in Eqn 15) *(Step 1)*:

$$\begin{aligned} \Gamma_t^{a,*} &\leftarrow \alpha^{a,*}(s) = R(s,a) \qquad (20) \\ \Gamma_t^{a,z} &\leftarrow \alpha_i^{a,z}(s) = \gamma \sum_{s' \in S} T(s,a,s') O(s',a,z) \alpha_i(s'), \forall \alpha_i \in \Gamma_{t-1}. \end{aligned}$$

Next, whereas performing an exact value update requires a cross-sum operation (Eqn 16), by operating over a finite set of points, we can instead use a simple summation. We construct $\Gamma_t^a, \forall a \in A$ *(Step 2)*:

$$\Gamma_t^a \leftarrow \alpha_b^a = \Gamma_t^{a,*} + \sum_{z \in Z} \operatorname*{argmax}_{\alpha \in \Gamma_t^{a,z}} (\sum_{s \in S} \alpha(s) b(s)), \forall b \in B. \qquad (21)$$

Finally, we find the best action for each belief point *(Step 3)*:

$$\alpha_b = \underset{\Gamma_t^a, \forall a \in A}{\operatorname{argmax}}\left(\sum_{s \in S} \Gamma_t^a(s) b(s)\right), \quad \forall b \in B. \tag{22}$$

$$\Gamma_t = \cup_{b \in B}\, \alpha_b \tag{23}$$

While these operations preserve only the best α-vector at each belief point $b \in B$, an estimate of the value function at *any* belief in the simplex (including $b \notin B$) can be extracted from the set Γ_t just as before:

$$V_t(b) = \max_{\alpha \in \Gamma_t} \sum_{s \in S} \alpha(s) b(s). \tag{24}$$

To better understand the complexity of updating the value of a set of points B, let $|S|$ be the number of states, $|A|$ the number of actions, $|Z|$ the number of observations, and $|\Gamma_{t-1}|$ the number of α-vectors in the previous solution set. As with an exact update, Step 1 creates $|A|\,|Z|\,|\Gamma_{t-1}|$ projections (in time $|S|^2\,|A|\,|Z|\,|\Gamma_{t-1}|$). Steps 2 and 3 then reduce this set to at most $|B|$ components (in time $|S|\,|A|\,|\Gamma_{t-1}|\,|Z|\,|B|$). Thus, a full point-based value update takes only polynomial time, and even more crucially, the size of the solution set Γ_t remains constant at every iteration. The point-based value backup algorithm is summarized in Table 1.

```
Γ_t=BACKUP(B, Γ_{t-1})                                                            1
    For each action $a \in A$                                                     2
        For each observation $z \in Z$                                            3
            For each solution vector $\alpha_i \in \Gamma_{t-1}$                  4
                $\alpha_i^{a,z}(s) = \gamma \sum_{s' \in S} T(s,a,s') O(s',a,z) \alpha_i(s'), \forall s \in S$   5
            End                                                                   6
            $\Gamma_t^{a,z} = \cup_i\, \alpha_i^{a,z}$                            7
        End                                                                       8
    End                                                                           9
    $\Gamma_t = \emptyset$                                                        10
    For each belief point $b \in B$                                               11
        $\alpha_b = \operatorname{argmax}_{a \in A} \left[\sum_{s \in S} R(s,a) b(s) + \sum_{z \in Z} \max_{\alpha \in \Gamma_t^{a,z}} \left[\sum_{s \in S} \alpha(s) b(s)\right]\right]$   12
        If($\alpha_b \notin \Gamma_t$)                                            13
            $\Gamma_t = \Gamma_t \cup \alpha_b$                                   14
    End                                                                           15
    Return $\Gamma_t$                                                             16
```

Table 1: Point-based value backup

Note that the algorithm as outlined in Table 1 includes a trivial pruning step (lines 12-13), whereby we refrain from adding to Γ_t any vector already included in it. As a result, it is often the case that $|\Gamma_t| \leq |B|$. This situation arises whenever multiple nearby belief points support the same vector. This pruning step can be computed rapidly (without solving linear programs) and is clearly advantageous in terms of reducing the set Γ_t.

The point-based value backup is found in many POMDP solvers, and in general serves to improve estimates of the value function. It is also an integral part of the PBVI framework.

3. Point-Based Value Iteration

In this section, we describe a new class of fast approximate POMDP algorithms known as *Point-Based Value Iteration* (PBVI). The algorithms in this class combine point-based value updates with various belief-point selection strategies to produce an anytime, approximate solution with bounded error, for any discrete POMDP domain.

We begin by discussing the general PBVI approach, including how value updates and belief-point selection can be interleaved, as well as the theoretical properties of this class of algorithms. We then proceed to discuss the various approaches to belief-point selection, which differentiates the multiple incarnations of PBVI-class algorithms.

3.1 Anytime PBVI Planning

PBVI-class algorithms offer an anytime solution to large-scale discrete POMDP domains. The key to achieving an anytime solution is to interleave two main components: the point-based update described in Table 1, with steps of belief set expansion.

PBVI-type algorithms start with a (small) initial set of belief points to which it applies a first series of backup operations. The set of belief points is then grown, a new series of backup operations are applied to all belief points (old and new), and so on, until a satisfactory solution is obtained. By interleaving value backup iterations with expansions of the belief set, PBVI offers a range of solutions, gradually trading off computation time and solution quality.

The full algorithm is presented in Table 2. The algorithm accepts as input an initial belief point set (B_{Init}), an initial value (Γ_0), the number of desired expansions (N), and the planning horizon (T). A common choice for B_{Init} is simply the initial belief b_0; alternately, a larger set could be used, especially in cases where sample trajectories are available. The initial value, Γ_0, is typically set to be purposefully low (e. g. $\alpha_0(s) = \frac{R_{min}}{1-\gamma}, \forall s \in S$). When we do this, we can prove that the point-based solution will always be a lower-bound on the exact solution (Lovejoy, 1991a).

For problems with a finite horizon, we run T value backups between each expansion of the belief set. In infinite-horizon problems, we select the horizon T so that

$$\gamma^T \left[R_{\max} - R_{\min}\right] < \epsilon, \tag{25}$$

where $R_{\max} = \max_{s,a} R(s,a)$ and $R_{\min} = \min_{s,a} R(s,a)$.

The complete algorithm terminates once a fixed number of expansions (N) have been completed. Alternately, the algorithm could terminate once the value function approximation reaches a given performance criterion. This is discussed further in Section 3.2.

The algorithm uses the BACKUP routine described in Table 1. We can assume for the moment that the EXPAND subroutine (line 8) selects belief points at random, uniformly distributed over the belief simplex. This performs reasonably well for small problems where it is easy to achieve good coverage of the entire belief simplex. However it scales very poorly to larger domains. More sophisticated approaches to selecting belief points are presented in Section 3.3. Overall, the PBVI framework described here offers a simple yet flexible approach to solving large-scale POMDPs.

```
Γ=PBVI-MAIN($B_{Init}$, $\Gamma_0$, $N$, $T$)      1
    $B$=$B_{Init}$                                  2
    $\Gamma = \Gamma_0$                             3
    For $N$ expansions                              4
        For $T$ iterations                          5
            $\Gamma$ =BACKUP($B$,$\Gamma$)          6
        End                                         7
        $B_{new}$ =EXPAND($B$,$\Gamma$)             8
        $B = B \cup B_{new}$                        9
    End                                             10
    Return $\Gamma$                                 11
```

Table 2: Algorithm for Point-Based Value Iteration (PBVI)

3.2 Convergence and Error Bounds

We know that for any belief set B and horizon t, PBVI can produce an estimate of the value function, denoted V_t^B. We now show that the error between V_t^B and the optimal value function V^* is bounded.

The bound depends on how densely B samples the belief simplex Δ; with denser sampling, V_t^B converges to V^*. Cutting off the PBVI iterations at any sufficiently large horizon, we can show that the difference between V_t^B and the optimal infinite-horizon V^* is not too large. The overall error in PBVI, $||V_t^B - V^*||_\infty$, is bounded (according to the triangle inequality) by:

$$||V_t^B - V_t^*||_\infty + ||V_t^* - V^*||_\infty. \tag{26}$$

The second term is bounded by $\gamma^t \|V_0^* - V^*\|$ (Bertsekas & Tsitsiklis, 1996). The remainder of this section states and proves a bound on the first term.

We begin by defining the density δ_B of a set of belief points B to be the maximum distance from any belief in the simplex Δ, to a belief in set B. More precisely:

$$\delta_B = \max_{b' \in \Delta} \min_{b \in B} \|b - b'\|_1. \tag{27}$$

Then, we can prove the following lemma:

Lemma 1. *The error introduced in PBVI when performing* ***one iteration*** *of value backup over B, instead of over* Δ*, is bounded by*

$$\epsilon \leq \frac{(R_{\max} - R_{\min})\delta_B}{1-\gamma}$$

Proof: Let $b' \in \Delta$ be the point where PBVI makes its worst error in value update, and $b \in B$ be the closest (1-norm) sampled belief to b'. Let α be the vector that is maximal at b, and α' be the vector that would be maximal at b'. By failing to include α' in its solution set, PBVI makes an error of at most $\alpha' \cdot b' - \alpha \cdot b'$. On the other hand, since α is maximal at b, then $\alpha' \cdot b \leq \alpha \cdot b$. So,

$$\begin{array}{rlll}
\epsilon & \leq & \alpha' \cdot b' - \alpha \cdot b' & \\
 & = & \alpha' \cdot b' - \alpha \cdot b' + (\alpha' \cdot b - \alpha' \cdot b) & \text{add zero} \\
 & \leq & \alpha' \cdot b' - \alpha \cdot b' + \alpha \cdot b - \alpha' \cdot b & \alpha \text{ optimal at } b \\
 & = & (\alpha' - \alpha) \cdot (b' - b) & \text{collect terms} \\
 & \leq & \|\alpha' - \alpha\|_\infty \|b' - b\|_1 & \text{Hölder inequality} \\
 & \leq & \|\alpha' - \alpha\|_\infty \delta_B & \text{definition of } \delta_B \\
 & \leq & \frac{(R_{\max} - R_{\min})\delta_B}{1-\gamma} & \text{see text}
\end{array}$$

The last inequality holds because each α-vector represents the reward achievable starting from some state and following some sequence of actions and observations. Therefore the sum of rewards must fall between $\frac{R_{\min}}{1-\gamma}$ and $\frac{R_{\max}}{1-\gamma}$. Of course this holds only for domains where the reward is bounded. $\square$

Lemma 1 states a bound on the approximation error introduced by one iteration of point-based value updates within the PBVI framework. We now look at the bound over multiple value updates.

Theorem 3.1. *For any belief set B and any horizon t, the error of the PBVI algorithm $\epsilon_t = \|V_t^B - V_t^*\|_\infty$ is bounded by*

$$\epsilon_t \leq \frac{(R_{\max} - R_{\min})\delta_B}{(1-\gamma)^2}$$

Proof:

$$\begin{array}{rlll}
\epsilon_t & = & ||V_t^B - V_t^*||_\infty & \\
 & = & ||\tilde{H}V_{t-1}^B - HV_{t-1}^*||_\infty & \tilde{H} \text{ denotes PBVI backup,} \\
 & & & \text{and } H \text{ denotes exact backup} \\
 & \leq & ||\tilde{H}V_{t-1}^B - HV_{t-1}^B||_\infty + ||HV_{t-1}^B - HV_{t-1}^*||_\infty & \text{triangle inequality} \\
 & \leq & \epsilon + ||HV_{t-1}^B - HV_{t-1}^*||_\infty & \text{definition of } \epsilon \\
 & \leq & \epsilon + \gamma ||V_{t-1}^B - V_{t-1}^*||_\infty & \text{contraction} \\
 & = & \epsilon + \gamma\epsilon_{t-1} & \text{definition of } \epsilon_{t-1} \\
 & \leq & \frac{(R_{\max} - R_{\min})\delta_B}{1-\gamma} + \gamma\epsilon_{t-1} & \text{lemma 1} \\
 & \leq & \frac{(R_{\max} - R_{\min})\delta_B}{(1-\gamma)^2} & \text{series sum} \quad \square
\end{array}$$

The bound described in this section depends on how densely B samples the belief simplex Δ. In the case where not all beliefs are reachable, PBVI does not need to sample all of Δ densely, but can replace Δ by the set of reachable beliefs $\bar{\Delta}$ (Fig. 2). The error bounds and convergence results hold on $\bar{\Delta}$.

As a side note, it is worth pointing out that because PBVI makes no assumption regarding the initial value function V_0^B, the point-based solution V^B is not guaranteed to improve with the addition of belief points. Nonetheless, the theorem presented in this section shows that the *bound on the error* between V_t^B (the point-based solution) and V^* (the optimal solution) is guaranteed to decrease (or stay the same) with the addition of belief points. In cases where V_t^B is initialized pessimistically (e. g. $V_0^B(s) = \frac{R_{min}}{1-\gamma}, \forall s \in S$, as suggested in Sec. 3.1), then V_t^B will improve (or stay the same) with each value backup and addition of belief points.

This section has thus far skirted the issue of belief point selection, however the bound presented in this section clearly argues in favor of dense sampling over the belief simplex. While randomly selecting points according to a uniform distribution may eventually accomplish this, it is generally inefficient, in particular for high dimensional cases. Furthermore, it does not take advantage of the fact that the error bound holds for dense sampling over *reachable* beliefs. Thus we seek more efficient ways to generate belief points than at random over the entire simplex. This is the issue explored in the next section.

3.3 Belief Point Set Expansion

In section 3.1, we outlined the prototypical PBVI algorithm, while conveniently avoiding the question of how belief points should be selected. There is a clear trade-off between including fewer beliefs (which would favor fast planning over good performance), versus including many beliefs (which would slow down planning, but ensure in expectation better performance). This brings up the question of *how many* belief points should be included. However the number of points is not the only consideration. It is likely that some collections of belief points (e. g. those frequently encountered) are more likely to produce a good value function than others. This brings up the question of *which* beliefs should be included.

This section looks at five strategies for selecting belief points, from the very fast and naive (e. g. uniform random over the belief simplex), to an approach that selects points that are likely to have the largest impact in reducing the error bound (Theorem 3.1). Most of the strategies focus on selecting *reachable* beliefs, rather than getting uniform coverage over the entire belief simplex. Therefore it is useful to begin this discussing by looking at how reachability is assessed.

While some exact POMDP value iteration solutions are optimal for *any* initial belief, PBVI (and other related techniques) assume a known initial belief b_0. As shown in Figure 2, we can use the initial belief to build a tree of reachable beliefs. In this representation, each path through the tree corresponds to a sequence in belief space, and increasing depth corresponds to an increasing plan horizon. When selecting a set of belief points for PBVI, including all reachable beliefs would guarantee optimal performance (conditioned on the initial belief), but at the expense of computational tractability, since the set of reachable beliefs, $\bar{\Delta}$, grows exponentially with the planning horizon. Therefore, it is best to select a subset $B \subset \bar{\Delta}$, which is sufficiently small for computational tractability, but sufficiently large for good value function approximation.[2]

2. All strategies discussed below assume that the belief point set, B, approximately doubles in size on each belief expansion. Each strategy could be used (with very little modification) to add a fixed number of new belief points. However since value iteration is much more expensive than belief computation, it seems appropriate to double the size of B at each expansion.

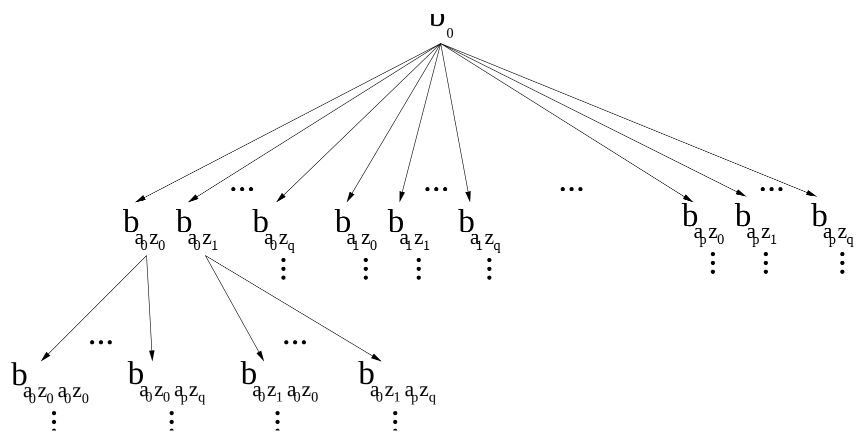


Figure 2: The set of reachable beliefs

We now discuss five strategies for selecting belief points, each of which can be used within the PBVI framework to perform expansion of the belief set.

3.3.1 Random Belief Selection (**RA**)

The first strategy is also the simplest. It consists of sampling belief points from a uniform distribution over the entire belief simplex. To sample over the simplex, we cannot simply sample each $b(s)$ as a random number over $[0, 1]$ (this would violate the constraint that $\sum_s b(s) = 1$). Instead, we use the efficient algorithm described in Table 3.

```
$B_{new}$=EXPAND$_{\mathrm{RA}}(B, \Gamma)$
  $B_{new}= B$
  Foreach $b \in B$
    $S$ := number of states
    $V$ = [0 1 ... S]
    For $i = 1 : S - 1$
      V[i]=rand$_{\mathrm{uniopen}}$()
    End
    Sort $V[]$ in ascending order
    For $i = 1 : S - 1$
      $b_{new}[i]=V[i+1] - V[i]$
    End
    $B_{new} = B_{new} \cup b_{new}$
  End
  Return $B_{new}$
```

Table 3: Algorithm for belief expansion with random action selection

This random point selection strategy, unlike the other strategies presented below, does not focus on reachable beliefs. For this reason, we do not necessarily advocate this approach. However we include it because it is an obvious choice, it is by far the simplest to implement, and it has been used in related work by Hauskrecht (2000) and Poon (2001). In smaller domains (e. g. <20 states), it performs reasonably well, since the belief simplex is relatively low-dimensional. In large domains (e. .g. 100+ states), it cannot provide good coverage of the belief simplex with a reasonable number of points, and therefore exhibits poor performance. This is demonstrated in the experimental results presented in Section 4.

All of the remaining belief selection strategies make use of the belief tree (Figure 2) to focus on *reachable* beliefs, rather than trying to cover the entire belief simplex.

3.3.2 Stochastic Simulation with Random Action (**SSRA**)

To generate points along the belief tree, we use a technique called stochastic simulation. It involves running single-step forward trajectories from belief points already in B. Simulating a single-step forward trajectory for a given $b \in B$ requires selecting an action and observation pair (a, z), and then computing the new belief $\tau(b, a, z)$ using the Bayesian update rule (Eqn 7). In the case of *Stochastic Simulation with Random Action* (SSRA), the action selected for forward simulation is picked (uniformly) at random from the full action set. Table 4 summarizes the belief expansion procedure for SSRA. First, a state s is drawn from the belief distribution b. Second, an action a is drawn at random from the full action set. Next, a posterior state s' is drawn from the transition model $T(s, a, s')$. Finally, an observation z is drawn from the observation model $O(s', a, z)$. Using the triple (b, a, z), we can calculate the new belief $b_{new} = \tau(b, a, z)$ (according to Equation 7), and add to the set of belief points B_{new}.

```
$B_{new}$=EXPAND$_{\text{SSRA}}(B, \Gamma)$                 1
    $B_{new}= B$                                  2
    Foreach $b \in B$                             3
        $s$=rand$_{\text{multinomial}}(b)$                 4
        $a$=rand$_{\text{uniform}}(A)$                     5
        $s'$=rand$_{\text{multinomial}}(T(s, a, \cdot))$    6
        $z$=rand$_{\text{multinomial}}(O(s', a, \cdot))$    7
        $b_{new} = \tau(b, a, z)$ (see Eqn 7)          8
        $B_{new} = B_{new} \cup b_{new}$                 9
    End                                          10
    Return $B_{new}$                              11
```

Table 4: Algorithm for belief expansion with random action selection

This strategy is better than picking points at random (as described above), because it restricts B_{new} to the belief tree (Fig. 2). However this belief tree is still very large, especially when the branching factor is high, due to large numbers of actions/observations. By being more selective about which paths in the belief tree are explored, one can hope to effectively restrict the belief set further.

3.3.3 Stochastic Simulation with Greedy Action (**SSGA**)

The procedure for generating points using *Stochastic Simulation with Greedy Action* (SSGA) is based on the well-known *epsilon-greedy* exploration strategy used in reinforcement learning (Sutton & Barto, 1998). This strategy is very similar to the SSRA procedure, except that rather than choosing an action randomly, SSEA will choose the *greedy* action (i. e. the current best action at the given belief b) with probability $1 - \epsilon$, and will chose a random action with probability ϵ (we use $\epsilon = 0.1$). Once the action is selected, we perform a single-step forward simulation as in SSRA to yield a new belief point. Table 5 summarizes the belief expansion procedure for SSGA.

```
$B_{new}$=EXPAND$_{\text{SSGA}}(B, \Gamma)$                         1
    $B_{new}$= $B$                                           2
    Foreach $b \in B$                                       3
        $s$=rand$_{\text{multinomial}}(b)$                           4
        If rand$_{\text{uniform}}[0,1] < \epsilon$                       5
            $a$=rand$_{\text{uniform}}(A)$                           6
        Else                                                7
            $a$=argmax$_{\alpha \in \Gamma} \sum_{s \in S} \alpha(s) b(s)$             8
        End                                                 9
        $s'$=rand$_{\text{multinomial}}(T(s, a, \cdot))$                  10
        $z$=rand$_{\text{multinomial}}(O(s', a, \cdot))$                  11
        $b_{new} = \tau(b, a, z)$ (see Eqn 7)                    12
        $B_{new} = B_{new} \cup b_{new}$                               13
    End                                                     14
    Return $B_{new}$                                          15
```

Table 5: Algorithm for belief expansion with greedy action selection

3.3.4 Stochastic Simulation with Exploratory Action (**SSEA**)

The error bound in Section 3.2 suggests that PBVI performs best when its belief set is uniformly dense in the set of reachable beliefs. The belief point strategies proposed thus far ignore this information. The next approach we propose gradually expands B by greedily choosing new reachable beliefs that improve the worst-case density.

Unlike SSRA and SSGA which select a single action to simulate the forward trajectory for a given $b \in B$, *Stochastic Sampling with Exploratory Action* (SSEA) does a one step forward simulation with *each action*, thus producing new beliefs $\{b_{a_0}, b_{a_1}, ...\}$. However it does not accept all new beliefs $\{b_{a_0}, b_{a_1}, ...\}$, but rather calculates the L_1 distance between each b_a and its closest neighbor in B. We then keep only that point b_a that is farthest away from any point already in B. We use the L_1 norm to calculate distance between belief points to be consistent with the error bound in Theorem 3.1. Table 6 summarizes the SSEA expansion procedure.

```
$B_{new}$=EXPAND$_{\text{SSEA}}(B, \Gamma)$                                  1
  $B_{new}= B$                                                              2
  Foreach $b \in B$                                                         3
    Foreach $a \in A$                                                       4
      $s$=rand$_{\text{multinomial}}(b)$                                    5
      $s'$=rand$_{\text{multinomial}}(T(s, a, \cdot))$                      6
      $z$=rand$_{\text{multinomial}}(O(s', a, \cdot))$                      7
      $b_a$=$\tau(b, a, z)$ (see Eqn 7)                                     8
    End                                                                     9
    $b_{new} = max_{a \in A}\ min_{b' \in B_{new}} \sum_{s \in S} |b_a(s) - b'(s)|$   10
    $B_{new} = B_{new} \cup b_{new}$ (see Eqn 7)                            11
  End                                                                       12
  Return $B_{new}$                                                          13
```

Table 6: Algorithm for belief expansion with exploratory action selection

3.3.5 Greedy Error Reduction (**GER**)

While the SSEA strategy above is able to improve the worst-case density of reachable beliefs, it does not directly minimize the expected error. And while we would like to directly minimize the error, all we can measure is a bound on the error (Section 3.2). We therefore propose a final strategy which greedily adds the candidate beliefs that will most effectively reduce this error bound. Our empirical results, as presented below, show that this strategy is the most successful one discovered thus far.

To understand how we expand the belief set in the GER strategy, it is useful to reconsider the belief tree, which we reproduce in Figure 3. Each node in the tree corresponds to a specific belief. We can divide these nodes into three sets. Set 1 includes those belief points already in B, in this case b_0 and b_{a_0,z_0}. Set 2 contains those belief points that are descendants of the points in B (i. e. the nodes in the grey zone). These are the candidates from which we will select the new points to be added to B. We call this set the envelope (denoted $\bar{B}$). Set 3 contains all other reachable beliefs.

We need to decide which belief b should be removed from the envelope $\bar{B}$ and added to the set of active belief points B. Every point that is added to B will improve our estimate of the value function. The new point will reduce the error bounds (as defined in Section 3.2) for points that were already in B; however, the error bound for the new point itself might be quite large. That means that the largest error bound for points in B will not monotonically decrease; however, for a particular point in B (such as the initial belief b_0) the error bound will be decreasing.

To find the point which will most reduce our error bound, we can look at the analysis of Lemma 1. Lemma 1 bounds the amount of additional error that a single point-based backup introduces. Write b' for the new belief which we are considering adding, and write b for some belief which is already in B. Write α for the value hyperplane at b, and write α'

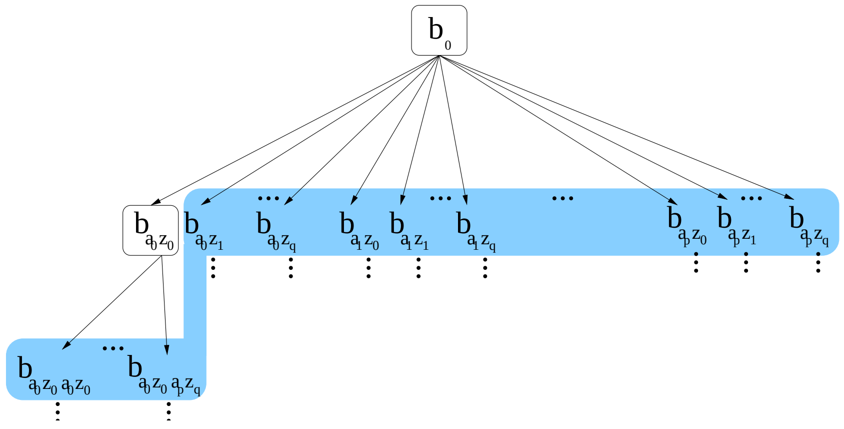


Figure 3: The set of reachable beliefs

for b'. As the lemma points out, we have

$$\epsilon \leq (\alpha' - \alpha) \cdot (b' - b)$$

Since we don't know what α' will be until we have done some backups at b', we will be conservative by choosing the worst-case value of α'. This worst-case value is the $\alpha' \in [R_{\min}/(1-\gamma), R_{\max}/(1-\gamma)]^{|S|}$ which maximizes the bound. Performing the maximization, we have

$$\epsilon(b') \leq \sum_i \begin{cases} (\frac{R_{\max}}{1-\gamma} - \alpha_i)(b'_i - b_i) & b'_i \geq b_i \\ (\frac{R_{\min}}{1-\gamma} - \alpha_i)(b'_i - b_i) & b'_i < b_i \end{cases} \tag{28}$$

Rather than directly picking the candidate $b' \in \bar{B}$ with the largest error bound, $\epsilon(b')$, we instead try to minimize the error over *reachable* beliefs. We note that the error at any given belief point b in the tree can be evaluated from the error of its immediate descendants:

$$\begin{aligned} \epsilon(b) &= \max_{a \in A} \sum_{z \in Z} O(b, a, z) \;\; \epsilon(\tau(b, a, z)) \qquad (29) \\ &= \max_{a \in A} \sum_{z \in Z} \left(\sum_{s \in S} \sum_{s' \in S} T(s, a, s') O(s', a, z) b(s) \right) \epsilon(\tau(b, a, z)), \end{aligned}$$

where $\tau(b, a, z)$ is the belief update operation defined in Equation 7, and $\epsilon(\tau(b, a, z))$ is evaluated according to Equation 28, unless $\tau(b, a, z) \in B$, in which case $\epsilon(\tau(b, a, z)) = 0$.

Using Equation 29, we find the existing point $b \in B$ with the largest error bound. We can now directly reduce its error by adding to our set one of its descendants. We select the next-step belief $\tau(b, a, z)$ which maximizes error bound reduction:

$$B = B \cup \tau(\tilde{b}, \tilde{a}, \tilde{z}), \tag{30}$$

$$\text{where} \quad \tilde{b}, \tilde{a} := \operatorname*{argmax}_{b \in B, a \in A} \sum_{z \in Z} O(b, a, z) \; \epsilon(\tau(b, a, z)) \tag{31}$$

$$\tilde{z} := \operatorname*{argmax}_{z \in Z} O(\tilde{b}, \tilde{a}, z) \; \epsilon(\tau(\tilde{b}, \tilde{a}, z)) \tag{32}$$

```
$B_{new}$=EXPAND$_{\text{GER}}(B, \Gamma)$                                      1
  $B_{new}$= $B$                                                                2
  $N$=$|B|$                                                                     3
  For $i = 1 : N$                                                               4
    $\tilde{b}, \tilde{a} := \text{argmax}_{b \in B, a \in A} \sum_{z \in Z} O(b, a, z)\ \epsilon(\tau(b, a, z))$   5
    $\tilde{z} := \text{argmax}_{z \in Z} O(\tilde{b}, \tilde{a}, z)\ \epsilon(\tau(\tilde{b}, \tilde{a}, z))$   6
    $b_{new} = \tau(\tilde{b}, \tilde{a}, \tilde{z})$                            7
    $B_{new} = B_{new} \cup b_{new}$                                             8
  End                                                                           9
  Return $B_{new}$                                                              10
```

Table 7: Algorithm for belief expansion

Table 7 summarizes this approach to belief point selection.

This concludes our presentation of belief selection techniques for the PBVI framework. In summary, there are three factors to consider when picking a belief point: (1) how likely is it to occur? (2) how far is it from other belief points already selected? (3) what is the current approximate value for that point? The simplest heuristic (RA) accounts for none of these, where as some of the others (SSRA, SSGA, SSEA) account for one or two of these factors. The last technique, GER, offers a more theoretically sound approach for selecting belief points which incorporates all three factors.

3.3.6 Belief Expansion Example

We consider a simple example, shown in Figure 4, to illustrate the difference between the various belief expansion techniques outlined above. The 1D POMDP (Littman, 1996) has four states, one of which is the goal (indicated by the star). The two actions, left and right, have the expected (deterministic) effect. The goal state is fully observable (*observation=goal*), while the other three states are aliased (*observation=none*). A reward of +1 is received for being in the goal state, otherwise the reward is zero. We assume a discount factor of $\gamma = 0.75$. The initial distribution is uniform over non-goal states, and the system resets to that distribution whenever the goal is reached.

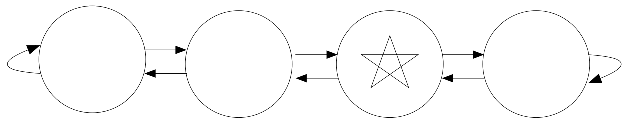


Figure 4: 1D POMDP

The belief set B is always initialized to contain the initial belief b_0. Figure 5 shows part of the belief tree, including the original belief set (top node), and its envelope (leaf nodes). We now consider what each belief expansion method might do.

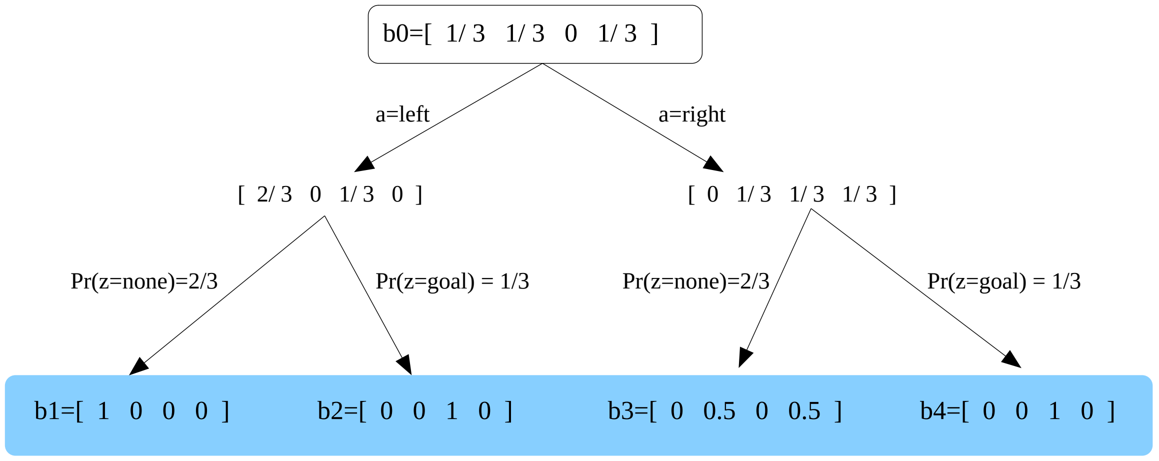


Figure 5: 1D POMDP belief tree

The **Random** heuristic can pick any belief point (with equal probability) from the entire belief simplex. It does not directly expand any branches of the belief tree, but it will eventually put samples nearby.

The **Stochastic Simulation with Random Action** has a 50% chance of picking each action. Then, regardless of which action was picked, there's a 2/3 chance of seeing observation *none*, and a 1/3 chance of seeing observation *goal*. As a result, the **SSRA** will select: $Pr(b_{new} = b1) = 0.5 * \frac{2}{3}$, $Pr(b_{new} = b2) = 0.5 * \frac{1}{3}$, $Pr(b_{new} = b3) = 0.5 * \frac{2}{3}$, $Pr(b_{new} = b4) = 0.5 * \frac{1}{3}$.

The **Stochastic Simulation with Greedy Action** first needs to know the policy at b_0. A few iterations of point-based updates (Section 2.4) applied to this initial (single point) belief set reveal that $\pi(b_0) = left$[3]. As a result, expansion of the belief will greedily select action $left$ with probability $1-\epsilon+\frac{\epsilon}{|A|} = 0.95$ (assuming $\epsilon = 0.1$ and $|A| = 2$). Action *right* will be selected for belief expansion with probability $\frac{\epsilon}{|A|} = 0.05$. Combining this along with the observation probabilities, we can tell that **SSGA** will expand as follows: $Pr(b_{new} = b1) = 0.95 * \frac{2}{3}$, $Pr(b_{new} = b2) = 0.95 * \frac{1}{3}$, $Pr(b_{new} = b3) = 0.05 * \frac{2}{3}$, $Pr(b_{new} = b4) = 0.05 * \frac{1}{3}$.

Predicting the choice of **Stochastic Simulation with Exploratory Action** is slightly more complicated. Four cases can occur, depending on the outcomes of random forward simulation from b_0:

1. If action *left* goes to b_1 ($Pr = 2/3$) and action *right* goes to b_3 ($Pr = 2/3$), then b_1 will be selected because $||b_0 - b_1||_1 = 4/3$ whereas $||b_0 - b_3||_1 = 2/3$. This case will occur with $Pr = 4/9$.

2. If action *left* goes to b_1 ($Pr = 2/3$) and action *right* goes to b_4 ($Pr = 1/3$), then b_4 will be selected because $||b_0 - b_4||_1 = 2$. This case will occur with $Pr = 2/9$.

3. If action *left* goes to b_2 ($Pr = 1/3$) and action *right* goes to b_3 ($Pr = 2/3$), then b_2 will be selected because $||b_0 - b_2||_1 = 2$. This case will occur with $Pr = 2/9$.

3. This may not be obvious to the reader, but it follows directly from the repeated application of equations 20-23

4. If action *left* goes to b_2 ($Pr = 1/3$) and action *right* goes to b_4 ($Pr = 1/3$), then either can be selected (since they are equi-distant to b_0). In this case each b_2 and b_4 has $Pr = 1/18$ of being selected.

All told, $Pr(b_{new} = b1) = 4/9$, $Pr(b_{new} = b2) = 5/18$, $Pr(b_{new} = b3) = 0$, $Pr(b_{new} = b4) = 5/18$.

Now looking at belief expansion using **Greedy Error Reduction**, we need to compute the error $\epsilon(\tau(b_0, a, z)), \forall a, z$. We consider Equation 28: since B has only one point, b_0, then necessarily $b = b_0$. To estimate α, we apply multiple steps of value backup at b_0 and obtain $\alpha = [0.94\ 0.94\ 0.92\ 1.74]$. Using b and α as such, we can now estimate the error at each candidate belief: $\epsilon(b_1) = 2.93$, $\epsilon(b_2) = 4.28$, $\epsilon(b_3) = 1.20$, $\epsilon(b_4) = 4.28$. Note that because B has only one point, the dominating factor is their distance to b_0. Next, we factor in the observation probabilities, as in Eqns 31-32, which allows us to determine that $\tilde{a} = left$ and $\tilde{z} = none$, and therefore we should select $b_{new} = b_1$.

In summary, we note that SSGA, SSEA and GER all favor selecting b_1, whereas SSRA picks each option with equal probability (considering that b_2 and b_4 are actually the same). In general, for a problem of this size, it is reasonable to expand the entire belief tree. Any of the techniques discussed here will be do this very quickly, except RA which will not pick the exact nodes in the belief tree, but will select equally good nearby beliefs. This example is provided simply to illustrate the different choices made by each strategy.

4. Experimental Evaluation

This section looks at a variety of simulated POMDP domains to evaluate the empirical performance of PBVI. The first three domains—Tiger-grid, Hallway, Hallway2—are extracted from the established POMDP literature (Cassandra, 1999). The fourth—Tag—was introduced in some of our earlier work as a new challenge for POMDP algorithms.

The main goal of these experiments is to establish the scalability of the PBVI framework, this is accomplished by showing that PBVI-type algorithms can successfully solve problems in excess of 800 states. We also demonstrate, in the course of these experiments, that PBVI algorithms compare favorably to alternative approximate value iteration methods. Finally, following on the example of Section 3.3.6, we study at a larger scale the impact of the belief selection strategy, which confirms the superior performance of the GER strategy.

4.1 Maze Problems

There exists a set of benchmark problems commonly used to evaluate POMDP planning algorithms (Cassandra, 1999). This section presents results demonstrating the performance of PBVI-class algorithms on some of those problems. While these benchmark problems are relatively small (at most 92 states, 5 actions, and 17 observations) compared to most robotics planning domains, they are useful from an analysis point of view and for comparison to previous work.

The initial performance analysis focuses on three well-known problems from the POMDP literature: Tiger-grid (also known as Maze33), Hallway, and Hallway2. All three are maze navigation problems of various sizes. The problems are fully described by Littman, Cassandra, and Kaelbling (1995a); parameterization is available from Cassandra (1999).

Figure 6a presents results for the Tiger-grid domain. Replicating earlier experiments (Brafman, 1997), test runs terminate after 500 steps (there's an automatic reset every time the goal is reached) and results are averaged over 151 runs.

Figures 6b and 6c present results for the Hallway and Hallway2 domains, respectively. In this case, test runs are terminated when the goal is reached or after 251 steps (whichever occurs first), and the results are averaged over 251 runs. This is consistent with earlier experiments (Littman, Cassandra, & Kaelbling, 1995b).

All three figures compare the performance of three different algorithms:

1. PBVI with Greedy Error Reduction (GER) belief point selection (Section 3.3.5).
2. QMDP (Littman et al., 1995b),
3. Incremental Pruning (Cassandra, Littman, & Zhang, 1997),

The *QMDP* heuristic (Littman et al., 1995b) takes into account partial observability at the current step, but assumes full observability on subsequent steps:

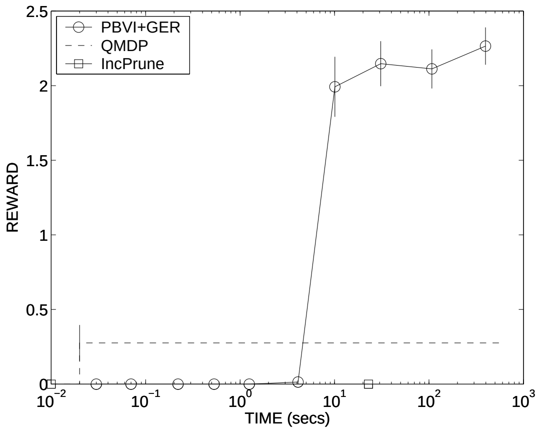
$$\pi_{QMDP}(b) \quad = \quad \operatorname*{argmax}_{a \in A} \sum_{s \in S} b(s) Q_{MDP}(s,a). \tag{33}$$

The resulting policy has some ability to resolve uncertainty, but cannot benefit from long-term information gathering, or compare actions with different information potential. QMDP can be seen as providing a good performance baseline. For the three problems considered, it finds a policy extremely quickly, but the policy is clearly sub-optimal.

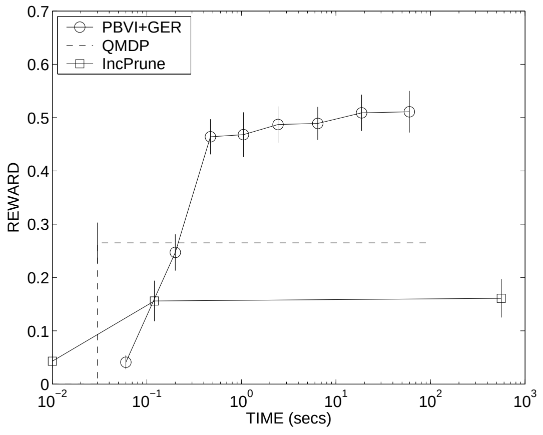
At the other end of the spectrum, the Incremental Pruning algorithm (Zhang & Liu, 1996; Cassandra et al., 1997) is a direct extension of the enumeration algorithm we described above. The principal insight is that the pruning of dominated α-vectors (Eqn 19) can be interleaved directly with the cross-sum operator (Eqn 16). The resulting value function is the same, but the algorithm is more efficient because it discards unnecessary vectors earlier on. While Incremental Pruning algorithm can theoretically find an optimal policy, for the three problems considered here it would take far too long. In fact, only a few iterations of exact value backups were completed in reasonable time. In all three problems, the resulting short-horizon policy was worse than the corresponding PBVI policy.

As shown in Figure 6, PBVI+GER provides a much better time/performance trade-off. The policies it finds are better than those obtained with QMDP, and it does so in a matter of seconds, thereby demonstrating that it does not suffer from the same paralyzing complexity as Incremental Pruning.

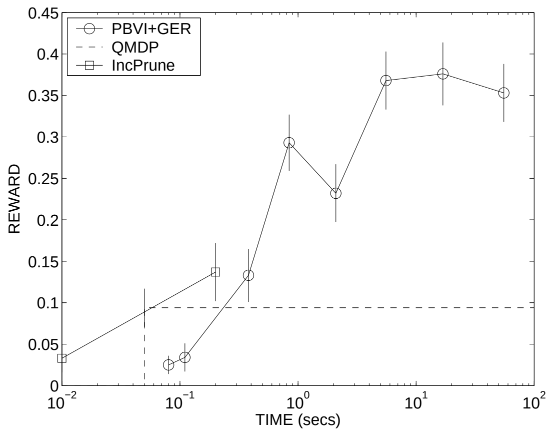
While those who take a closer look at these results may be surprised to see that the performance of PBVI actually *decreases* at some points (e.g. the "dip" in Fig. 6c), this is not unexpected. It is important to remember that the theoretical properties of PBVI only guarantee a bound on the estimate of the value function, but as shown here, this does not necessarily imply that the policy needs to improve monotonically. Nonetheless, as the value function converges, so will the policy (albeit at a slower rate).



(a) Tiger-grid



(b) Hallway



(c) Hallway2

Figure 6: PBVI performance on well-known POMDP problems

4.2 Tag Problem

While the previous section establishes the good performance of PBVI on some well-known simulation problems, these are quite small and do not fully demonstrate the scalability of the algorithm. To provide a better understanding of PBVI's effectiveness for large problems, this section presents results obtained when applying PBVI to the *Tag* problem, a robot version of the popular game of lasertag. In this problem, the agent must navigate its environment with the goal of searching for, and tagging, a moving target (Rosencrantz, Gordon, & Thrun, 2003). Real-world versions of this problem can take many forms, and in Section 5 we present a similar problem domain where an interactive service robot must find an elderly patient roaming the corridors of a nursing home. This scenario is an order of magnitude larger (870 states) than most other POMDP problems considered thus far in the literature (Cassandra, 1999), and was recently proposed as a new challenge for fast, scalable, POMDP algorithms (Pineau, Gordon, & Thrun, 2003b; Roy, 2003).

This scenario can be formulated as a POMDP problem, where the robot learns a policy optimized to quickly find the person. Meanwhile the person is assumed to move stochastically according to a fixed policy. The spatial configuration of the environment used throughout this experiment is illustrated in Figure 7.

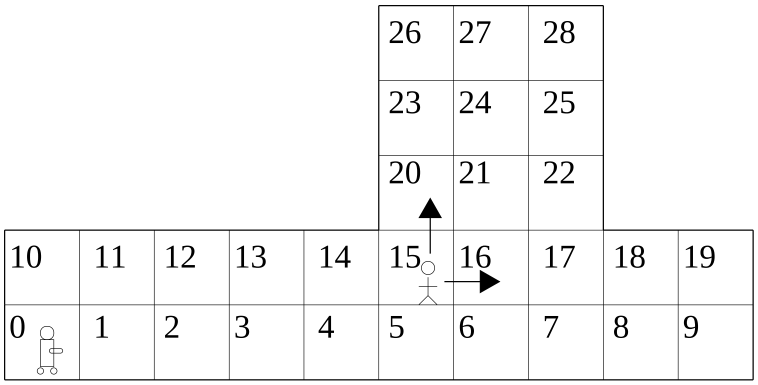


Figure 7: Spatial configuration of the domain

The state space is described by the cross-product of two position features, $Robot = \{s_0, \ldots, s_{29}\}$ and $Person = \{s_0, \ldots, s_{29}, s_{found}\}$. Both start in independently-selected random positions, and the scenario finishes when $Person = s_{found}$. The robot can select from five actions: {*North, South, East, West, Tag*}. A reward of -1 is imposed for each motion action; the *Tag* action results in a $+10$ reward if the robot and person are in the same cell, or -10 otherwise. Throughout the scenario, the Robot's position is fully observable, and a *Move* action has the predictable deterministic effect, e. g.:

$$Pr(Robot = s_{10} \mid Robot = s_0, North) = 1,$$

and so on for each adjacent cell and direction.

The position of the person, on the other hand, is completely unobservable unless both agents are in the same cell. Meanwhile at each step, the person (with omniscient knowledge) moves away from the robot with $Pr = 0.8$ and stays in place with $Pr = 0.2$, e. g.:

$$Pr(Person = s_{16} \mid Person = s_{15} \& Robot = s_0) = 0.4$$

$$Pr(Person = s_{20} \mid Person = s_{15} \& Robot = s_0) = 0.4$$
$$Pr(Person = s_{15} \mid Person = s_{15} \& Robot = s_0) = 0.2.$$

Figure 8 shows the performance of PBVI with Greedy Error Reduction on the Tag domain. Results are averaged over 1000 runs, using different (randomly chosen) start positions for each run. The QMDP approximation is also tested to provide a baseline comparison. The results show a gradual improvement in PBVI's performance as samples are added (each shown data point represents a new expansion of the belief set with value backups). It also confirms that computation time is directly related to the number of belief points. PBVI requires fewer than 100 belief points to overcome QMDP, and the performance keeps on improving as more points are added. Performance appears to be converging with approximately 250 belief points. These results show that a PBVI-class algorithm can effectively tackle a problem with 870 states.

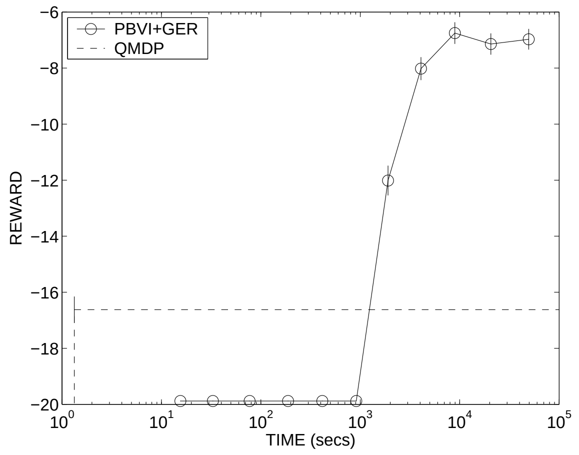


Figure 8: PBVI performance on Tag problem

This problem is far beyond the reach of the Incremental Pruning algorithm. A single iteration of optimal value iteration on a problem of this size could produce over 10^{20} α-vectors before pruning. Therefore, it was not applied.

This section describes one version of the Tag problem, which was used for simulation purposes in our work and that of others (Braziunas & Boutilier, 2004; Poupart & Boutilier, 2004; Smith & Simmons, 2004; Vlassis & Spaan, 2004). In fact, the problem can be reformulated in a variety of ways to accommodate different environments, person motion models, and observation models. Section 5 discusses variations on this problem using more realistic robot and person models, and presents results validated onboard an independently developed robot simulator.

4.3 Empirical comparison of PBVI-class algorithms

Having establish the good performance of PBVI+GER on a number of problems, we now consider empirical results for the different PBVI-class algorithms. This allows us to compare the effects of the various belief expansion heuristics. We repeat the experiments on the

Tiger-grid, Hallway, Hallyway2 and Tag domains, as outlined above, but in this case we compare the performance of five different PBVI-class algorithms:

1. PBVI+RA: PBVI with belief points selected randomly from belief simplex (Section 3.3.1).
2. PBVI+SSRA: PBVI with belief points selected using stochastic simulation with random action (Section 3.3.2).
3. PBVI+SSGA: PBVI with belief points selected using stochastic simulation with greedy action (Section 3.3.3).
4. PBVI+SSEA: PBVI with belief points selected using stochastic simulation with exploratory action (Section 3.3.4).
5. PBVI+GER: PBVI with belief points selected using greedy error reduction (Section 3.3.5).

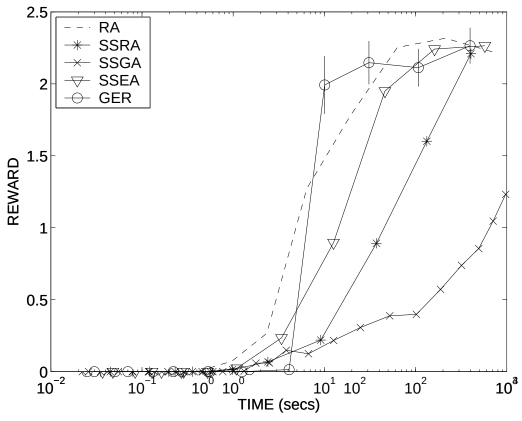
All PBVI-class algorithms can converge to the optimal value function given a sufficiently large set of belief points. But the rate at which they converge, depends on their ability to generally pick useful points, and leave out the points containing less information. Since the computation time is directly proportional to the number of belief points, the algorithm with the best performance is generally the one which can find a good solution with the least number of belief points.

Figure 9 shows a comparison between the performance of each of the five PBVI-class algorithms enumerated above on each of the four problem domains. In Tiger-grid and Hallway, PBVI+GER reaches near-optimal performance somewhat sooner than the other algorithms. In Hallway2, it is unclear which of the five algorithms is best, though GER sseems to converge earlier.

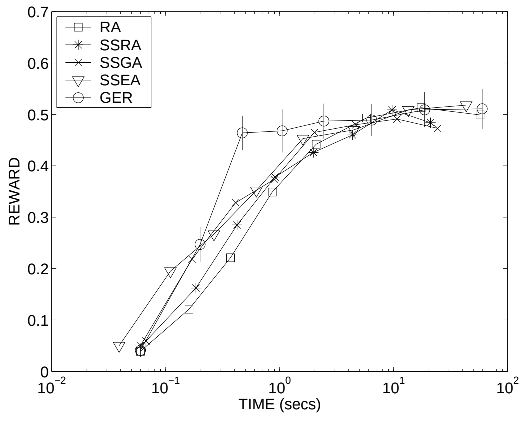
In the larger Tag domain, the situation is different. The PBVI+GER combination is clearly superior to the others. There is reason to believe that PBVI+SSEA could match its performance, but would require on the order of twice as many points to do so. Nonetheless, PBVI+SSEA performs better than either PBVI+SSRA or PBVI+SSGA. These results suggest that the choice of belief points is crucial when dealing with large problems. GER—and SSEA to a lesser degree—seem more effective than the other heuristics at getting good coverage over the large dimensional beliefs featured in this domain. With the random heuristic (PBVI+RA), the reward did not improve regardless of how many belief points were added (4000+), and therefore we do not include it in the results.

As a side note, we were surprised by SSGA's poor performance (in comparison with SSRA) on the Tiger-grid and Tag domains. This could be due to a poorly tuned greedy bias ϵ, which did not investigate at length. Future investigations using problems with a larger number of actions may shed better light on this issue.

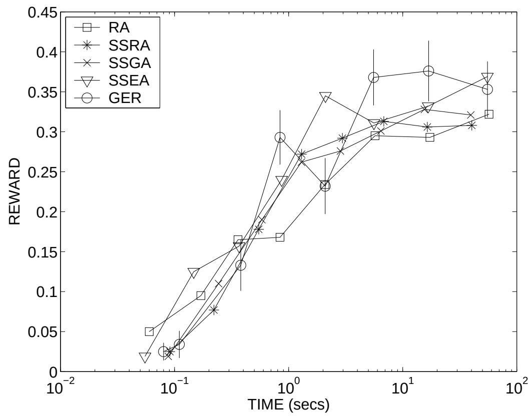
In terms of computational requirement, GER is the most expensive to compute, followed by SSEA. However in all cases, the time to perform the belief expansion step is generally negligible ($< 1\%$), compared to the cost of the value update steps. Therefore it seems best to use the more effective (though more expensive) heuristic.



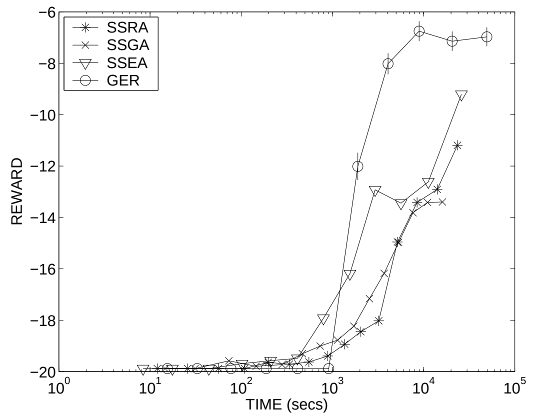
(a) Tiger-grid



(b) Hallway



(c) Hallway2



(d) Tag

Figure 9: Belief expansion results

The PBVI framework can accommodate a wide variety of strategies, past what is described in this paper. For example, one could extract belief points directly from sampled experimental traces. This will be the subject of future investigations.

4.4 Comparative analysis

While the results outlined above show that PBVI-type algorithms are able to handle a wide spectrum of large-scale POMDP domains, it is not sufficient to compare the performance of PBVI only to QMDP and Incremental Pruning—the two ends of the spectrum—as done in Section 4.1. In fact there has been significant activity in recent years in the development of fast approximate POMDP algorithms, and so it is worthwhile to spend some time comparing the PBVI framework to these alternative approaches. This is made easy by the fact that many of these have been validated using the same set of problems as described above.

Table 8 summarizes the performance of a large number of recent POMDP approximation algorithms, including PBVI, on the four target domains: Tiger-grid, Hallway, Hallway2, and Tag. The algorithms listed were selected based on the availability of comparable published results or available code, or in some cases because the algorithm could be re-implemented easily.

We compare their empirical performance, in terms of execution performance versus planning, on a set of simulation domains. However as is often the case, these results show that there is not a single algorithm that is best for solving all problems. We therefore also compile a summary of the attributes and characteristics of each algorithm, in an attempt to tell which algorithm may be best for what types of problems. Table 8 includes (whenever possible) the goal completion rates, sum of rewards, policy computation time, number of required belief points, and policy size (number of α-vectors, or number of nodes in finite state controllers).

The results marked [*] were computed by us on a 3GHz Pentium 4; other results were likely computed on different platforms, and therefore time comparisons may be approximate at best. Nonetheless the number of samples and the size of the final policy are both useful indicators of computation time. The results reported for PBVI correspond to the earliest data point from Figures 6 and 8 where PBVI+GER achieves top performance.

Algorithms are listed in order of performance, starting with the algorithm(s) achieving the highest reward. All results assume a standard (not *lookahead*) controller (see Hauskrecht (2000) for definition).

Overall, the results indicate that some of the algorithms achieve sub-par performance in terms of expected reward. In the case of QMDP, this is because of fundamental limitations in the algorithm. While Incremental Pruning and the exact value-directed compression can theoretically reach optimal performance, they would require longer computation time to do so. The grid method (see Tiger-grid results), BPI (see Tiger-grid, Hallway and Tag results) and PBUA (see Tag results) suffer from a similar problem, but offer much more graceful performance degradation. It is worth noting that none of these approaches assumes a known initial belief, so in effect they are solving harder problems. The results for BBSLS are not sufficiently extensive to comment at length, but it appears to be able to find reasonable policies with very small controllers (see Tag results).

The remaining algorithms—HSVI, Perseus, and our own PBVI+GER—all offer comparable performance on these relatively large POMDP domains. HSVI seems to offer good control performance on the full range of tasks, but requires bigger controllers, and is therefore probably slower, especially on domains with high stochasticity (e. g. Tiger-grid, Hallway, Hallway2). The trade-offs between Perseus and PBVI+GER are less clear: the planning time, controller size and performance quality are quite comparable, and in fact the two approaches are very similar. Perseus uses the same point-based backups as in PBVI (see Section 2.4), but it differs in both how the core set of belief points is selected (Perseus uses a non-incremental version of the SSRA heuristic), and the order in which the value at those points is updated (this is randomized). The effect of these differences is hard to narrow, but the results certainly suggest that the randomization of value updates may be an effective strategy to accelerate planning. We did experiment with similar random value updates in the context of PBVI, but did not achieve any significant speed-up; it is possible that the performance gain from randomized updates is simply not the same when dealing with smaller belief sets.

Method	Goal%	Reward ± Conf.Int.	Time(s)	$\|B\|$	$\|\pi\|$
Tiger-Grid (Maze33)					
HSVI (Smith & Simmons, 2004)	n.a.	2.35	10341	n.v.	4860
Perseus (Vlassis & Spaan, 2004)	n.a.	2.34	104	10000	134
PBUA (Poon, 2001)	n.a.	2.30	12116	660	n.v.
PBVI+GER[*]	n.a.	2.27 ± 0.13	397	512	508
BPI (Poupart & Boutilier, 2004)	n.a.	1.81	163420	n.a.	1500
Grid (Brafman, 1997)	n.a.	0.94	n.v.	174	n.a.
QMDP (Littman et al., 1995b)[*]	n.a.	0.276	0.02	n.a.	5
IncPrune (Cassandra et al., 1997)[*]	n.a.	0.0	24hrs+	n.a.	n.v.
Exact VDC (Poupart & Boutilier, 2003)[*]	n.a.	0.0	24hrs+	n.a.	n.v.
Hallway					
PBUA (Poon, 2001)	100	0.53	450	300	n.v.
HSVI (Smith & Simmons, 2004)	100	0.52	10836	n.v.	1341
PBVI+GER[*]	100	0.51 ± 0.03	19	64	64
Perseus (Vlassis & Spaan, 2004)	n.v.	0.51	35	10000	55
BPI (Poupart & Boutilier, 2004)	n.v.	0.51	249730	n.a.	1500
QMDP (Littman et al., 1995b)[*]	51	0.265	0.03	n.a.	5
Exact VDC (Poupart & Boutilier, 2003)[*]	39	0.161	24hrs+	n.a.	n.v.
IncPrune (Cassandra et al., 1997)[*]	39	0.161	24hrs+	n.a.	n.v.
Hallway2					
PBVI+GER[*]	100	0.37 ± 0.04	6	32	31
Perseus (Vlassis & Spaan, 2004)	n.v.	0.35	10	10000	56
HSVI (Smith & Simmons, 2004)	100	0.35	10010	n.v.	1571
PBUA (Poon, 2001)	100	0.35	27898	1840	n.v.
BPI (Poupart & Boutilier, 2004)	n.v.	0.28	274280	n.a.	1500
Grid (Brafman, 1997)	98	n.v.	n.v.	337	n.a.
QMDP (Littman et al., 1995b)[*]	22	0.109	1.44	n.a.	5
Exact VDC (Poupart & Boutilier, 2003)[*]	48	0.137	24hrs+	n.a.	n.v.
IncPrune (Cassandra et al., 1997)[*]	48	0.137	24hrs+	n.a.	n.v.
Tag					
HSVI (Smith & Simmons, 2004)	100	-6.37	10113	n.v.	1657
PBVI+GER[*]	100	-6.75 ± 0.39	8946	256	203
Perseus (Vlassis & Spaan, 2004)	n.v.	-6.85	3076	10000	205
BBSLS (Braziunas & Boutilier, 2004)	n.v.	-8.31	100054	n.a.	30
BPI (Poupart & Boutilier, 2004)	n.v.	-9.18	59772	n.a.	940
QMDP (Littman et al., 1995b)[*]	19	-16.62	1.33	n.a.	5
PBUA (Poon, 2001)[*]	0	-19.9	24hrs+	4096	n.v.
IncPrune (Cassandra et al., 1997)[*]	0	-19.9	24hrs+	n.a.	n.v.

n.a.=not applicable n.v.=not available [*]=results computed by us

Table 8: Results of PBVI for standard POMDP domains

4.5 Error estimates

The results presented thus far suggest that the PBVI framework performs best when using the Greedy Error Reduction (GER) technique for selecting belief points. Under this scheme, to decide which belief points will be included, we estimate an error bound at a set of candidate points and then pick the one with the largest error estimate. The error bound is estimated as described in Equation 28. We now consider the question of how this estimate evolves as more and more points are added. The natural intuition is that with the first few points, error estimates will be very large, but as the density of the belief set increases, the error estimates will become much smaller.

Figure 10 reconsiders the four target domains: Tiger-grid, Hallway, Hallway2 and Tag. In each case, we present both the reward performance as a function of the number of belief points (top row graphs), and the error estimate of each point selected according the order in which points were picked (bottom row graphs). Because new belief points are only selected from the envelope of reachable beliefs, it is not surprising to occasionally see large jumps in the graphs. Overall, it seems that there is reasonably good correspondence between an improvement in performance and a decrease in the error estimates.

We note that the error profiles for the Hallway2 and Tag domains exhibit a few large jumps in the later stages. These could point towards an improvement to GER, for example by maintaining a deeper envelope of candidate belief points. Currently the envelope contains those points that are 1-step forward simulations from the points already selected. It may be useful to consider points 2-3 steps ahead. We predict this would reduce the jaggedness seen in Figure 10, and more importantly, also reduce the number of points necessary for good performance. Of course, the tradeoff between the time spent selecting points and the time spent planning would have to be re-evaluated under this light.

We can conclude from this figure and the performance results in Figure 6 that in fact the PBVI error bound is quite informative in guiding exploration of the belief simplex.

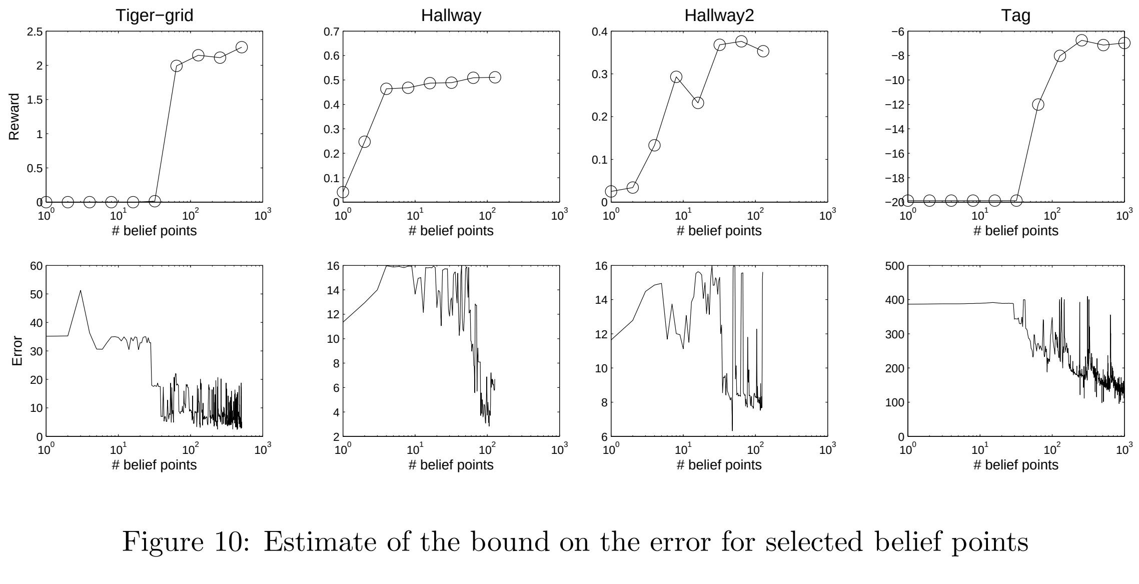


Figure 10: Estimate of the bound on the error for selected belief points

5. Robotic applications

The overall motivation behind the work described in this paper is the desire to provide high-quality robust planning for real-world autonomous systems, and in particular for robots. On the practical side, our search for a robust robot controller has been in large part guided by the Nursebot project (Pineau, Montermerlo, Pollack, Roy, & Thrun, 2003c). The overall goal of the project is to develop personalized robotic technology that can play an active role in providing improved care and services to non-institutionalized elderly people. Pearl, shown in Figure 11, is the main robotic platform used for this project.

Figure 11: Pearl the Nursebot, interacting with elderly people at a nursing facility

From the many services a nursing-assistant robot could provide (Engelberger, 1999; Lacey & Dawson-Howe, 1998), much of the work to date has focused on providing timely cognitive reminders (e. g. medications to take, appointments to attend, etc.) to elderly subjects (Pollack, 2002). An important component of this task is finding the patient whenever it is time to issue a reminder. This task shares many similarities with the Tag problem presented in Section 4.2. In this case, however, a robot-generated map of a real physical environment is used as the basis for the spatial configuration of the domain. This map is shown in Figure 12. The white areas correspond to free space, the black lines indicate walls (or other obstacles) and the dark gray areas are not visible or accessible to the robot. One can easily imagine the patient's room and physiotherapy unit lying at either end of the corridor, with a common area shown in the upper-middle section.

The overall goal is for the robot to traverse the domain in order to find the missing patient and then deliver a message. The robot must systematically explore the environment, reasoning about both spatial coverage and human motion patterns, in order to find the person.

5.1 POMDP Modeling

The problem domain is represented jointly by two state features: *RobotPosition, Person-Position*. Each feature is expressed through a discretization of the environment. Most of

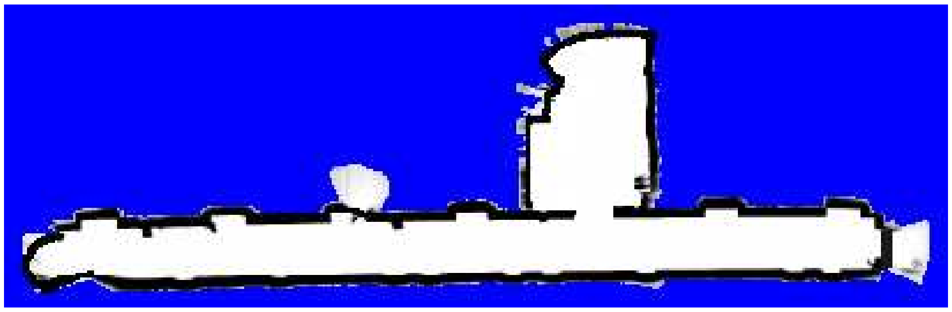


Figure 12: Map of the environment

the experiments below assume a discretization of 2 meters, which means 26 discrete cells for each feature, for a total of 676 states.

It is assumed that the person and robot can move freely throughout this space. The robot's motion is deterministically controlled by the choice of action (*North, South, East, West*). The robot has a fifth action (*DeliverMessage*), which concludes the scenario when used appropriately (i. e. when the robot and person are in the same location).

The person's motion is stochastic and falls in one of two modes. Part of the time, the person moves according to Brownian motion (e. g. moves in each cardinal direction with $Pr = 0.1$, otherwise stays put). At other times, the person moves directly away from the robot. The Tag domain of Section 4.2 assumes that the person always moves always moves away the robot. This is not realistic when the person cannot see the robot. The current experiment instead assumes that the person moves according to Brownian motion when the robot is far away, and moves away from the robot when it is closer (e. g. < 4m). The person policy was designed this way to encourage the robot to find a robust policy.

In terms of state observability, there are two components: what the robot can sense about its own position, and what it can sense about the person's position. In the first case, the assumption is that the robot knows its own position at all times. While this may seem like a generous (or optimistic) assumption, substantial experience with domains of this size and maps of this quality have demonstrated very robust localization abilities (Thrun et al., 2000). This is especially true when planning operates at relatively coarse resolution (2 meters) compared to the localization precision (10 cm). While exact position information is assumed for *planning* in this domain, the *execution* phase (during which we actually measure performance) does update the belief using full localization information, which includes positional uncertainty whenever appropriate.

Regarding the detection of the person, the assumption is that the robot has no knowledge of the person's position unless s/he is within a range of 2 meters. This is plausible given the robot's sensors. However, even in short-range, there is a small probability ($Pr = 0.01$) that the robot will miss the person and therefore return a false negative.

In general, one could make sensible assumptions about the person's likely position (e. g. based on a knowledge of their daily activities), however we currently have no such information and therefore assume a uniform distribution over all initial positions. The person's subsequent movements are expressed through the motion model described above (i. e. mix of Brownian motion and purposeful avoidance).

The reward function is straightforward: $R = -1$ for any motion action, $R = 10$ when the robot decides to $DeliverMessage$ and it is in the same cell as the person, and $R = -100$ when the robot decides to $DeliverMessage$ in the person's absence. The task terminates when the robot successfully delivers the message (i. e. $a = DeliverMessage$ and $s_{robot} = s_{person}$). We assume a discount factor of 0.95.

The initial map (Fig. 12) of the domain was collected by a mobile robot, and slightly cleaned up by hand to remove artifacts (e.g. people walking by). We then assumed the model parameters described here, and applied PBVI planning to the problem as such. Finally, the resulting control policy was then implemented and tested onboard the publically available CARMEN robot simulator (Montemerlo, Roy, & Thrun, 2003). These results are described in the next section.

5.2 Experimental Results

The subtask described here, with its 626 states, is beyond the capabilities of exact POMDP solvers. Furthermore, as will be demonstrated below, MDP-type approximations are not equipped to handle uncertainty of the type exhibited in this task. The main purpose of this section is therefore to evaluate the effectiveness of the PBVI approach described in this paper to address this problem. While the results on the Tag domain (Section 4.2) hint at the fact that PBVI may be able to handle this task, the more realistic map and modified motion model provide new challenges.

PBVI is applied to the problem as stated above, alternating value updates and belief point expansions until (in simulation) the policy is able to find the person on $> 90\%$ of trials (trials were terminated when the person is found or after 100 execution steps). The planning phase required 40000 seconds (approx. 11 hours) on a 1.2 GHz Pentium II.

The resulting policy is illustrated in Figure 13. This figure shows six snapshots obtained from a single run. In this particular scenario, the person starts at the far end of the left corridor. The person's location is not shown in any of the figures since it is not observable by the robot. The figure instead shows the *belief* over person positions, represented by a distribution of point samples (grey dots in Fig. 13). Each point represents a plausible hypothesis about the person's position. The figure shows the robot starting at the far right end of the corridor (Fig. 13a). The robot moves toward the left until the room's entrance (Fig. 13b). It then proceeds to check the entire room (Fig. 13c). Once relatively certain that the person is nowhere to be found, it exits the room (Fig. 13d), and moves down the left branch of the corridor, where it finally finds the person at the very end of the corridor (Fig. 13e).

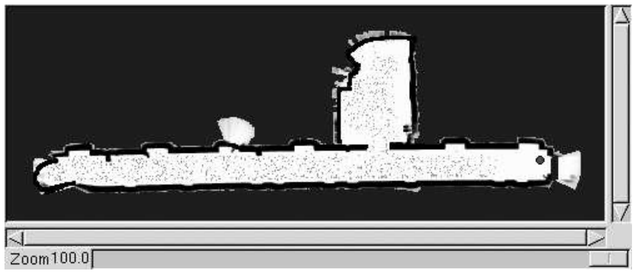
This policy is optimized for any start positions (for both the person and the robot). The scenario shown in Figure 13 is one of the longer execution traces since the robot ends up searching the entire environment before finding the person. It is interesting to compare the choice of action between snapshots (b) and (d). The robot position in both is practically identical. Yet in (b) the robot chooses to go up into the room, whereas in (d) the robot chooses to move toward the left. This is a direct result of planning over *beliefs*, rather than over *states*. The belief distribution over person positions is clearly different between those two cases, and therefore the policy specifies a very different course of action.

The sequence illustrated in Figure 13 is the result of planning with over 3000 belief points. It is interesting to consider what happens with fewer belief points. Figure 14 shows such a case. The scenario is the same, namely the person starts at the far left end of the corridor and the robot start at the far right end (Fig. 14a). The robot then navigates its way to the doorway (Fig. 14b). It enters the room and looks for the person in a portion of the room (Fig. 14c). Unfortunately an incomplete plan forces it into a corner (Fig. 14d) where it stays until the scenario is forcibly terminated. Using this policy (and assuming uniform random start positions for both robot and person), the person is only found in 40% of trials, compared to 90% using the policy shown in Figure 13. Planning in this case was done with 443 belief points, and required approximately 5000 seconds.

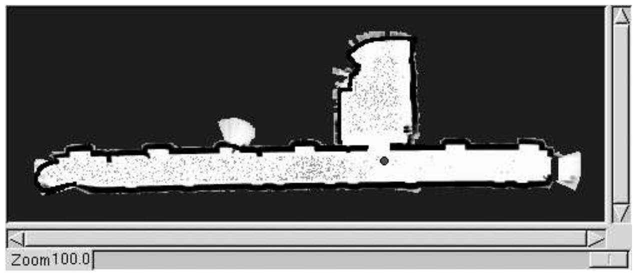
Figure 15 looks at the policy obtained when solving this same problem using the QMDP heuristic. Once again, six snapshots are offered from different stages of a specific scenario, assuming the person started on the far left side and the robot on the far right side (Fig. 15a). After proceeding to the room entrance (Fig. 15b), the robot continues down the corridor until it *almost* reaches the end (Fig. 15c). It then turns around and comes back toward the room entrance, where it stations itself (Fig. 15d) until the scenario is forcibly terminated. As a result, the robot cannot find the person when s/he is at the left edge of the corridor. What's more, because of the running-away behavior adopted by the subject, even when the person starts elsewhere in the corridor, as the robot approaches the person will gradually retreat to the left and similarly escape from the robot. Planning with the QMDP heuristic required 200 seconds.

Even though QMDP does not explicitly plan over *beliefs*, it can generate different policy actions for cases where the state is identical but the belief is different. This is seen when comparing Figure 15 (b) and (d). In both of these, the robot is identically located, however the belief over person positions is different. In (b), most of the probability mass is to the left of the robot, therefore it travels in that direction. In (d), the probability mass is distributed evenly between the three branches (left corridor, room, right corridor). The robot is equally pulled in all directions and therefore stops there. This scenario illustrates some of the strength of QMDP. Namely, there are many cases where it is not necessary to explicitly reduce uncertainty. However, it also shows that more sophisticated approaches are needed to handle some cases.

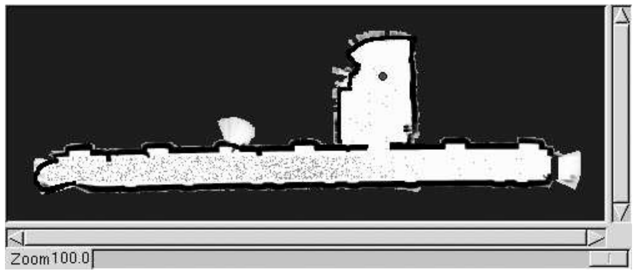
These results show that PBVI can perform outside the bounds of simple maze domains, and is able to handle realistic problem domains. In particular, throughout these experiments, the robot simulator was in no way constrained to behave as described in our POMDP model (Sec. 5.1). This means that the robot's actions often had stochastic effects, the robot's position was not always fully observable, and that belief tracking had to be performed asynchronously (i. e. not always a straightforward ordering of actions and observations). Despite this misalignment between the model assumed for planning, and the execution environment, the control policy optimized by PBVI could successfully be used to complete the task.



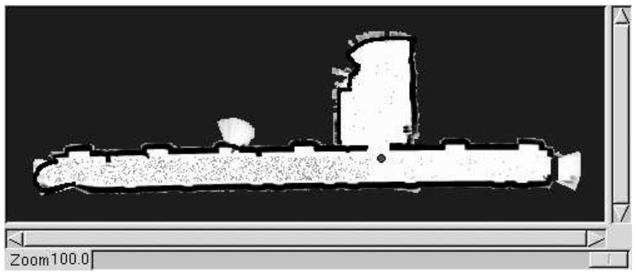
(a) t=1



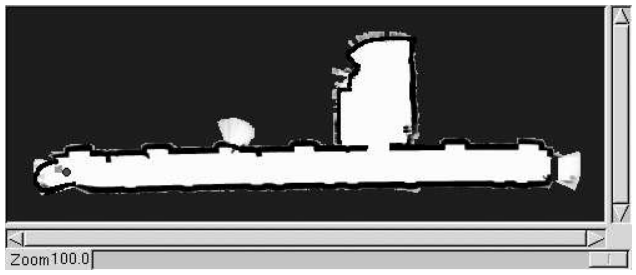
(b) t=7



(c) t=12

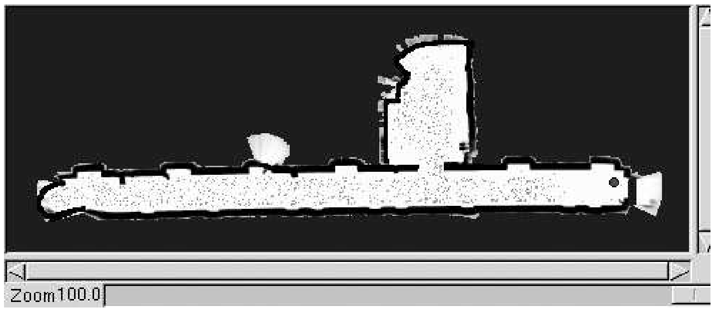


(d) t=17

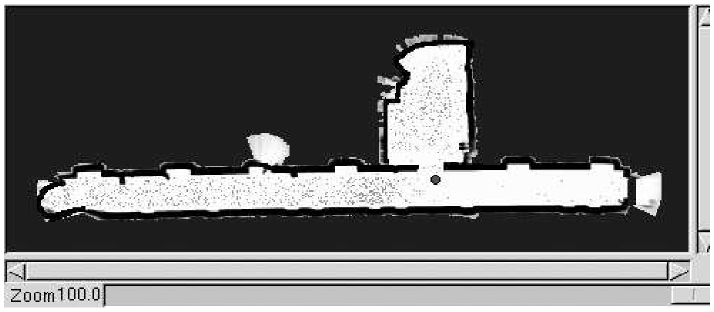


(e) t=29

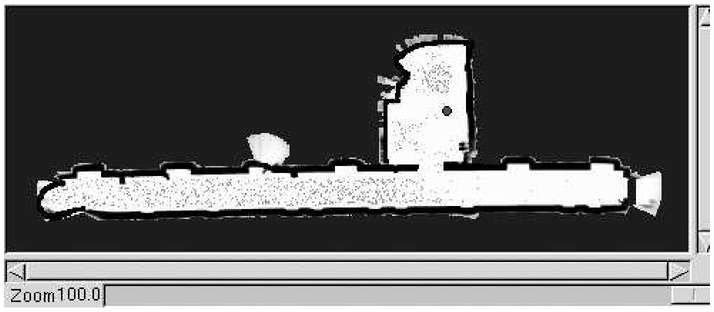
Figure 13: Example of a PBVI policy successfully finding the person (using 3000+ belief points)



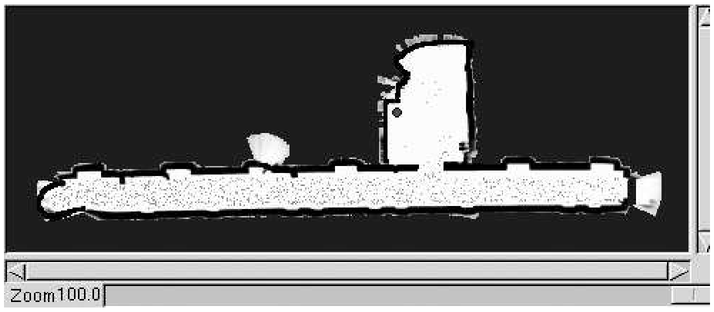
(a) t=1



(b) t=7

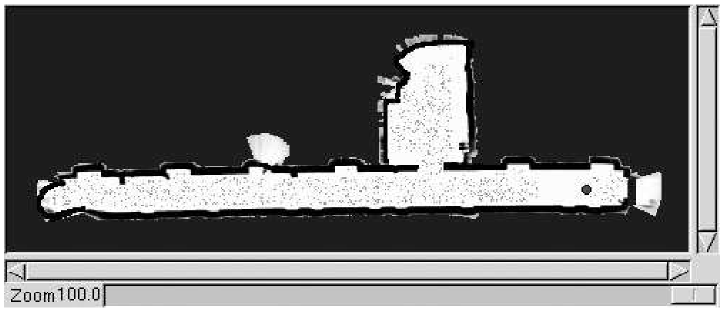


(c) t=10

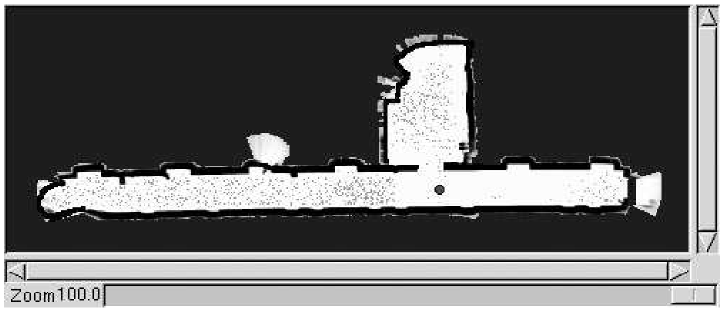


(d) t=12

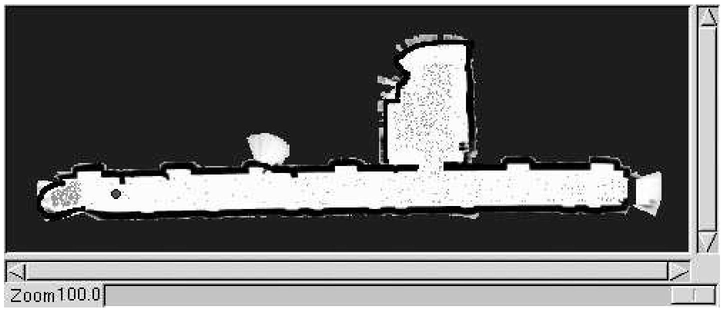
Figure 14: Example of a PBVI policy failing to find the person (using 443 belief points)



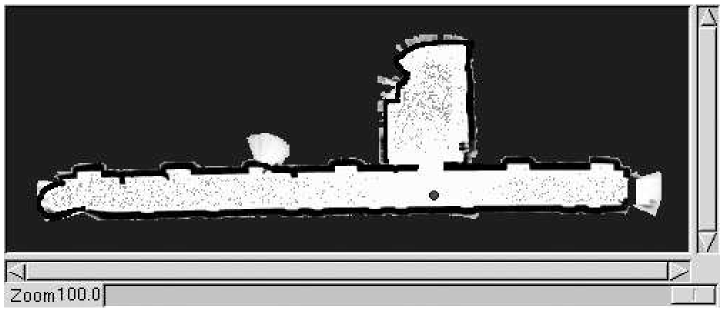
(a) t=1



(b) t=7



(c) t=17



(d) t=27

Figure 15: Example of a QMDP policy failing to find the person

6. Related Work

This paper describes a new class of point-based algorithms for POMDP solving. There is a large body work in approximate POMDP solving which shares similarities with this research.

Grid-based methods generally optimize an approximate value function by iteratively updating the values of discrete belief points. These methods differ in how they partition the belief space into a grid, and in how they update the value function. Some methods update only the value at each point (Brafman, 1997; Zhou & Hansen, 2001), and use an interpolation-type rule to estimate the value at other points..

More similar to the PBVI-class of algorithms are those approaches that update both the value and gradient at each grid point (Lovejoy, 1991a; Hauskrecht, 2000; Poon, 2001). These methods are able to preserve the piecewise linearity and convexity of the value function, and define a value function over the entire belief simplex. The actual point-based value update is essentially the same as in PBVI. However the overall algorithms differ in other aspects, in particular in how points are selected. Many earlier algorithms used random beliefs, or required the inclusion of a large number of fixed beliefs such as the corners of the probability simplex. In contrast, most PBVI-class algorithms (the only exception is PBVI+RA) select only reachable beliefs, and in particular those belief points that improve the error bounds as quickly as possible. While earlier approaches (Hauskrecht, 2000; Poon, 2001) did consider using stochastic simulation to generate new points, neither found simulation to be superior to random point placements. We hypothesize this may be due to the smaller size of their test domains. Our empirical results clearly show that with a large domain, such as Tag, PBVI's belief-selection is an important factor in the algorithm's performance.

New approaches building directly on the PBVI framework have been proposed, subsequent to publication of the first PBVI-class algorithms (Pineau et al., 2003b). This includes the Perseus algorithm (Vlassis & Spaan, 2004) in which point-based value updates are not systematically applied to all points at each iteration. Instead, points are sampled randomly (and updated) until the value of all points has been improved; updating the α-vector at one point often also improves the value estimate of other nearby points.

The HSVI (Smith & Simmons, 2004) algorithm also uses a different approach to ordering value updates: whenever a belief point is expanded from the belief tree, HSVI updates only the value of its direct ancestors (parents, grand-parents, etc., all the way back to the initial belief in the head node). This is in contrast to PBVI which performs a batch of belief point expansions, followed by a batch of value updates over all points. HSVI also uses both a lower and an upper bound to select belief points (while PBVI in principle also uses both a lower and upper, its upper bound is very loose, i. e. $\frac{R_{max}}{1-\gamma}$). In other respects, HSVI and PBVI share many similarities: both offer anytime performance, theoretical guarantees, scalability.

7. Discussion

The main contributions pertaining to the PBVI framework are summarized in this section.

Scalability. The primary contribution of the PBVI framework is to provide scalable POMDP solutions. This is achieved by bounding the policy size.

Anytime planning. PBVI-class algorithms alternates between steps of value updating and steps of belief point selection. As new points are added, the solution improves, at the expense of increased computational time. The trade-off can be controlled by adjusting the number of points. The algorithm can be terminated either when a satisfactory solution is found, or when planning time is elapsed.

Bounded error. An important contribution of the PBVI framework is the theoretical guarantees it provides: the theoretical properties described in Section 3.2 are more widely applicable and provide stronger error bounds than what was available prior to this work. Furthermore, these theoretical results are important because they directly lead to the development of the most powerful PBVI-type algorithm to date: PBVI+GER, where estimates of the error bound are used directly to select belief points.

Exploration. We proposed a set of new point selection heuristics, which explore over the tree of reachable beliefs to select useful belief points. The most successful technique described, Greedy Error Reduction (GER), uses an estimate of the error bound on candidate belief points to select the most useful points.

Improved empirical performance. PBVI has demonstrated the ability to reduce planning time for a number of well-known POMDP problems, including Tiger-grid, Hallway, and Hallway2. By operating on a set of discrete points, PBVI algorithms can perform polynomial-time value updates, thereby overcoming the curse of history that paralyzes exact algorithms. The GER technique used to select points allows us to solve large problems with fewer belief points than alternative approaches.

New problem domain. PBVI was applied to a new POMDP planning domain (Tag), for which it generated an approximate solution that outperformed baseline algorithms QMDP and Incremental Pruning. This new domain has since been adopted as a test case for other algorithms (Vlassis & Spaan, 2004; Smith & Simmons, 2004; Braziunas & Boutilier, 2004; Poupart & Boutilier, 2004). This fosters an increased ease of comparison between new techniques.

Demonstrated performance. PBVI was used in the context of a robotic search-and-rescue type scenario, where a mobile robot is required to search its environment and find a non-stationary individual. PBVI's performance was evaluated using a realistic, independently-developed, robot simulator.

Extensions to the PBVI framework discussed here, whereby value updates are applied to groups of belief points according to their spatial distribution, are described in (Pineau, Gordon, & Thrun, 2003a).

7.1 Future work

This paper describes a new class of POMDP algorithms which features both good empirical performance, in simulation and robotic tasks, as well as solid theoretical guarantees.

While we have demonstrated the ability to solve problems on the order of 10^3 states, many real-world domains far exceed this. In particular, it is not unusual for a problem to be expressed through a number of multi-valued state *features*, in which case the number of states grows exponentially with the number of features. This is of concern because each belief point and each α-vector has dimensionality $|S|$ (where $|S|$ is the number of states) and all dimensions are updated simultaneously. This is an important issue to address to improve the scalability of point-based value approaches.

There are various existing attempts at overcoming the curse of dimensionality in POMDPs. Some of these—e. g. the belief compression techniques of Roy and Gordon (2003)—cannot be incorporated within the PBVI framework without compromising its theoretical properties (as discussed in Section 3.2). Others, in particular the exact compression algorithm of (Poupart & Boutilier, 2003), can be combined with PBVI. However, preliminary experiments in this direction have yielded little performance improvement. There is reason to believe that approximate value compression would yield better results, but again at the expense of forgoing PBVI's theoretical properties. The challenge therefore is to devise function-approximation techniques that both reduce the dimensionality effectively, while maintaining the convexity properties of the solution.

A secondary (but no less important) issue concerning the scalability of PBVI pertains to the number of belief points necessary to obtain a good solution. While problems addressed thus far can usually be solved with $O(|S|)$ number of belief points, this need not be true. In the worse case, the number of belief points necessary may be exponential in the plan length. The PBVI framework can accommodate a wide variety of strategies for generating belief points, and the Greedy Error Reduction technique seems particularly effective. However this is unlikely to be the definitive answer to belief point selection. In more general terms, this relates closely to the well-known issue of exploration versus exploitation, which arises across a wide array of problem-solving techniques.

Acknowledgments

The authors wish to thank Craig Boutilier, Michael Littman, Andrew Moore and Matthew Mason for many thoughtful comments and discussions regarding this work. We also thank Darius Braziunas, Pascal Poupart, Trey Smith and Nikos Vlassis, for conversations regarding their algorithms and results. The contributions of Michael Montemerlo and Nicholas Roy in conducting the empirical robot evaluations are gratefully acknowledged. This work was funded through the DARPA MARS program and NSF's ITR program.

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