COMP 102: Excursions in Computer Science

Lecture 19: Complexity and Tractability

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Class web page: www.cs.mcgill.ca/~jpineau/comp102

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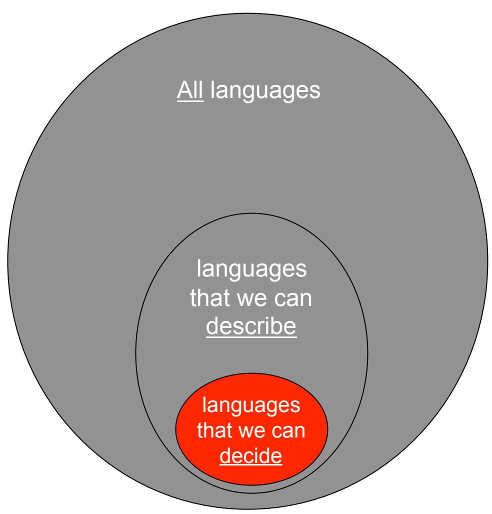
Computability Theory



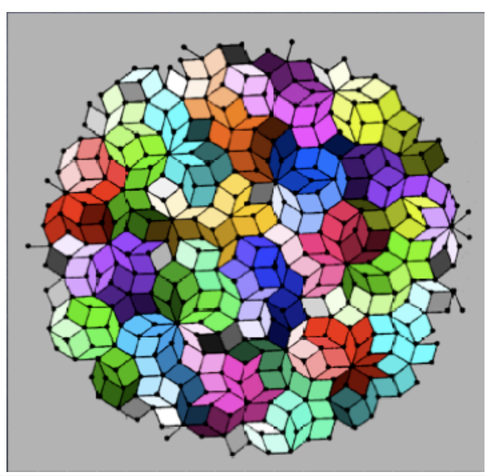
The map-colouring problem

Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?

Not all decidable problems are equally hard to solve!

The map-colouring problem

Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?



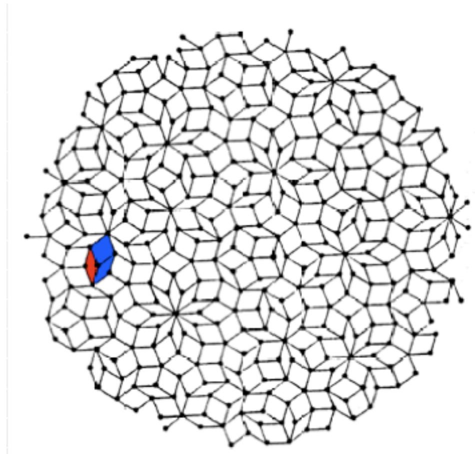
Yes! If you can use as many colours as you like.

The map-colouring problem

Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?

Rarely possible with only <u>two</u> colours!

But we find this out quickly.

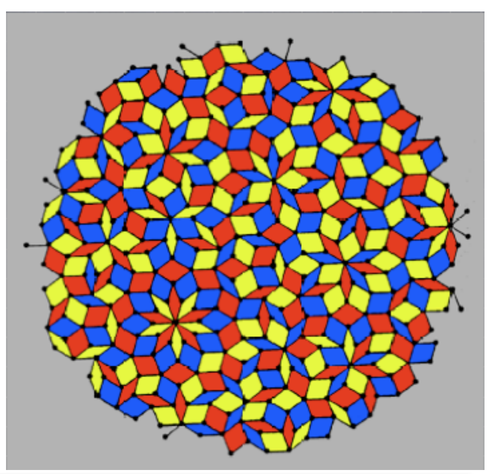


The map-colouring problem

Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?

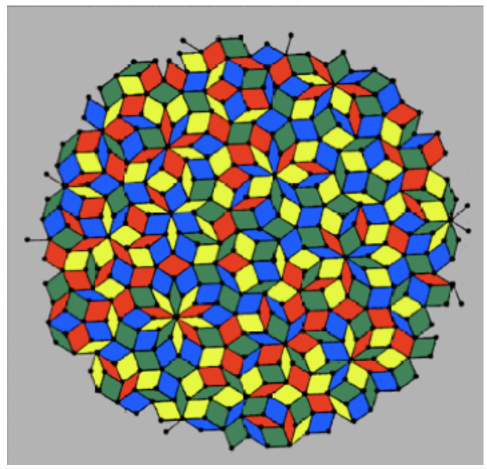
Somewhat easier with <u>three</u> colours, though not always possible.

But can take long time to decide.



The map-colouring problem

Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?



Much easier with

<u>four</u> colours!

But is it always possible?

K-colouring of maps (planar graphs)

Question: Can we colour any given map with at most K colours?

- K=1, Easy to decide. Only maps with zero or one region are 1-colourable.
- K=2, Easy to decide. Impossible as soon as 3 regions touch each other.
- K=3, No known <u>efficient</u> algorithm to <u>decide</u>. However it is easy to verify a solution.
- K≥4, All maps are K-colourable (hard proof). Does not imply it is easy to find a K-colouring for a given map.

All these are <u>decidable</u> problems. But of different difficulty.

3-colouring of maps

- Seems hard to solve in general.
  - If I give you a map, it's hard to find a correct colouring, or be able to tell that no colouring is possible.
- It's easy to verify when a solution is given.
  - If I give you a map and a colouring, it's easy for you to check whether the colouring is correct (o.e. no adjacent cell has same colour.)
- This is a special type of problem, called **NP-complete**.
  - Hard to solve in general, but easy to verify whether a given solution is correct.

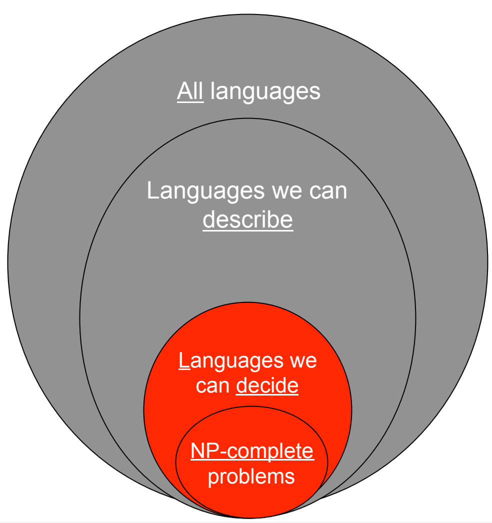
NP-complete problems

- **NP** = **N**ondeterministic **P**olynomial Time
- Many practical problems are NP-complete.
  - Some books list hundreds of such problems.
- If any of them is easy, they are all easy. (Remember what we said about reducing a problem to another when we discussed decidability.)
- In practice, some of them may be solved efficiently in some special cases.

Examples of NP-complete problems

- Boolean satisfiability: Given a Boolean expressions, is there an assignment of the boolean variables making the formula evaluate to true ?
- 3-SAT: Similar to Boolean satisfiability, but the expression must contain only 3-variable sub-expressions, separated by AND statements.
- Travelling Salesman: Given a set of cities and distances between them, what is the shortest route to visit each city once.
- KnapSack: Given items with various weights, is there of subset of them of total weight K.

Computability Theory



Boolean satisfiability example

- Question: Is there an assignment of the boolean input variables {A, B, C, D} that makes the output E evaluate to true ?

  E = (A OR (NOT B) OR (NOT C)) AND (A OR B OR D)

- Complexity of Boolean satisfiability:
  - How many operations to find a solution?
  - How many operations to verify a given solution?
  - How does this change as a function of the number of input variables?

Reduction

How do we know if a new problem is NP-complete?

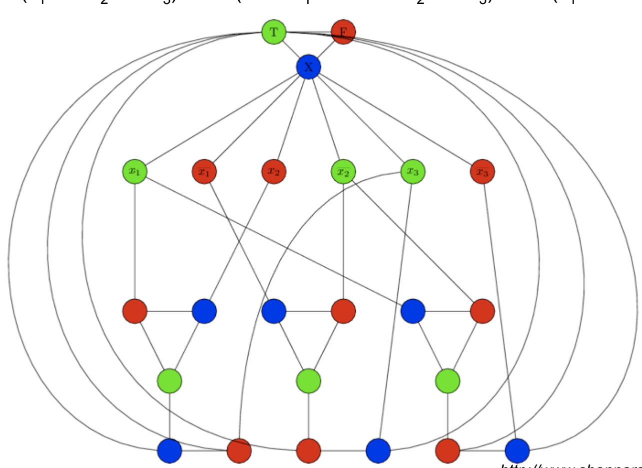
Reduce a known NP-complete problem to this new problem.

Reduction from 3-SAT to graph colouring

- Construct a graph that will be 3-colorable if and only if the 3-SAT instance is satisfiable.
- Graph construction:
  - Create 3 special nodes T, F, X, joined in a triangle. (Note: T=True, F=False, X=other)
  - For each variable in the SAT problem, create 2 nodes x_i and NOT(x_i), connected by an edge. Connect all of these nodes to X.
  - Add 5 nodes and 10 edges to represent each 3-variable sub-clause.
- Colouring rules:
  - Each x_i, NOT(x_i) pair must be a different colour.
  - Each must be a different colour than X.
  - X, T, and F must get different colors.
- Any 3-colouring of this graph gives a valid solution to the boolean expression.

Reduction from 3-SAT to graph colouring

E = (x_1 OR x_2 OR x_3) AND (NOT x_1 OR NOT x_2 OR x_3) AND (x_1 OR NOT x_2 OR NOT x_3)



http://www.shannarasite.org/kb/kbse60.html

Tractable Problems (P)

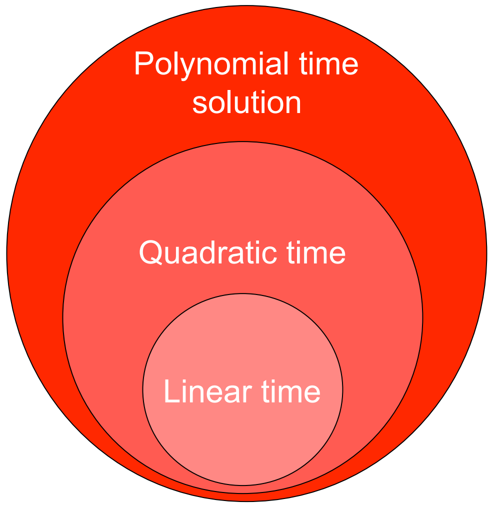
- Fortunately, many practical problems are tractable.
- The name P stands for Polynomial- Time computable.
  - Polynomial = constant, linear, quadratic, … But not exponential!
- Computer scientists spend most of their time finding efficient solutions to tractable problems.
  - But also lots of useful work in figuring out how to get approximate solutions for non-tractable (i.e. NP-complete) problems.

Examples of Tractable Problems (P)

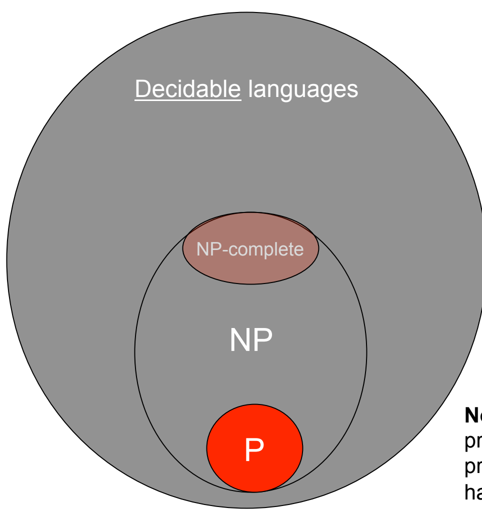
- 2-colorability of maps.
- Primality testing.
- Solving NxNxN Rubik's cube.
- Finding a word in a dictionary.
- Sorting elements in a list.

These are "easy" problems!

Complexity of tractable problems



Complexity Theory



P = NP?

Note: NP-complete problems are those problems which are the hardest in the NP class.

P=NP?

- THE fundamental question of theoretical computer science:

  For all problems for which a computer can *verify* a given solution quickly (i.e. in polynomial time), can the computer also *find* that solution quickly.

- This is the million dollar question! (Quite literally…)

  http://www.claymath.org/millennium/P_vs_NP/

- In 2002, 100 researchers were asked "Do you think P=NP?"
  - 61 believed the answer is no,
  - 9 believed the answer is yes
  - 22 were unsure
  - 8 believed the question may be impossible to prove or disprove.

Some progress?

P $\neq$ NP

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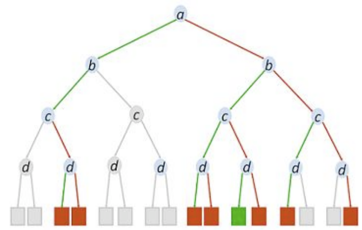
Approximating NP-hard problems

- Boolean satisfiability: given a boolean expression, is there an assignment of the boolean variables making the formula evaluate to true ?
- Small expressions can be solved by searching through all assignments.



- Large expressions can (sometimes) be solved by random search. E.g. Genetic algorithms.

Beyond NP-completeness

- **PSPACE Completeness**: Problems that require a polynomial (P) amount of memory (i.e. space) to be solved.
- There are many such problems.
- We currently don't know if **P = PSPACE**.
- Many other complexity classes exist (in addition to P, NP, PSPACE), e.g. EXPTIME, EXPSPACE, Co-NP, PP, etc.

PSPACE completeness

- Geography Game:

  Given a set of country names:

  Angola, Canada, Cuba, France, Italy, Japan, Korea, Vietnam

- A two player game:

  One player chooses a name. The other player must choose a name that starts with the last letter of the previous name and so on. A player wins when his opponent cannot play any name.

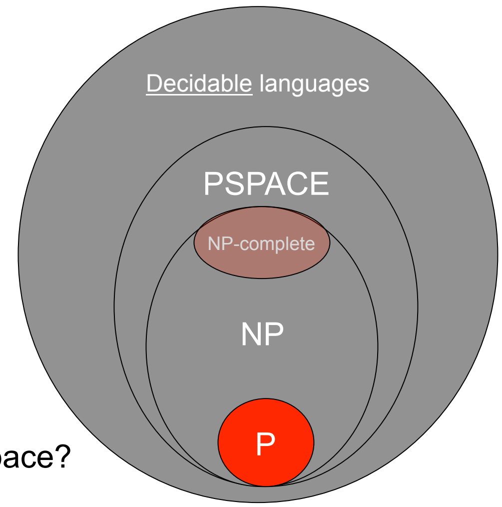
Generalized Geography

- Given an arbitrary set of names: w_1, ..., w_n.

- Is there a winning strategy for the first player to the previous game ?

- Can answer this by storing only a list of all the names.
  - So memory (space) requirement of the algorithm is polynomial in the size of the problem (n).

Complexity Theory



NP = P-space?

Theoretical Computer Science

- Challenges of theoretical computer science:
  - FIND efficient solutions to many problems.
  - PROVE that certain problems are NOT computable within a certain time or space.

- Consider new models of computation.

  (Such as a Quantum Computing – more on this in a few weeks.)

Take-home message

- Know the difference between problems that are P (known to be "easy") and those that are NP (possibly "hard").
- Be able to name some examples of NP-complete problems.
- Be able to name some examples of tractable (polynomial-time) problems.
- Understand the idea of reduction, as used in complexity theory.