COMP 102: Excursions in Computer Science
Lecture 18: Computability

Instructor: Joelle Pineau (jpineau@cs.mcgill.ca)

Class web page: www.cs.mcgill.ca/~jpineau/comp102

David Hilbert

- In 1900, German mathematician David Hilbert presented 23 unsolved problems in mathematics.
  - Several of them turned out to be very influential for 20th century mathematics.
  - See the full list: *http://en.wikipedia.org/wiki/Hilbert's_problems*
- Ten of these problems were the subject of a famous lecture on 8 August 1900, at the 2nd International Congress of Mathematicians, at La Sorbonne in Paris.

Problem #2

Can we prove all the mathematical statements that we can formulate?

Certainly, there are many mathematical problems that we know how to state, but do not know how to solve.

Is this just because we are not smart enough to find a solution ?

Or, is there something deeper going on ?

Consider this

This statement is false.

This statement is unprovable.

Kurt Gödel

- In 1931, he proved that any formalization of mathematics contains some statements that cannot be proved or disproved.

  => **Godel's incompleteness theorem.**

Computer science version of this question

- If my boss / supervisor / teacher formulates a problem to be solved urgently, can I write a program to solve this problem in an efficient manner ?

- Issue #1 (Computability) : Are there some problems that cannot be solved at all ? (related to Hilbert's problem #2)

- Issue #2 (Complexity): Are there problems that cannot be solved efficiently ? (related to Hilbert's problem #10)

Alan Turing

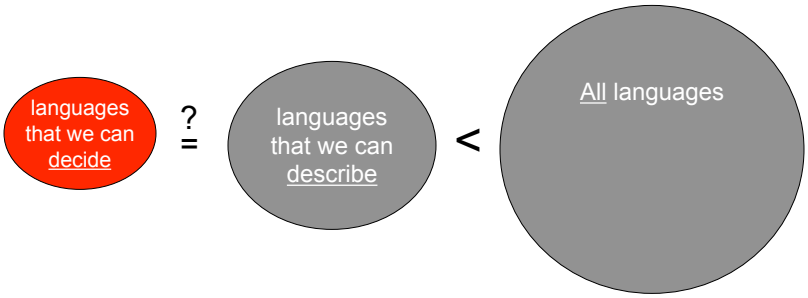
- In 1934, he formalized the notion of decidability of a language by a computer.
- What else do we know about Turing?

  (Yet more to come…)

A language

- Let Σ be a finite alphabet. (ex: {0,1})
- Let Σ^* be all sequences of elements from this alphabet. (ex: 0, 1, 00000, 0101010101,...)
- A language L is any subset of Σ^*.
- An algorithm decides a language if it answers *Yes* whenever a given input x is in L, and *No* otherwise.

Comparing cardinalities

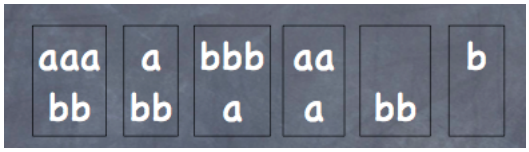


Emil Post

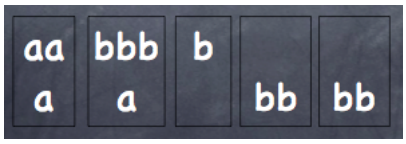
- In 1946, he gave a very natural example of an undecidable language.

Post Correspondence Problem (PCP)

- An instance of PCP with 6 tiles.

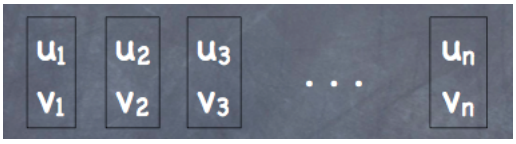


- A solution to PCP.

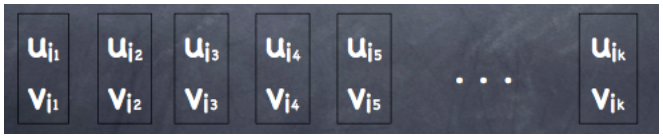


Post Correspondence Problem (PCP)

- Given n tiles, u_1/v_1 ... u_n/v_n where each u_i or v_i is a sequence of letters.



- Is there a k and a sequence $< i_1, i_2, i_3, ..., i_k >$ (with each $1<i_j<n$) such that $u_{i1} \mid u_{i2} \mid u_{i3} \mid ... \mid u_{ik} = v_{i1} \mid v_{i2} \mid v_{i3} \mid ... \mid v_{ik}$?



Post Correspondence Problem (PCP)

- Theorem:

  The Post Correspondence Problem cannot be **decided** by any algorithm (or computer program).

  In particular, **no algorithm can identify in a finite amount of time** the instances that have a **negative outcome**.

  However, if a solution exists, we can find it.

What's a theorem? *An idea that has been demonstrated as true.*

Post Correspondence Problem (PCP)

- Theorem:

  The Post Correspondence Problem cannot be **decided** by any algorithm (or computer program).

  In particular, **no algorithm can identify in a finite amount of time** the instances that have a **negative outcome**.

  However, if a solution exists, we can find it.

- Proof: Reduction technique - if PCP was decidable, then another problem would be decidable.

The Halting Problem

- Notice the following:
  - an algorithm is a piece of text.
  - an algorithm can receive text as input.
  - an algorithm can receive an algorithm as input.

The Halting Problem:

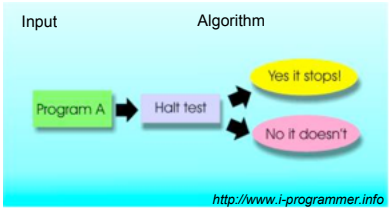
Given two versions A,A', consider A as an algorithm and A' as a text. Will algorithm A halt (as opposed to loop forever) on input A'?

The Halting Problem

- Theorem: No algorithm can decide the Halting Problem.
- Proof (by contradiction):

  Assume for a moment that an algorithm `Halting(A)` exists that can correctly decide the Halting Problem.

  Consider this algorithm:



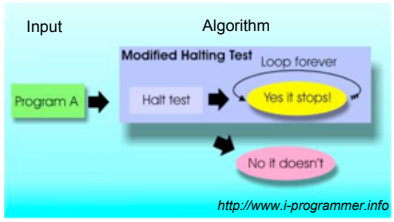
It starts off innocently enough – just create a program that tells you if another program stops or loops forever when reading itself as input

The Halting Problem

- Theorem: No algorithm can decide the Halting Problem.
- Proof (by contradiction):

  Assume for a moment that an algorithm `Halting(A)` exists that can correctly decide the Halting Problem.

  Consider this algorithm:



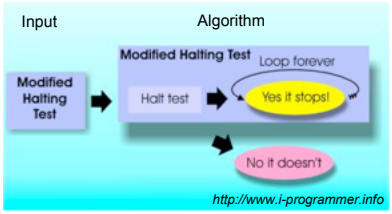
A slight modification is to add a loop that never ends if the program being tested ends

The Halting Problem

- Theorem: No algorithm can decide the Halting Problem.
- Proof (by contradiction):

  Assume for a moment that an algorithm `Halting(A)` exists that can correctly decide the Halting Problem.

  Consider this algorithm:



Now we hit the problem. Feed the modified program complete with a test subject into itself. Now if it stops it doesn't, but if it doesn't it does!

The Halting Problem

- If `Halting(Halting)` does not loop forever, it is because `HaltTest=False`, which means `Halting(Halting)` loops forever.

Contradiction!

- If `Halting(Halting)` loops forever it is because `HaltTest=True`, which means `Halting(Halting)` does not loop forever.

Contradiction!

Conclusion: **Halting() cannot exist.**

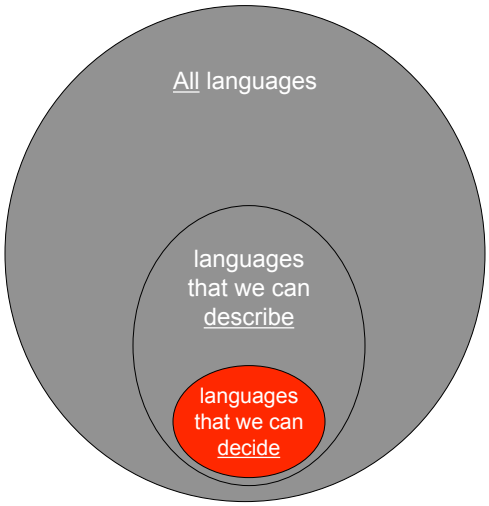
Alan Turing

- In 1934, he formalized the notion of decidability of a language by a computer.
- In 1936, he proved that an algorithm to solve the Halting problem cannot exist.

The Halting Problem and PCP

- Any algorithm to decide PCP can be converted to an algorithm to decide the Halting Problem.
- Conclusion: PCP cannot be decided either.

Computability Theory



Decidable Programs

Can we always tell if a program is decidable?

Sometimes we just don't know!

Syracuse Conjecture

For any integer $n>0$, define the following sequence:

$$s_1 = n$$

$$s_{i+1} = \begin{cases} s_i / 2 & \text{if } s_i \text{ is even} \\ 3s_i + 1 & \text{if } s_i \text{ is odd} \end{cases}$$

Then:

$$Syracuse(n) = \begin{cases} \text{least } i \text{ such that } s_1=n, \ldots, s_i=1, \text{ if it exists} \\ 0 \quad \text{if } s_i \neq 1 \text{ for all } i. \end{cases}$$

Example

- Syracuse(9) = 20

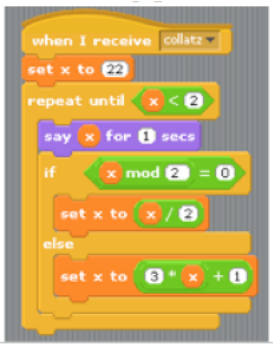
  S_1=9, S_2=28, S_3=14, S_4=7, S_5=22, S_6=11, S_7=34, S_8=17, S_9=52, S_{10}=26, S_{11}=13, S_{12}=40, S_{13}=20, S_{14}=10, S_{15}=5, S_{16}=16, S_{17}=8, S_{18}=4, S_{19}=2, S_{20}=1

- Easy cases: Syracuse(2^k) = k+1 for any integer k ≥ 0

- But not so easy for numbers which are not powers of 2!

Program to calculate Syracuse(n)

- Example for n=22:



Note: "n" is called "x" in this program.

Syracuse Conjecture

- Observation:
  - For all n that we have computed so far, Syracuse(n) > 0.
- Conjecture:
  - For all n>0, Syracuse(n)>0

But currently, no one knows if this program always stops!

- Problem:

  If there exists n such that Syracuse(n) = 0, we might not be able to prove it.

Syracuse Conjecture

- The Syracuse conjecture is believed to be true but no proof of that statement was discovered so far.
- It is an **open** problem.
- Even worse, it might be decidable, but there might be no proof that it is decidable !!!

Summary

- There are many problems that turn out to be undecidable.
  - All involve computations that might take an infinite number of operations to solve and you're never quite sure when to stop.
- It is useful to know which programs you should run, and which programs you shouldn't run!
- Showing that a problem is decidable often involves showing that this problem is analogous to another problem which we already know is decidable or not.

  E.g. PCP is not decidable because it is analogous to the Halting Problem.

Take-home message

- Know the difference between:
  - Languages that we can describe.
  - Languages that we can decide.
- Be familiar with the Post Correspondence Problem, and why it is not decidable.
- Understand the general idea of the Halting Problem.
- Be familiar with the Syracuse Conjecture.