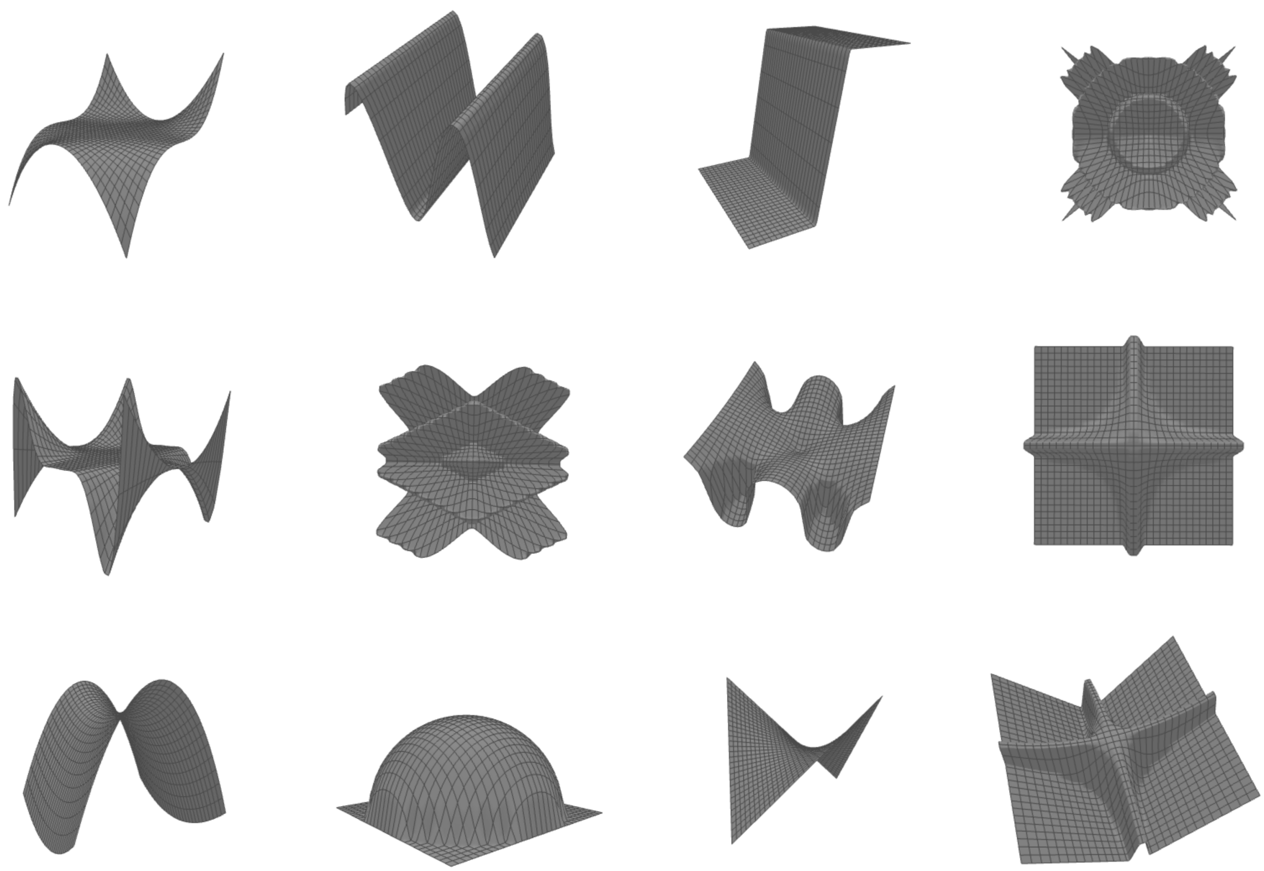
Semester: Fall 2022
Instructor: Kamran, Niky

Honours Vector Calculus



Course Content. Partial derivatives and differentiation of functions in several variables; Jacobians; maxima and minima; implicit functions. Scalar and vector fields; orthogonal curvilinear coordinates. Multiple integrals; arc length, volume and surface area. Line and surface integrals; irrotational and solenoidal fields; Green's theorem; the divergence theorem. Stokes' theorem; and applications.

Summary of Contents

The Geometry of Euclidean Space

n-Dimensional Euclidean Space

Let $\mathbb{R}^n$ be the vector space of n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with entries from $\mathbb{R}$, defined under the operations of coordinate-wise addition and multiplication. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have that,

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$$
$$c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n)$$

We will consider the Euclidean inner product on $\mathbb{R}^n$ defined by,

$$\langle \mathbf{x}, \mathbf{y} \rangle \longmapsto \mathbf{x} \cdot \mathbf{y}$$
$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{n} x_i \cdot y_i$$

For $\mathbf{x} \in \mathbb{R}^n$, we define the norm of $\mathbf{x}$ to be,

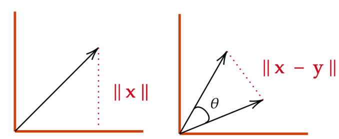
$$\|x\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

The Euclidean distance between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as,

$$d(\mathbf{x}, \mathbf{y}) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

The geometric significance of the norm $\| \cdot \|$ in $\mathbb{R}^2$ is shown below. Recall that,

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$



Remark. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we have,

1. $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
2. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
3. $\mathbf{x} \cdot \mathbf{x} \geq 0$
4. $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$

Theorem 1 (Cauchy-Schwartz Inequality)**.** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. If either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, then the statement holds trivially. Assume that this is not the case. Then,

$$\begin{aligned} p(\lambda) &:= (\mathbf{x} + \lambda\mathbf{y}) \cdot (\mathbf{x} + \lambda\mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \lambda \cdot \mathbf{y} \cdot \mathbf{x} + \lambda \cdot \mathbf{x} \cdot \mathbf{y} + \lambda^2(\mathbf{y} \cdot \mathbf{y}) \\ &= \underbrace{\|\mathbf{x}\|^2}_{c} + \lambda \cdot \underbrace{2(\mathbf{y} \cdot \mathbf{x})}_{b} + \lambda^2 \underbrace{\|\mathbf{y}\|^2}_{a} \geq 0 \end{aligned}$$

with the second equality holding by the commutativity of the dot product. We have a quadratic polynomial with discriminant,

$$4(\mathbf{x}\cdot\mathbf{y}) - 4\|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2$$

which must be non-positive because,

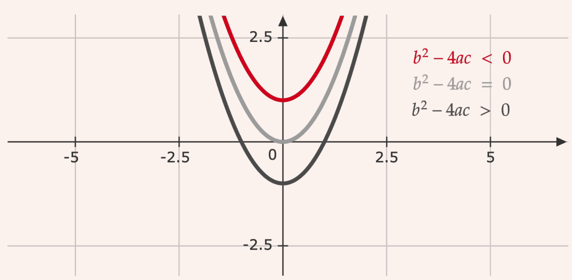
$$p(\lambda) \geq 0$$

Simplifying gives that $(\mathbf{x}\cdot\mathbf{y})^2 \leq \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2$ and therefore,

$$|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

□

Example 1: Characterizing $p(\lambda)$



Corollary (Triangle Inequality)**.** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then,

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Proof. We will consider the case where $\lambda = 1$.

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2|\mathbf{y}\cdot\mathbf{x}| + \lambda^2\|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\cdot\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2\end{aligned}$$

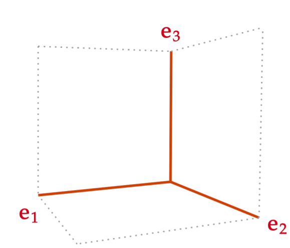
since $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\cdot\|\mathbf{y}\|$ by Cauchy-Schwartz. This gives that,

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$

which implies the desired result: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. □

An orientation is a choice of ordering for our basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. By convention,

$$\begin{aligned} \mathbf{e_1}\times\mathbf{e_2} &= \mathbf{e_3} & \mathbf{e_1}\times\mathbf{e_1} &= 0 \\ \mathbf{e_3}\times\mathbf{e_1} &= \mathbf{e_2} & \mathbf{e_2}\times\mathbf{e_2} &= 0 \\ \mathbf{e_2}\times\mathbf{e_3} &= \mathbf{e_1} & \mathbf{e_3}\times\mathbf{e_3} &= 0 \end{aligned}$$



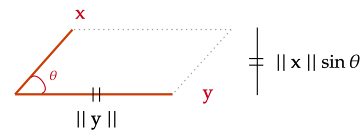
Understanding the Cross Product

Definition (Cross-Product). Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the standard basis of $\mathbb{R}^3$. The **cross-product** $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the map defined by,

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

for two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$.

$\mathbf{x} \times \mathbf{y}$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. Moreover, $\|\mathbf{x} \times \mathbf{y}\|$ is the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$.



Remark. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^3$. Then,

$$\mathbf{x} \times \mathbf{y} = (x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}) \times (y_1\mathbf{e_1} + y_2\mathbf{e_2} + y_3\mathbf{e_3})$$

Expanding and using the fact that,

$$\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}$$
$$\mathbf{e_1} \times \mathbf{e_3} = -\mathbf{e_2}$$
$$\mathbf{e_2} \times \mathbf{e_3} = \mathbf{e_1}$$

gives the following equality,

$$\mathbf{x} \times \mathbf{y} = \mathbf{e_3}(x_1y_2 - x_2y_1) - \mathbf{e_2}(x_1y_3 - x_3y_1) + \mathbf{e_1}(x_2y_3 - x_3y_2)$$

but this is the determinant of the matrix,

$$\begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

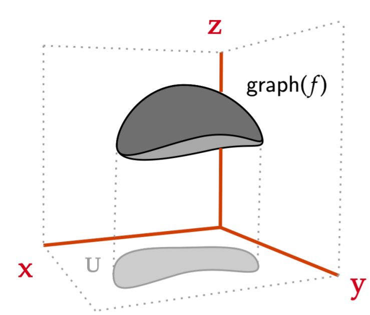
Corollary. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are linearly independent if and only if their cross-product is non-zero. This result does not hold in higher dimensions because the normal $\mathbf{n}$ satisfying,

$$\mathbf{x} \times \mathbf{y} = (\|\mathbf{x}\|\|\mathbf{y}\| \sin \Theta) \cdot \mathbf{n}$$

is no longer unique.

Graphs and Level-Sets

The graph of a function of two variables taking values in $\mathbb{R}$ is,



Definition (Graphs of Functions). The **graph** of a function,

$$f : U \subseteq \mathbb{R}^n \to \mathbb{R}$$

is the subset of $\mathbb{R}^{n+1}$ given by,

$$\texttt{graph}(f) := \left\{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in U\right\}$$

The level set is called a **level curve** if $n = 2$ and a **level surface** if $n = 3$.

Definition (Level Set). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. The **level set** of value c is the subset of $\mathbb{R}^n$ given by,

$$\{\mathbf{x} \in U \mid f(\mathbf{x}) = c\} = f^{-1}(\{c\})$$

Remark. If $c_1, c_2 \in \text{range}(f)$ are such that $c_1 \neq c_2$, then,

$$f^{-1}(\{c_1\}) \cap f^{-1}(\{c_2\}) = \emptyset$$

Examples of Graphs in $\mathbb{R}^3$

Example 2: Paraboloid of Revolution

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

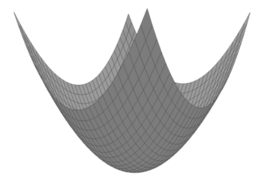
$$f(x, y) = x^2 + y^2$$

is a **paraboloid of revolution**. It has $\text{range}(f) = [0, \infty)$ and

$$f^{-1}(\{0\}) = \{(0, 0)\} \text{ and } f^{-1}(\{c\}) \text{ is a circle of radius } \sqrt{c}$$

Paraboloid of Revoluation

$f(x, y) = x^2 + y^2$



Example 3: Paraboloid of Translation

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

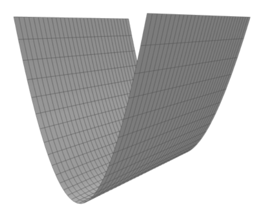
$$f(x, y) = x^2 + 1$$

is a **paraboloid of translation**. It has $\text{range}(f) = [1, \infty)$ and,

$$f^{-1}(\{c\}) \text{ is a pair of lines at } \pm\sqrt{c - 1}$$

Paraboloid of Translation

$f(x, y) = x^2 + 1$



Example 4: Lower Hemisphere

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

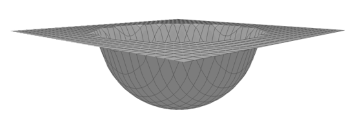
$$f(x, y) = -\sqrt{1 - (x^2 + y^2)}$$

is a **lower hemisphere**. It has $\text{range}(f) = [-1, 0]$ and,

$$f^{-1}(\{c\}) \text{ is a circle of radius } \sqrt{1 - c^2}$$

Lower Hemisphere

$f(x, y) = -\sqrt{1 - (x^2 + y^2)}$



Example 5: Connected Components of the Level Sets

Level sets of a single value need not belong to a single connected component. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = xy$$

has $\text{range}(f) = (-\infty, \infty)$. Geometrically,

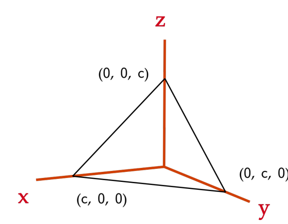
We can also analyze functions taking values from $\mathbb{R}^3$. For instance,

$$f(x, y, z) = x + y + z$$

has $\text{range}(f) = \mathbb{R}$ and

$$f^{-1}(\{c\}) = \{(x, y, z) \mid x + y + z = c\}$$

is a plane intersecting the x-axis at c. This can be visualized as follows,



Remark. We will briefly review the six **quadratic surfaces**. These are,

1. The **ellipsoid**, which is called a sphere when $a = b = c$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

2. The **elliptic paraboloid**, which is along the z-axis,

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

3. The **hyperbolic paraboloid**,

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

4. The **cone**,

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

5. The **hyperboloid of one sheet**, with sheets along z,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

6. The **hyperboloid of two sheets**, with sheets along z,

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Example 6: Quadratic Surfaces

The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = x^2 + y^2 + z^2$$

has range$(f) = (\infty, \infty)$ and,

$$f^{-1}(\{c\}) = 0 \text{ and } f^{-1}(\{c\}) \text{ is a sphere for } c > 0$$

If we had instead considered the function,

$$f(x,y,z) = x^2 - y^2 + z^2$$

then we would have had that,

$$f^{-1}(\{c\}) \text{ is a } \begin{cases} \text{Hyperboloid of Two Sheets} & \text{for } c < 0 \\ \text{Circular Cone} & \text{for } c = 0 \\ \text{Hyperboloid of One Sheet} & \text{for } c > 0 \end{cases}$$

Limits and Continuity

Limits of Functions

Definition (Open Disk). Let $\mathbf{x} \in \mathbb{R}^n$. Given $r > 0$,

$$D_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| < r\}$$

is the **open ball** of radius r centered at $\mathbf{x}$.

Definition (Open Subset). A subset $U \subset \mathbb{R}^n$ is **open** if,

$$\exists r > 0 \text{ such that } D_r(\mathbf{x}_0) \subseteq U \text{ for all } \mathbf{x}_0 \in U$$

Proposition 1. $D_r(\mathbf{x}_0)$ is **open** according to the preceding definition.

Proof. Let $\mathbf{x}_0$ be arbitrary. We need to show that there exists $s > 0$ such that $D_s(\mathbf{x}_0) \subseteq D_r(\mathbf{x}_0)$. Choose $s := r - \|\mathbf{x} - \mathbf{x}_0\|$. Now,

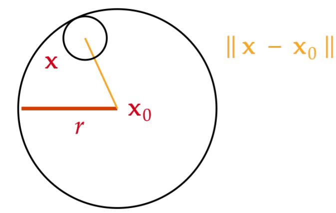
$$\|\mathbf{y} - \mathbf{x}_0\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{x}_0\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < s + \|\mathbf{x} - \mathbf{x}_0\|$$

since $\mathbf{y} \in D_s(\mathbf{x})$. By our choice of s, it follows that,

$$\|\mathbf{y} - \mathbf{x}_0\| < r$$

and consequently $D_r(\mathbf{x}_0)$ is **open**. □

Choosing s to prove that $D_r(\mathbf{x}_0)$ is open.



Definition (Boundary Point). We call $\mathbf{x} \in \mathbb{R}^n$ a **boundary point** of an open set A if every neighborhood of $\mathbf{x}$ contains a point in A and a point in A^c. We write ∂A for the set of boundary points of A.

We say that a **neighborhood** of $\mathbf{x} \in \mathbb{R}^n$ is an open set U such that $\mathbf{x} \in U$.

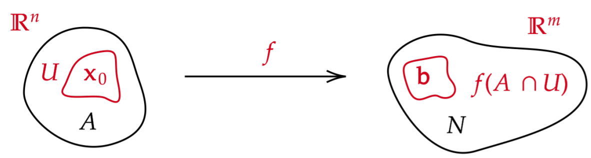
Corollary. $\partial D_r(\mathbf{x}_0) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\| = r\}$

Definition (Limit of a Function). Let $f : A \subseteq R^n \to \mathbb{R}^m$ be a function defined on an open subset A of $\mathbb{R}^n$. Let $\mathbf{x}_0 \in A \cup \partial A$. Then,

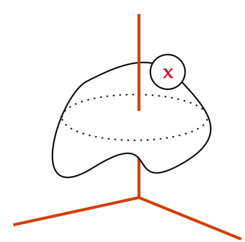
1. f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$ if $\exists U$, a neighborhood of $\mathbf{x}_0$, such that if $\mathbf{x} \neq \mathbf{x}_0$ and $\mathbf{x} \in A \cap U$, then $f(\mathbf{x}) \in N$
2. $f(\mathbf{x})$ approaches $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ if, given any neighborhood N of $\mathbf{b}$, f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$

where $\mathbf{b} \in \text{range}(f) \subseteq \mathbb{R}^m$. In either case, we write,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$



Example of a boundary point x.



The following are properties of limits of functions,

Remark (Uniqueness of Limits). Suppose that,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$$
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2$$

Then $\mathbf{b}_1 = \mathbf{b}_2$. That is, if f has a limit at $\mathbf{x}_0$, then that limit is unique.

Remark (Limit Properties). Suppose that $A \subset \mathbb{R}^n$, $\mathbf{x}_0 \in A \cup \partial A$, and f and g are functions on A taking values in $\mathbb{R}^m$. If we have that,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(x) = \mathbf{b}_1 \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2$$

Then the following properties hold,

1. $\lim_{\mathbf{x}\to\mathbf{x}_0} cf(x) = c\mathbf{b}_1$ for $c \in \mathbb{R}$
2. $\lim_{\mathbf{x}\to\mathbf{x}_0}(f(x) + g(x)) = \mathbf{b}_1 + \mathbf{b}_2$
3. If $m = 1$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(x) \cdot g(x) = \mathbf{b}_1 \cdot \mathbf{b}_2$
4. If $m = 1$ and $f(x) \neq 0\ \forall x \in A$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} 1/f(x) = 1/\mathbf{b}_1$

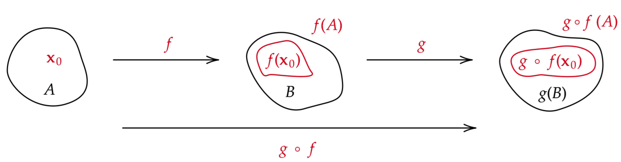
Continuity

Definition (Continuity). A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is called **continuous** on A if it is **continuous at every point $\mathbf{x}_0$** $\in A$.

Theorem 2 (Continuity of Compositions). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be two functions with $f(A) \subseteq B$. If f is continuous at $\mathbf{x}_0$ and g is continuous at $f(\mathbf{x}_0)$, then the composition,

$$g \circ f : A \subseteq \mathbb{R}^n \to \mathbb{R}^l$$

is continuous at $\mathbf{x}_0$.



We want to formulate the property of continuity precisely.

Theorem 3. Consider $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Let $\mathbf{x}_0 \in A$ or $\mathbf{x}_0 \cup \partial A$.

1. If $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ implies that,

   $$\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$$

   then we have that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$
2. f is continuous at $\mathbf{x}_0 \in A$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$

Example 7: Continuity via Composition

We can prove that the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by,

$$f(x,y,z) = e^{-(x^2+y^2+z^2)}$$

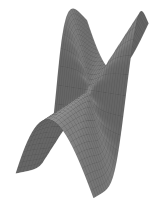
is continuous for all $\mathbf{x} \in \mathbb{R}^3$ using continuity of compositions:

1. $f_1(t) = e^{-t}$ is continuous for all $t \in \mathbb{R}$
2. $f_2(x,y,z) = -(x^2+y^2+z^2)$ is continuous for all $\mathbf{t} \in \mathbb{R}^3$

Thus, $f = f_1 \circ f_2 = e^{-(x^2+y^2+z^2)}$ is continuous for all $\mathbf{x} \in \mathbb{R}^3$.

The graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$f(x,y) = \frac{4x^2y}{x^2+y^2}$$



The graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

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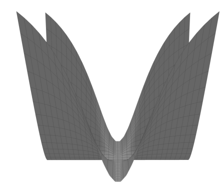


Example 8: $f(x,y) = x+y$

We will prove that the function,

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = x+y$$

is continuous. Let $\epsilon > 0$ be arbitrary and define $\delta(\epsilon) := \frac{\epsilon}{2}$. Suppose that $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Then,

$$|x+y-(x_0+y_0)| \le \underbrace{|x-x_0|}_{<\delta(\epsilon)} + \underbrace{|y-y_0|}_{<\delta(\epsilon)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since,

$$\delta > \|\mathbf{x}-\mathbf{x}_0\| = \sqrt{(x-x_0)^2+(y-y_0)^2} \ge \sqrt{(x-x_0)^2} = |x-x_0|$$

and

$$\delta > \|\mathbf{x}-\mathbf{x}_0\| = \sqrt{(x-x_0)^2+(y-y_0)^2} \ge \sqrt{(x-x_0)^2} = |y-y_0|$$

Example 9: $f(x,y) = 4x^2y/x^2+y^2$

We will prove that the function,

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \frac{4x^2y}{x^2+y^2}$$

approaches $\mathbf{0}$ as $(x,y) \to (0,0)$. Let $\epsilon > 0$ be arbitrary and define $\delta(\epsilon) = \frac{\epsilon}{4}$. Suppose that $\|\mathbf{x} - \mathbf{0}\| = \|\mathbf{x}\| < \delta$.

$$\left|\frac{4x^2y}{x^2+y^2}\right| \le \left|\frac{4x^2y}{x^2}\right| = 4|y| \le 4\|\mathbf{x}\| < \epsilon$$

Example 10: $f(x,y) = x^2y^2/x^2+y^2$

We will prove that the function,

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \frac{x^2y^2}{x^2+y^2}$$

approaches **0** as $(x,y) \to (0,0)$. Let $\epsilon > 0$ be arbitrary and define $\delta(\epsilon) = \sqrt{\epsilon}$. Suppose that $\|\mathbf{x} - \mathbf{0}\| = \|\mathbf{x}\| < \delta$.

$$\left|\frac{x^2y^2}{x^2+y^2}\right| = |x|^2 \cdot \underbrace{\left|\frac{y^2}{x^2+y^2}\right|}_{<1} \leq |x|^2 \leq \|\mathbf{x}\|^2 < \epsilon$$

Differentiation

Defining the Derivative

Given a function $f : U \subseteq \mathbb{R} \to \mathbb{R}$. The **derivative** of f is,

$$\frac{df}{dx}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We want to generalize this to functions of more than one variable.

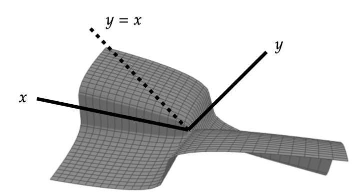
Definition (Partial Derivative)**.** Let $U \subseteq \mathbb{R}^n$ be an open set. Given a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, the **partial derivative** of f is,

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_n) = \lim_{h\to 0} \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

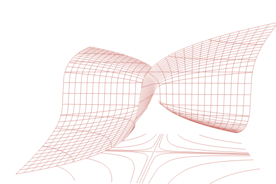
The graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$f(x,y) = x^{1/3} \cdot y^{1/3}$$

does not have a tangent plane at $(0,0)$,



Its contour plot is shown below,



Remark. If the partial derivative of a function, e.g., $f : \mathbb{R}^2 \to \mathbb{R}$, is being evaluated at a point, e.g., $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$, we write,

$$f_x(x_0,y_0) \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0,y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0}$$

Example 11: Existence of the Partial Derivatives

Let $f(x,y) = x^{1/3} \cdot y^{1/3}$. By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Thus, $f_y(0,0) = f_x(0,0) = 0$. Consider the restriction of $f(x,y)$ to the line $y = x$. We obtain $f(x,x) = x^{2/3}$, which we know from one-variable calculus is not differentiable at $(0,0)$.

Example 12: Computing Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = e^{x^2 y}$. We compute,

$$\frac{\partial f}{\partial x} = e^{x^2 y} \cdot 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x^2 y} \cdot x^2$$

Definition (Linear Approximation). The **linear approximation** $Lf|_{\mathbf{x}_0}$ to $\mathtt{graph}(f)$ at the point $\mathbf{x}_0 = (x_0, y_0)$ is given by,

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

Definition (Differentiability). A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **differentiable** at $\mathbf{x}_0$ if the partial derivatives f_x and f_y exist and we have that,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - Lf|_{\mathbf{x}_0}}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

Corollary. If f is differentiable at $\mathbf{x}_0$, then $Lf|_{\mathbf{x}_0}$ is called the **tangent plane** of $\mathtt{graph}(f)$ at the point $(x_0, y_0, f(x_0, y_0)) \in \mathbb{R}^3$.

Example 13: Finding the Equation of the Tangent Plane

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

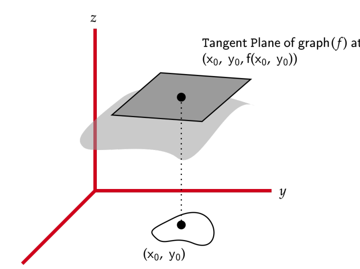
$$f(x,y) = 1 - x^2 - y^2$$

The equation of the tangent plane at,

$$(x_0, y_0) = (1/\sqrt{3}, 1/\sqrt{3})$$

is given by,

$$z - 1/3 = -\frac{2}{\sqrt{3}}\left(x - \frac{1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\left(y - \frac{1}{\sqrt{3}}\right)$$



Definition (Jacobian). Let $U \subseteq \mathbb{R}^n$ be an open set. Given a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, the **derivative** $(\mathbf{D}f)(\mathbf{x})$ is,

$$(\mathbf{D}f)(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

which we call the **Jacobian matrix**.

Remark. The Jacobian matrix can be thought of as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. When $m = 1$, the **gradient** ∇f is the $1 \times n$ matrix,

$$\nabla f := (\mathbf{D}f) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}_0$ if the partial derivatives exist at $\mathbf{x}_0$ and,

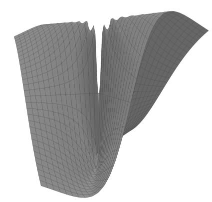
$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\| f(\mathbf{x}) - (Lf|_{\mathbf{x}_0}) \|}{\| \mathbf{x} - \mathbf{x}_0 \|} = 0$$

where,

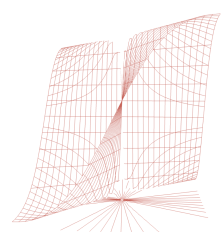
$$Lf|_{\mathbf{x}_0} = \underbrace{f(\mathbf{x}_0)}_{\in \mathbb{R}^m} + \underbrace{(\mathbf{D}f)(\mathbf{x}_0)}_{\mathbb{R}^n \to \mathbb{R}^m} \cdot \underbrace{(\mathbf{x} - \mathbf{x}_0)}_{\in \mathbb{R}^m}$$

The graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$



Its contour plot is shown below,



Theorem 4. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then,

1. If f is differentiable at $\mathbf{x}_0$, then f is continuous at $\mathbf{x}_0$
2. If the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous in a neighborhood of $\mathbf{x}_0$, then f is differentiable at $\mathbf{x}_0$

Example 14: Continuity of the Partial Derivatives

The existence of the partial derivatives does not guarantee continuity. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

The partial derivatives of f exist,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$$

but f is not continuous at the origin,

$$\lim_{x\to 0} f(x,x) = \frac{1}{2} \neq f(0,0)$$

$\mathtt{graph}(f)$ is shown in the margin.

Sums, Products, and Quotients

Theorem 5 (Sums, Products, and Quotients)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in U$.

1. **(Constant Rule)** Let $c \in \mathbb{R}$. $cf(x)$ is differentiable at $\mathbf{x}_0$, and,

$$(\mathbf{D}cf)(\mathbf{x}_0) = c(\mathbf{D}f)(\mathbf{x}_0)$$

2. **(Sum Rule)** $f(x) + g(x)$ is differentiable at $\mathbf{x}_0$, and,

$$(\mathbf{D}(f+g))(\mathbf{x}_0) = (\mathbf{D}f)(\mathbf{x}_0) + (\mathbf{D}g)(\mathbf{x}_0)$$

The following two properties hold when $m = 1$,

1. **(Product Rule)** $g(x)f(x)$ is differentiable at $\mathbf{x}_0$, and,

$$\mathbf{D}(gf)(\mathbf{x}_0) = g(\mathbf{x}_0)(\mathbf{D}f)(\mathbf{x}_0) + f(\mathbf{x}_0)(\mathbf{D}g)(\mathbf{x}_0)$$

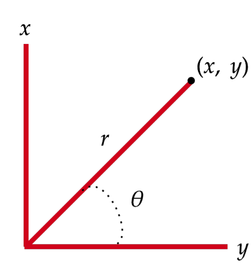
2. **(Quotient Rule)** $f(x)/g(x)$ is differentiable at $\mathbf{x}_0$, and,

$$\mathbf{D}(f/g)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)(\mathbf{D}f)(\mathbf{x}_0) + f(\mathbf{x}_0)(\mathbf{D}g)(\mathbf{x}_0)}{(g(\mathbf{x}_0))^2} \quad \text{and} \quad g(\mathbf{x}_0) \neq 0$$

Chain Rule

Theorem 6 (Chain Rule)**.** Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : U \subset \mathbb{R}^m \to \mathbb{R}^p$. If g is differentiable at $\mathbf{x}_0$ and f is differentiable at $g(\mathbf{x}_0)$, then,

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = (\mathbf{D}f)(\mathbf{y}_0)(\mathbf{D}g)(\mathbf{x}_0)$$



Example 15: Polar Coordinates

We can relate a set of **polar coordinates** (r, θ) to each point $(x, y) \in \mathbb{R}^2$ expressed in **Cartesian coordinates**. Observe,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$. Consider the composition,

$$(r, \theta) \overset{f}{\longmapsto} (x = r\cos\theta, y = r\sin\theta) \overset{g}{\longrightarrow} g(x, y)$$

Fix a point $\mathbf{x}_0 = (r, \theta)$. We will compute $(\mathbf{D}g)(\mathbf{x}_0)$:

$$(\mathbf{D}f)(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$(\mathbf{D}g)\left(f\left(\mathbf{x}_0\right)\right) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} (x = r\cos\theta, y = r\sin\theta)$$

Therefore,

$$\begin{aligned} \mathbf{D}(g \circ f) &= \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \left(\frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta - \frac{\partial g}{\partial x} r\sin\theta + \frac{\partial g}{\partial y} r\cos\theta \right) \end{aligned}$$

by the Chain Rule.

Example 16: Simple Function Composition

Consider $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$\begin{aligned} f(t) &= (t, t^2) \\ g(x, y) &= x^2 + y^2 \end{aligned}$$

We begin by computing the partial derivatives of f and g,

$$\nabla g = (2x \quad 2y) \quad \text{and} \quad \nabla f = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

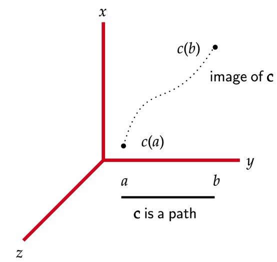
We can compute $(\mathbf{D}g \circ f)(t)$ as follows,

$$\begin{aligned} (\mathbf{D}g \circ f)(t) &= (\mathbf{D}g)(f(t)) \cdot (\mathbf{D}f)(t) \\ &= \left(2t, 2t^2\right) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} \\ &= 2t + 2t \\ &= 4t \end{aligned}$$

since $(\mathbf{D}g)(f(t)) = \left(2t, 2t^2\right)$.

Paths and Curves

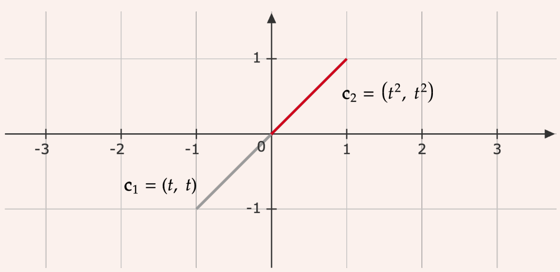
Definition (Curve). A **curve** is the image of a set of real numbers, called a **path**. We write t for the independent variable, so that $\mathbf{c}(t)$ is its position. The path $\mathbf{c}$ is said to **parameterize** the curve.



Example 17: Two Simple Curves

Define two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ on the interval $[0,1]$ as follows,

$$\mathbf{c_1}(t) = (t,t)$$
$$\mathbf{c_2}(t) = (2t,2t)$$



Observe that $\mathbf{c}_1([0,1]) = \mathbf{c}_2([0,1])$ but $\mathbf{c}_1'(t) \neq \mathbf{c}_2'(t)$,

$$\mathbf{c}_1'(t) = (1,1) \qquad \mathbf{c}_2'(t) = (2t,2t)$$

$\|\mathbf{c}_1(t)\|$ is constant, but $\|\mathbf{c}_1(t)\|$ is not.

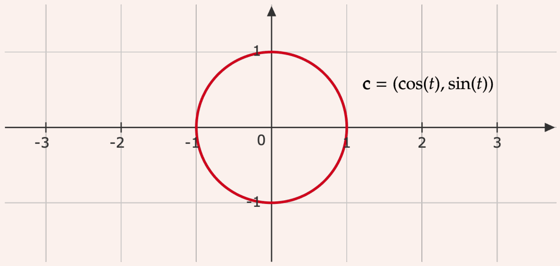
$$\|\mathbf{c}_1(t)\| = \|(1,1)\| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\|\mathbf{c}_2(t)\| = \sqrt{4t^2+4t^2} = 2\sqrt{2}t$$

Example 18: Unit Circle

Define a curve $\mathbf{c}$ on the interval $[0,2\pi]$ as follows,

$$\mathbf{c}(t) = (\cos t, \sin t)$$

The unit circle $\{(x,y) \mid x^2+y^2=1\}$ is parameterized by $\mathbf{c}$.



Theorem 7 (Differentiation of Paths)**.** If a path $\mathbf{c}$ with component functions $x_1(t), \cdots, x_n(t)$ is differentiable, then its derivative is

$$z'(t) = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

Proposition 2. $\mathbf{c}'(t)$ is the **tangent vector** at the point $\mathbf{c}(t)$.

We can apply the typical differentiation rules to the components of $\mathbf{c}(t)$.

Directional Derivatives

Definition (Directional Derivative)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$. The **directional derivative** of f at $\mathbf{x}_0$ along the vector $\mathbf{v}$ is,

$$(\nabla_{\mathbf{v}} f)(\mathbf{x}_0) = \frac{d}{dt}\bigg|_{t=0} f(\mathbf{x}_0 + t\mathbf{v}) = \lim_{h \to 0} \frac{1}{h}\left(f(\mathbf{x}_0 + h\mathbf{v}) - f(\mathbf{x}_0)\right)$$

Remark. Conventionally, we take $\mathbf{v}$ so that $\|\mathbf{v}\| = 1$.

Example 19: Computing Rate of Change in a Direction

We will compute the rate of change of

$$f(x, y) = \left(x^2 + y^2\right) \cdot e^{-\left(x^2+y^2+10\right)}$$

at $(2, 1)$ in the direction pointing towards $(0, 0)$. To do this, we will find $(\nabla_{\mathbf{v}} f)(2, 1)$, where $\mathbf{v}$ is the unit vector pointing from $(2, 1)$ towards $(0, 0)$. We require the partial derivatives,

$$f_x = 2x \cdot e^{-\left(x^2+y^2+10\right)} + \left(x^2 + y^2\right) \cdot e^{-\left(x^2+y^2+10\right)}(-2x)$$
$$f_y = 2y \cdot e^{-\left(x^2+y^2+10\right)} + \left(x^2 + y^2\right) \cdot e^{-\left(x^2+y^2+10\right)}(-2y)$$

Evaluating f_x and f_y at $(2,1)$,

$$f_x(2,1) = -16e^{-15}$$
$$f_y(2,1) = -8e^{-15}$$

We obtain the final result,

$$(\nabla_{\mathbf{v}} f)(2,1) = \frac{256}{\sqrt{5}} \cdot e^{-15}$$

Theorem 8. $(\nabla_{\mathbf{v}} f)(\mathbf{x}_0) = \mathbf{v} \cdot (\nabla f)(\mathbf{x}_0)$

Proof. Observe that,

$$\begin{aligned} \frac{d}{dt} f(\mathbf{x}_0 + t\mathbf{v}) &= (\nabla f)(\mathbf{x}_0 + t\mathbf{v}) \cdot \frac{d}{dt}(\mathbf{x}_0 + t\mathbf{v}) \\ &= (\nabla f)(\mathbf{x}_0 + t\mathbf{v}) \cdot \mathbf{v} \end{aligned}$$

Plugging in $t = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} f(\mathbf{x}_0 + t\mathbf{v}) = (\nabla f)(\mathbf{x}_0) \cdot \mathbf{v}$$

□

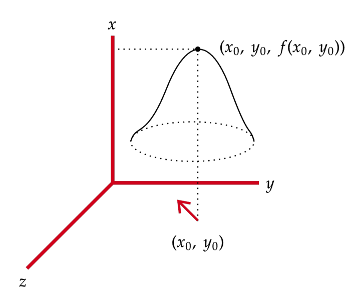
Corollary. If $\mathbf{x}_0 \in U$ is such that $(\nabla f)(\mathbf{x}_0) \neq \mathbf{0}$, then $(\nabla f)(\mathbf{x}_0)$ indicates the direction of steepest increase for f at $\mathbf{x}_0$.

Proof. Using Theorem 7,

$$\begin{aligned} (\nabla_{\mathbf{v}} f)(\mathbf{x}_0) &= \mathbf{v} \cdot (\nabla f)(\mathbf{x}_0) \\ &= \|\mathbf{v}\| \cdot \|(\nabla f)(\mathbf{x}_0)\| \cos\theta \\ &= \|(\nabla f)(\mathbf{x}_0)\| \cos\theta \text{ since } \mathbf{v} \text{ is a unit vector} \end{aligned}$$

This expression is maximized when $\theta = 0$, which occurs when $\mathbf{v}$ and $(\nabla f)(\mathbf{x}_0)$ are parallel. That is, when $\mathbf{v}$ points to $(\nabla f)(\mathbf{x}_0)$. □

$(\nabla f)(\mathbf{x}_0)$ indicates the direction of steepest increase for the function f.



Remark. The gradient points in the direction in which the values of f change most rapidly, whereas a level surface lies in the directions in which they do not change at all. Hence, for f reasonably behaved, the gradient and the level surface will be perpendicular.

Example 20: $f(x,y) = 1 - x^2 - y^2$

Consider the curve $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = 1 - x^2 - y^2$$

The gradient of f is given by,

$$(\nabla f)(\mathbf{x}) = (-2x - 2y)$$

which at $(x_0, y_0, z_0) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{3}\right)$ gives the tangent plane,

$$z - \frac{1}{3} = -\frac{2}{\sqrt{3}}(x - \frac{1}{\sqrt{3}}) - \frac{2}{\sqrt{3}}(y - \frac{1}{\sqrt{3}})$$

Higher-Order Derivatives

Iterated Partial Derivatives

The **second-order iterated derivatives** for a function $f : \mathbb{R}^2 \to \mathbb{R}$ are,

$$\underbrace{\frac{\partial f}{\partial x^2}}_{fxx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \underbrace{\frac{\partial f}{\partial xy}}_{fxy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\underbrace{\frac{\partial f}{\partial yx}}_{fyx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \underbrace{\frac{\partial f}{\partial y^2}}_{fyy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

where f is assumed to be of class C^2.

We say that $f \in C^k$ if

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

all exist and are continuous in U.

Example 21: Computing Iterated Partials

Consider the following function,

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = x^3 + x^2 y + y^2$$

We will compute the iterated partials of f,

$$f_{xx} = 6x + 2y \quad \text{and} \quad f_{yy} = 2$$
$$f_{yx} = f_{xy} = 2x$$

Theorem 9. If f_{xy} and f_{yx} are continuous in U, then they are equal.

Taylor's Theorem

We can generalize **Taylor's Theorem** to functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$ in n variables. The first-order formula is given by,

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + R_1(\mathbf{x}_0, \mathbf{h})$$

where $R_1(\mathbf{x}_0, \text{ h}) / \|\text{h}\| \to 0$ as $\mathbf{h} \to \mathbf{0}$ in $\mathbb{R}^n$ and f is assumed to be differentiable. The second-order formula is given be,

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{x}_0, \mathbf{h})$$

where $R_2(\mathbf{x}_0, \text{ h}) / \|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$ and f is assumed to have continuous partial derivatives of third order.

> **Remark.** We can obtain an explicit formula for $R_k(\mathbf{x}_0, \mathbf{h})$ by repeatedly applying Integration by Parts,
>
> $$R_k(\mathbf{x}_0, \mathbf{h}) := \int_{\mathbf{x}_0}^{\mathbf{x}_0 + \mathbf{h}} \frac{1}{k!} (\mathbf{x}_0 + \mathbf{h} - z)^k f^{(k+1)}(z) dz$$

Example 22: Computing the 2nd Order Taylor Polynomial

We will compute the 2nd order Taylor polynomial for,

$$f(x, y) = e^{x^2 + y}$$

at the point $\mathbf{x}_0 = (1, 1)$. The partial derivatives of f are,

$$f_x = 2x \cdot e^{x^2+y} \implies f_x(\mathbf{x}_0) = 2e^2$$
$$f_y = e^{x^2+y} \implies f_y = (\mathbf{x}_0) = e^2$$

The iterated partial derivatives of f are,

$$f_{xx} = 2e^{x^2+y} + 4x^2 \cdot e^{x^2+y} \implies f_{xx}(\mathbf{x}_0) = 6e^2$$
$$f_{xy} = 2x \cdot e^{x^2+y} \implies f_{xy}(\mathbf{x}_0) = 2e^2$$
$$f_{yx} = 2x \cdot e^{x^2+y} \implies f_{yx}(\mathbf{x}_0) = 2e^2$$
$$f_{yy} = e^{x^2+y} \implies f_{yy}(\mathbf{x}_0) = e^2$$

This gives the following 2nd order approximation,

$$\underbrace{e^2}_{f(\mathbf{x}_0)} + \underbrace{2e^2 \cdot h_1 + e^2 \cdot h_2}_{\sum \frac{\partial f}{\partial x_i}(\mathbf{x}_0) h_i} + \frac{1}{2} e^2 (6 \cdot h_1^2 + \underbrace{2 \cdot 2 \cdot h_1 h_2}_{f_{xy} \cdot h_1 h_2 + f_{yx} \cdot h_2 h_1} + \underbrace{1 \cdot h_2^2}_{f_{yy} \cdot h_2^2})$$

Defining Extreme, Critical, and Saddle Points

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where U is an open set.

Definition (Local Maxima and Minima). We say that,

1. f has a **local maximum** at $\mathbf{x}_0$ if there exists an open neighborhood N of $\mathbf{x}_0$ such that $f(x) \geq f(\mathbf{x}_0)$ for all $x \in N$
2. f has a **local minimum** at $\mathbf{x}_0$ if there exists an open neighborhood N of $\mathbf{x}_0$ such that $f(x) \leq f(\mathbf{x}_0)$ for all $x \in N$

The local extremum are called **strict** if the inequalities are strict.

Extrema can be **local** or **global**. Depending on the choice of U, these extrema may or may not be captured by the first-derivative test.

Definition (Critical Points). There are (3) types of critical points,

- $\mathbf{x}_0 \in U$ is **extreme** if $\mathbf{x}_0$ is a local minimum or maximum
- $\mathbf{x}_0 \in U$ is **critical** if,
  - f is not differentiable at $\mathbf{x}_0$
  - f is differentiable at $\mathbf{x}_0$ and

$$(\mathbf{D}f)(\mathbf{x}_0) = 0 \iff (\nabla_{\mathbf{v}} f)(\mathbf{x}_0) = 0$$

- $\mathbf{x}_0 \in U$ is a **saddle point** if $\mathbf{x}_0$ is critical but not extreme

First-Derivative Test for Local Extrema

Theorem 10 (First-Derivative Test for Local Extrema). Let $\mathbf{x}_0$ be a local maximum or minimum. If f is differentiable at $\mathbf{x}_0$, then $\mathbf{D}f\,(\mathrm{x}_0) = 0$.

Proof. Suppose that f achieves a local maximum at $\mathbf{x}_0$.

1. If $m = 1$, then for any $\mathbf{h} \in \mathbb{R}^n$, the function $g(t) = f(\mathbf{x}_0, t\mathbf{h})$ has a local maximum at $t = 0$. From one-variable calculus,

$$g'(0) = 0$$

   By the chain rule,

$$g'(0) = [(\mathbf{D}f)(\mathbf{x}_0)] \cdot \mathbf{h} = 0$$

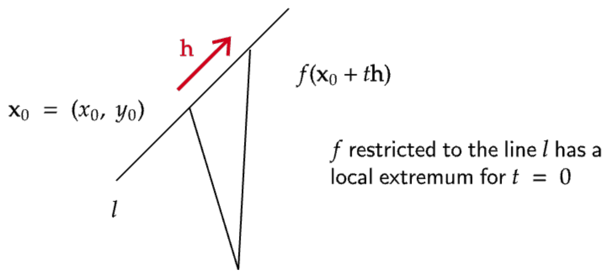
   This implies that $\mathrm{D}f(\mathbf{x}_0) = \mathbf{0}$.

2. If $m > 1$, then we can use the same idea. Given $\mathbf{x}_0$ and $\mathbf{h}$ fixed,

$$\frac{d}{dt} f(\mathbf{x_0} + t\mathbf{h}) = (\nabla f)(\mathbf{x_0} + t\mathbf{h}) \cdot \mathbf{h}$$

Evaluated at $t = 0$,

$$0 = (\nabla \mathbf{f})(\mathbf{x_0}) \cdot \mathbf{h} \Rightarrow (\nabla \mathbf{f})(\mathbf{x_0}) = 0$$



The case where f achieves a local minimum is analogous. □

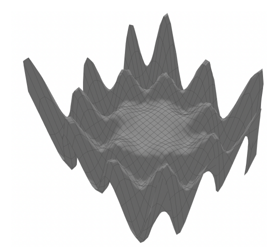
Example 23: Critical Points which are not Local Extremum

The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

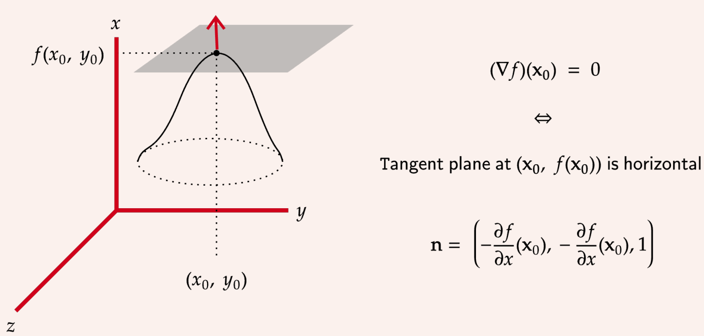
$$f(x, y) = xy$$

has $(0, 0)$ as a critical point, but it is not a local extremum.

Functions can have many critical points:



Example 24: Geometric Interpretation of Critical Points



Second Derivative Test

We will establish an analog of the **second derivative test**. At a critical point $\mathbf{x}_0$, Taylor's Theorem tells us that,

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x_0}) h_i + \frac{1}{2} \cdot \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i x_j}(\mathbf{x_0}) h_i h_j + R_2(\mathbf{x_0}, \mathbf{h})$$

implying in particular that,

$$f\left(\mathbf{x}_0+\mathbf{h}\right)-f\left(\mathbf{x}_0\right)=\frac{1}{2}\sum_{i,j}\underbrace{\frac{\partial^2 f}{\partial x_i x_j}\left(\mathbf{x_0}\right)h_i h_j}_{(*)}+R_2\left(\mathbf{x_0},\mathbf{h}\right)$$

where $(*)$ is quadratic in $\mathbf{h}$ and the remainder decays faster than quadratically. We require the following **algebraic terminology**:

Definition (Quadratic Function). A function $g:\mathbb{R}^n\to\mathbb{R}$ defined

$$g\left(h_1,\ldots,h_n\right)=\sum_{i,j=1}^{n}a_{ij}h_i h_j$$

for an $n\times n$ matrix $\mathbf{A}$ with entries a_{ij} is called **quadratic**.

Example 25: Quadratic Function: $n=3$

$$\begin{aligned}g\left(h_1,h_2,h_3\right)&=\left[h_1,h_2,h_3\right]\begin{bmatrix}1&-1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}h_1\\h_2\\h_3\end{bmatrix}\\&=h_1^2-2h_1h_2+h_3^2\end{aligned}$$

Proposition 3 (Properties of Quadratic Forms). Observe that,

1. g is homogeneous of degree 2,

$$g(\lambda h_1,\cdots\lambda h_n)=\lambda^2 g(h_1,\cdots,h_n)$$

2. We may assume that the matrix $\mathbf{A}$ is symmetric, i.e., $a_{ij}=aji$ for all i,j. If not, then we can write

$$a_{ij}=\frac{1}{2}\underbrace{\left(a_{ij}+a_{ji}\right)}_{b_{ij}}+\frac{1}{2}\underbrace{\left(a_{ij}-a_{ji}\right)}_{c_{ij}}$$

where $b_{ij}=b_{ji}$ (symmetric) and $c_{ij}=-c_{ji}$ (skew-symmetric).

$$\sum a_{ij}\cdot h_i h_j=\sum b_{ij}\cdot h_i h_j+\overbrace{\sum c_{ij}\cdot h_i h_j}^{=0}$$

and choose the symmetric matrix $\mathbf{B}$.

Every matrix can be written as a function of a symmetric matrix and a skew symmetric matrix.

Definition (Positive and Negative Definite). A quadratic form is

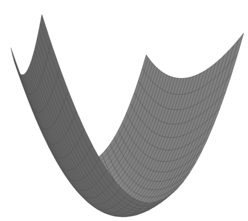
- **Positive definite** if $g(\mathbf{h}) \geq 0$ for all $\mathbf{h} \in \mathbb{R}^n$
- **Negative definite** if $g(\mathbf{h}) \leq 0$ for all $\mathbf{h} \in \mathbb{R}^n$

with the added condition that $g(\mathbf{h}) = 0$ if and only if $\mathbf{h} = 0$.

Definition (Hessian). Suppose that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ has second-order continuous derivatives at $\mathbf{x}_0 \in U$. The **Hessian** of f at $\mathbf{x}_0$ is

$$\begin{aligned}(\mathbf{H}f)(\mathbf{x}_0)(\mathbf{h}) &= \frac{1}{2} \cdot \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\, h_i h_j \\ &= \frac{1}{2}[h_1, \ldots, h_n] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}\end{aligned}$$

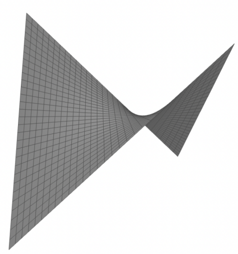
which is a quadratic function by equality of the mixed partials.



$g(\mathbf{h}) = h_1^2 + h_2^2$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

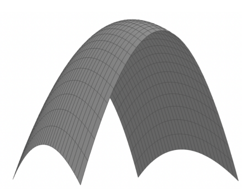
Positive Definite



$g(\mathbf{h}) = h_1^2 \cdot h_2^2$

$A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

Neither



$g(\mathbf{h}) = -h_1^2 - h_2^2$

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Negative Definite

Proposition 4. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive definite quadratic form. There exists a constant $M > 0$ such that,

$$g(\mathbf{h}) \geq M \cdot \|\mathbf{h}\|^2$$

Theorem 11 (Second Derivative Test). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of class C^3. Consider a critical point $\mathbf{x}_0$ of f. Then,

$$(\mathbf{H}(f))(\mathbf{x}_0)(\mathbf{h}) = \begin{cases} \text{Positive Definite} & \Rightarrow \mathbf{x}_0 \text{ is a local minimum} \\ \text{Negative Definite} & \Rightarrow \mathbf{x}_0 \text{ is a local maximum} \end{cases}$$

Proof. If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is of class C^3, then,

$$f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) = (\mathbf{H}f)(\mathbf{x}_0)(\mathbf{h}) + R_2(\mathbf{x}_0, \mathbf{h})$$

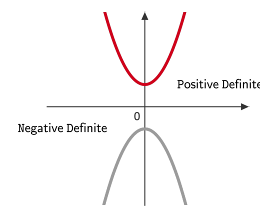
by Taylor's Theorem, where $R_2(\mathbf{x}_0, \mathbf{h}/\|\mathbf{h}\|^2 \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$ and $\mathbf{x}_0 \in U$ is a critical point. Since $(\mathbf{H}f)(\mathbf{x}_0)(\mathbf{h})$ is positive definite,

$$(\mathbf{H}f)(\mathbf{x}_0)(\mathbf{h}) \geq M \cdot \|\mathbf{h}\|^2$$

for some $M > 0$. There exists $\delta > 0$ such that for $0 < \|\mathbf{h}\| < \delta$,

$$|R_2(\mathbf{x}_0, \mathbf{h}| < M \cdot \|\mathbf{h}\|^2$$

Thus, $0 < (\mathbf{H}f)(\mathbf{x}_0)(\mathbf{h}) + R_2(\mathbf{x}_0, \mathbf{h}) = f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)$ for $0 < \|\mathbf{h}\| < \delta$. It follows that $\mathbf{x}_0$ is a strict relative minimum. The negative definite case follows by applying this argument to $-f$. □



Proposition 5. For a quadratic form $g(h_1, h_2)$,

$$a > 0 \text{ and } ac - b^2 > 0 \iff \text{Positive Definite}$$
$$a < 0 \text{ and } ac - b^2 > 0 \iff \text{Negative Definite}$$

Proof. Take

$$g(h_1, h_2) = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Expanding this expression gives that,

$$g(h_1, h_2) = ah_1^2 + 2bh_1h_2 + ch_2^2$$

Completing the square:

$$g(h_1, h_2) = \frac{1}{2}a\left(h_1 + \frac{b}{a}h_2\right)^2 + \frac{1}{2}\left(c - \frac{b^2}{a}\right)h_2^2$$

Suppose that g is positive definite.

$$
\begin{aligned}
h_2 &= 0 \Rightarrow \frac{1}{2} a h_1^2 \Rightarrow a > 0 \\
h_1 &= -\frac{b}{a} h_2 \Rightarrow \frac{1}{2} \underbrace{\left(c - \frac{b^2}{a} \right)}_{>0} h_2^2
\end{aligned}
$$

Conversely,

$$a > 0 \text{ and } ac - b^2 > 0 \implies g(h_1, h_2) > 0$$

since we sum over positive numbers. □

Remark (Determinant Test)**.** Let $n \geq 0$ and consider

$$g(\mathbf{h}) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

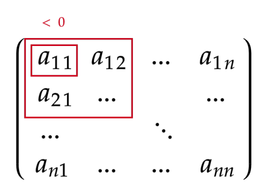
with entries taken from the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & a_{nn} \end{pmatrix}$$

With reference to the margin figure,

1. g is positive definite if the determinants of every diagonal embedded minor are positive
2. g is negative definite if the determinants of every diagonal embedded minor alternate signs

Diagonal embedded minors of $\mathbf{A}$



Theorem 12 (Second-Derivative Test)**.** Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$.

- $\mathbf{x}$ is a **local minimum** of f if,
  1. $f_x(\mathbf{x}) = f_y(\mathbf{x}) = 0$
  2. $f_{xx}(\mathbf{x}) > 0$
  3. $(f_{xx} \cdot f_{yy} - f_{xy}^2)(\mathbf{x}) > 0$
- $\mathbf{x}$ is a **local maximum** of f if,
  1. $f_x(\mathbf{x}) = f_y(\mathbf{x}) = 0$
  2. $f_{xx}(\mathbf{x}) < 0$

3. $(f_{xx} \cdot f_{yy} - f_{xy}^2)(\mathbf{x}) > 0$

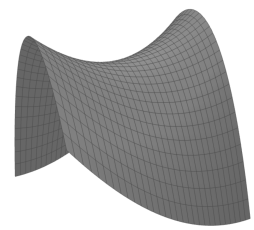
- $\mathbf{x}$ is a **saddle type** of f if,

1. $(f_{xx} \cdot f_{yy} - f_{xy}^2)(\mathbf{x}) < 0$

with the **indeterminant** case occuring when,

$$(f_{xx} \cdot f_{yy} - f_{xy}^2)(\mathbf{x}) = 0$$

$f(x, y) = x^2 - y^2 + xy$



Example 26: $f(x, y) = x^2 - y^2 + xy$

Consider the function,

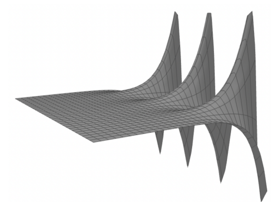
$$f(x, y) = x^2 - y^2 + xy$$
$$f_x = 2x + y = 0 \quad \text{and} \quad f_y = -2y + x = 0$$

$(0, 0)$ is the unique critical point. Computing the Hessian,

$$\left|\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\right| = \left|\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\right| = \left(f_{xx} \cdot f_{yy} - f_{xy}^2\right)(0, 0) = -5$$

shows that $(0, 0)$ is a saddle point.

$f(x, y) = e^x \cdot \cos y$



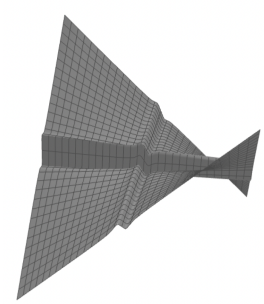
Example 27: $f(x, y) = e^x$

Consider the function,

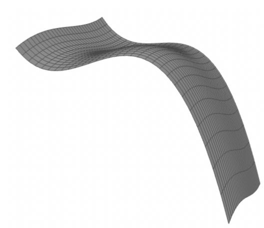
$$f(x, y) = e^x \cdot \cos y$$
$$f_x = e^x \cos y \quad \text{and} \quad f_y = e^x \sin y$$

f has no critical points.

$\frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{1}{2}x^2 - \frac{5}{2}y^2 + xy + 10$



$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$



Example 28: $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

Consider the function,

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$
$$f_x = y - \frac{1}{x^2} \quad \text{and} \quad f_y = x - \frac{1}{y^2}$$

$(1, 1)$ is the unique critical point. Computing the Hessian,

$$\left|\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}(x_0, y_0)\right| = \left|\begin{pmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{pmatrix}(1, 1)\right| = \left|\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right| = 3$$

shows that $(1, 1)$ is a local minimum.

Example 29: $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{1}{2}x^2 - \frac{5}{2}y^2 + xy + 10$

Consider the function,

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{1}{2}x^2 - \frac{5}{2}y^2 + xy + 10$$
$$f_x = x^2 - x \quad \text{and} \quad f_y = y^2 - 5y + 6$$

$(0,2)$,$(0,3)$, $(1,2)$, and $(1,3)$ are critical points.

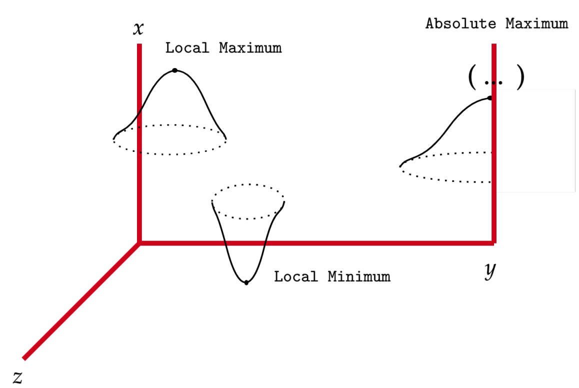
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2x-1 & 0 \\ 0 & 2y-5 \end{pmatrix}$$

shows that,

$(0,2)$ is a local minimum
$(0,3)$ is a saddle point
$(1,2)$ is a saddle point
$(1,3)$ is a local minimum

Classifying Global Extrema

The theorems that we saw previously allowed us to classify **local extrema**. We want to identify **global extrema**.



Definition (Global Extrema). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. $\mathbf{x}_0 \in A$ is a,

$$\begin{cases} \text{Global Maximum} & \text{if } f(\mathbf{x}) \leq f(\mathbf{x}_0) \text{ for all } \mathbf{x} \in A \\ \text{Global Minimum} & \text{if } f(\mathbf{x}) \geq f(\mathbf{x}_0) \text{ for all } \mathbf{x} \in A \end{cases}$$

A set $D \in \mathbb{R}^n$ is **bounded** if there is a number $M > 0$ such that $\|x\| < M$ for all $x \in D$. It is **closed** if it contains all of its boundary points. For example, the level sets of a continuous function are always closed.

Definition (Compact). A set is **compact** if it is closed and bounded.

Example 30: Compact Sets

The following two sets are compact,

1. $\{(x,y) \mid x^2 + y^2 \leq a^2\}$
2. $\{(x,y) \mid a \leq |x| \leq b\}$

Theorem 13. If $D \subseteq \mathbb{R}^n$ is compact, then $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ admits a global maximum and minimum, reached at some points of D.

Example 31: Finding Global Maxima and Minima

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined on a compact set D. To find the global maximum and minimum,

1. Locate all critical points of f in $\text{int}(D)$
2. Locate all critical points of f on ∂D
3. Compute the value of f on each critical point
4. Compare these values to determine the largest and smallest

Example 32: Finding Global Maxima and Minima

We want to find the absolute maximum and minimum of,

$$f : A \subseteq \mathbb{R}^2 \to \mathbb{R} \text{ defined by } f(x,y) = x^2 + xy + y^2$$

on the set $A = \{(x,y) \mid x^2 + y^2 \leq 1\}$. We have that,

$$\partial A = \{(x,y) \mid x^2 + y^2 = 1\}$$

Let $x := \cos\theta$ and $y := \sin\theta$ for $0 \leq \theta < 2\pi$. Then,

$$\begin{aligned} f|_{\partial A} = f(\cos\theta, \sin\theta) &= 1 + \cos\theta\sin\theta \\ &= 1 + \frac{1}{2}\sin(2\theta) =: g(\theta) \end{aligned}$$

Differentiating $g(\theta)$ gives that,

$$g'(\theta) = \cos(2\theta) \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

This gives two points,

$$f(\mathbf{p}_0) = f(0,0) = 0$$
$$f(\mathbf{p}_1) = f\left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right) = \frac{3}{2} = \frac{3}{2}$$
$$f(\mathbf{p}_1) = f\left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right) = \frac{3}{2} = f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$$

There is a global minimum at $\mathbf{p}_0$ and a global maximum at $\mathbf{p}_1$.

Example 33: Finding Global Maxima and Minima

We want to find the absolute maximum and minimum of,

$$f : A \subseteq \mathbb{R}^2 \to \mathbb{R} \text{ defined by } f(x,y) = \sin x + \cos x$$

on the set $A = \{(x,y) \mid x \in [0, 2\pi] \text{ and } y \in [0, 2\pi]\}$. Write,

$$\partial A = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$$

If we consider the restriction,

$$f|_{\gamma_1} = f(x,0) = \sin x + 1 := g_1(x)$$

on $x \in (0, 2\pi)$, then we obtain that,

$$g'(x) = \cos x \implies x = \pi/2 \text{ and. } x = 3\pi/2$$

so the critical points are $(\pi/2, 0)$ and $(3\pi/2, 0)$. We can repeat this for each γ_i to find the global maximum and minimum.

Constrained Extrema and Lagrange Multipliers

We want to find the local extrema of a function f restricted to a level set $g(\mathbf{x}_0) = c$. We call this a **constrained** extremum.

Theorem 14. Suppose that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are of the class C^1. f has a constrained extremum at $g(\mathbf{x}_0) = c$ if,

$$(\nabla f)(\mathbf{x}_0) = \lambda(\nabla g)(\mathbf{x}_0)$$

where $\lambda \in \mathbb{R}$ s called a **Lagrange multiplier**.

Remark. The point $\mathbf{x}_0$ is a **critical point** of $f|U$. If $f|U$ has a maximum or minimum at $\mathbf{x}_0$, then $(\nabla f)(\mathbf{x}_0)$ is perpendicular to U at $\mathbf{x}_0$.

Remark. λ is an additional variable in the auxiliary function,

$$L(x_1, \cdots, x_n, \lambda) = f(x_1, \cdots, x_n) - \lambda \cdot (g(x_1, \cdots, x_n) - c)$$

To find the extreme points of $f|S$ we find the critical points of L,

$$\begin{aligned} 0 &= h_{x_1} = f_{x_1} - \lambda g_{x_1} \\ &\vdots \\ 0 &= h_{x_n} = f_{x_n} - \lambda g_{x_n} \\ 0 &= h_\lambda = g(x_1, \cdots, x_n) - c \end{aligned}$$

Example 34: Constrained Extrema

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by,

$$f(x, y, z) = xy + z^2$$

on the sphere $x^2 + y^2 + z^2 = 1$. Define the Lagrange function $L := f + \lambda g = xy + z^2 + \lambda\left(x^2 + y^2 + z^2\right)$. Now,

$$\nabla L = 0 \implies \begin{cases} y + 2\lambda x = 0 \\ x + 2\lambda y = 0 \\ 2z + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 0 \end{cases}$$

We can then solve the system.

Example 35: Applications of Lagrange Multipliers

We want to find the points on the curve

$$g(x, y) = 17x^2 + 12xy + 8y^2 = 100$$

which are closest to and farthest from the origin $(0, 0)$. To do this, define the squared distance function $f(x, y) = x^2 + y^2$.

Remark. Given k constraints,

$$g_1(\mathbf{x}) = c_1, \ldots, g_k(\mathbf{x}) = c_k$$

We have that $(\nabla f)(\mathbf{x}_0) = \lambda_1 (\nabla g_0)(\mathbf{x}_0) + \ldots + \lambda_k (\nabla g_k)(\mathbf{x}_0)$.

The Implicit Function Theorem

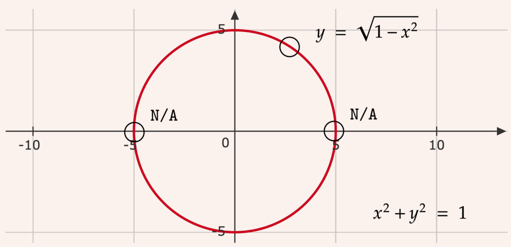
Example 36: Motivating Example

We can find neighborhoods around points of the circle

$$x^2 + y^2 = 1$$

for which they correspond to the graph of the function

$$f(x) = \pm\sqrt{1 - x^2}$$



This does not hold at $(1, 0), (-1, 0)$.

The **Implicit Function Theorem** provides conditions under which a relationship of the form $f(x, y) = 0$ can be re-written as a function $y = f(x)$ locally.

Theorem 15 (Implicit Function Theorem)**.** Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be of class C^1. Denote points in $\mathbb{R}^{n+1}$ by (x, z), where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. If,

$$f(x_0, z_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial z}(x_0, z_0) \neq 0$$

for a point $(x_0, y_0) \in \mathbb{R}^{n+1}$, then there exists,

1. A ball U containing x_0 in $\mathbb{R}^n$
2. A neighborhood V of z_0 in $\mathbb{R}$

such that there is a unique implicit function $z = g(x)$ satisfying that,

1. z is defined for x in U and z is in V
2. $f(x, g(x)) = 0$

Moreover, if $x \in U$ and $z \in Z$ satisfy $f(x, z) = 0$, then $z = g(x)$. Finally, $z = g(x)$ is continuously differentiable and,

$$(\mathbf{D}g)(\mathbf{x}) = -\frac{1}{\frac{\partial f}{\partial z}(\mathbf{x}, z)} \cdot \mathbf{D}_x f(\mathbf{x}, z)\bigg|_{z=g(x)}$$

where $\mathbf{D}_x f$ is the partial derivative of f with respect to x. That is,

$$\frac{\partial g}{\partial x_i} = -\frac{\partial f / \partial x_i}{\partial f / \partial z}$$

for all $i = 1, 2, \cdots, n$.

Theorem 16 (General Implicit Function Theorem)**.** Suppose that $\mathbf{F}_i$ is C^1 for $1 \leq i \leq m$. Consider the determinant Δ of the matrix,

$$\begin{bmatrix} \frac{\partial \mathbf{F}_1}{\partial z_1} & \cdots & \frac{\partial \mathbf{F}_1}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \mathbf{F}_m}{\partial z_1} & \cdots & \frac{\partial \mathbf{F}_m}{\partial z_m} \end{bmatrix}$$

evaluated at a point (x_0, z_0). If $\Delta \neq 0$, then

$$\begin{aligned} \mathbf{F}_1 (x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ \mathbf{F}_2 (x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ \mathbf{F}_m (x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \end{aligned}$$

defines a unique set of smooth functions,

$$z_i = z_i (x_1, \ldots, x_n) \quad (i = 1, \ldots, m)$$

near the point (x_0, z_0).

The derivatives of z_i can be computed by implicit differentiation.

Example 37: Applications of the Implicit Function Theorem

Consider the functions $F_1, F_2 : \mathbb{R}^4 \to \mathbb{R}$ defined in the system,

$$\begin{aligned} F_1(x, y, u, v) &= x^2 + xy - y^2 - u = 0 \\ F_2(x, y, u, v) &= 2xy + y^2 - v = 0 \end{aligned}$$

We want to show that x and y can be solved for as C^1 functions of u and v near the point $(x_0, y_0, u_0, v_0) = (2, -1, 1, -3)$.

1. $(2, -1, 1, -3)$ satisfies the constraints,

$$\begin{aligned} F_1(2, -1, 1, -3) &= 4 - 2 - 1 - 1 = 0 \\ F_2(2, -1, 1, -3) &= -4 + 1 + 3 = 0 \end{aligned}$$

2. Computing the determinant of the matrix,

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x + y & x - 2y \\ 2y & 2x + 2y \end{pmatrix}$$

   at our point gives $3 \cdot 2 - (-2) \cdot 4 = 6 + 8 \neq 0$.

To compute the partial derivatives via implicit differentiation,

$$\begin{aligned} D_u F_1 &: 2xx_u + x_u y + xy_u - 2yy_u - 1 = 0 \\ &\implies x_u(2x+y) + y_u(x-2y) - 1 = 0 \end{aligned}$$

$$\begin{aligned} D_u F_2 &: 2x_u y + 2xy_u + 2yy_u = 0 \\ &\implies x_u(2y) + y_u(2x+2y) = 0 \end{aligned}$$

Evaluated at $(2, -1, 1, -3)$, this is,

$$\begin{aligned} 3x_u + 4y_u - 1 &= 0 \\ -2x_u + 2y_u &= 0 \end{aligned}$$

which implies that $x_u = y_u = 1/7$.

Example 38: Applications of the Implicit Function Theorem

Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined on the level surface,

$$F(x,y,z) = x + y - z + \cos(xyz) = 0$$

We want to compute $F_x(0,0)$.

1. $(0,0,1)$ satisfies the constraints,

$$0 + 0 - 1 + \cos(0) = 0$$

To compute the partial derivatives by implicit differentiation,

$$\begin{aligned} D_x F &: 1 - z \cdot z_x - [yz + xyz_x z] \sin(xyz) \\ &\implies 1 - z_x = 0 \end{aligned}$$

$$\begin{aligned} D_y F &: 1 - z \cdot z_y - [xz + xyz_y z] \sin(xyz) \\ &\Rightarrow 1 - z_y = 0 \end{aligned}$$

Vector-Valued Functions

Vector Fields

Definition (Vector Field). A **vector field** in $\mathbb{R}^n$ is a map

$$\mathbf{V} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

that assigns each x in its domain U a vector $\mathbf{V}(x)$.

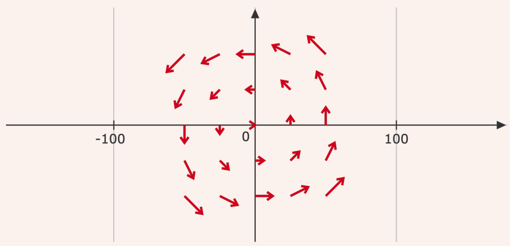
A map $\mathbf{V} : U \subseteq \mathbb{R}^n \to \mathbb{R}$ assigning a number to each point is a **scalar field**.

A vector field on $\mathbb{R}^n$ has n components. If each component is a C^k function, then the vector field is said to be of class C^k.

Example 39: Describing Rotary Motion using a Vector Field

Rotary motion can be described by the vector field,

$$\mathbf{V}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

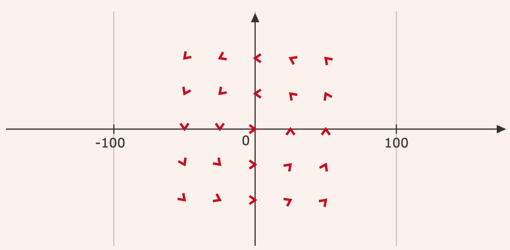


Example 40: Unit Length Vector Fields

The vector field $\mathbf{V}$ defined by,

$$\mathbf{V}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \cdot \mathbf{i} + \frac{-y}{\sqrt{x^2+y^2}} \cdot \mathbf{j}$$

has unit length. It is not defined at the origin,



Example 41: Gradient Vector Fields

The gradient of a C^1 function is given by,

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) \cdot \mathbf{i} + \frac{\partial f}{\partial y}(x,y,z) \cdot \mathbf{j} + \frac{\partial f}{\partial z}(x,y,z) \cdot \mathbf{k}$$

We can think of this as an example of a vector field $\mathbf{V}$.

Example 42: Identifying Gradient Vector Fields

- $\mathbf{V}(x,y) = -y\mathbf{i} + x\mathbf{j}$ is not a gradient vector field because the mixed partials $\mathbf{V}_{xy}$ and $\mathbf{V}_{yx}$ are not equal,

$$\mathbf{V}_x = -y \quad \text{and} \quad \mathbf{V}_y = x$$

- $\mathbf{V}(x,y) = y\mathbf{i} + x\mathbf{j}$ is a conservative because the mixed partials $\mathbf{V}_{xy}$ and $\mathbf{V}_{yx}$ are equal to 1,

$$\mathbf{V}_x = y \quad \text{and} \quad \mathbf{V}_y = x$$

Example 43: Equipotential Surfaces

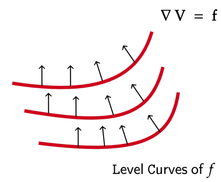
Given the gradient of V,

$$\nabla V = (x, 2y, 3z) = x\mathbf{i} + 2\mathbf{j} + 3\mathbf{z}$$

we can recover the original function $V : \mathbb{R}^2 \to \mathbb{R}$,

$$V = \frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2$$

The level curves of V are called **equipotential surfaces**.



Definition (Flow Line). A **flow line** $\mathbf{c}(t)$ for a vector field $\mathbf{V}$ has

$$\mathbf{c}'(t) = \mathbf{V}(\mathbf{c}(t))$$

Example 44: Rays

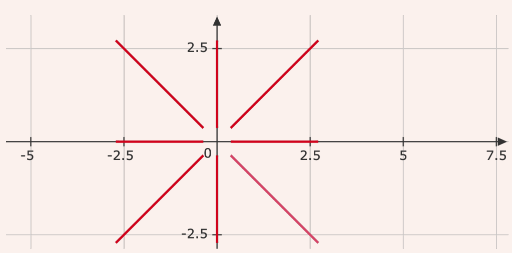
Consider the vector field $\mathbf{V}(x,y) = x\mathbf{i} + y\mathbf{j}$, where $\|\mathbf{V}\| = r$.

$$\underbrace{x'(t)\cdot\mathbf{i} + y'(t)\cdot\mathbf{j}}_{\mathbf{c}'(t)} = \underbrace{x(t)\cdot\mathbf{i} + y(t)\cdot\mathbf{j}}_{\mathbf{F}(\mathbf{c}(t))}$$

gives the following differential equation,

$$x'(t) = x(t) \implies x(t) = c_1 \cdot e^t$$
$$y'(t) = y(t) \implies y(t) = c_2 \cdot e^t$$

implying that the flow lines are rays through the origin.



Example 45: Concentric Circles

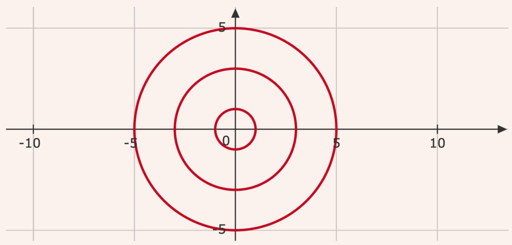
Consider the vector field $\mathbf{V}(x,y) = x\mathbf{i} - y\mathbf{j}$, where $\|\mathbf{V}\| = r$.

$$\underbrace{x'(t)\cdot\mathbf{i} - y'(t)\cdot\mathbf{j}}_{c'(t)} = \underbrace{x(t)\cdot\mathbf{i} + y(t)\cdot\mathbf{j}}_{\mathbf{F}(\mathbf{c}(t))}$$

gives the following differential equation,

$$x'(t) = x(t) \implies x(t) = c_1 \cdot \cos t$$
$$y'(t) = y(t) \implies y(t) = c_2 \cdot \sin t$$

implying that the flow lines are concentric circles at the origin.



Exercise: What are the flow lines of:

$$\mathbf{V} = \frac{-y}{\sqrt{x^2+y^2}}\cdot\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\cdot\mathbf{j}$$

Divergence and Curl

Definition (∇). The **del operator** in n-space is,

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$$

The gradient of f is obtained by taking the ∇ operator and applying it to f.

Definition (div $\mathbf{V}$). The **divergence** of a vector field $\mathbf{V}$ on $\mathbb{R}^n$ is,

$$\text{div}\,\mathbf{V} = \nabla\cdot\mathbf{F} = \sum_{i=1}^{n}\frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}$$

Definition (Solenoidal). If a vector field $\mathbf{V}$ on $\mathbb{R}^n$ has div $\mathbf{V}(\mathbf{x}) = \mathbf{0}$, then $\mathbf{V}$ is called **solenoidal**.

Remark. We evaluate the divergence at a point $\mathbf{x}$.

1. If div $\mathbf{V}(\mathbf{x}) < 0$, then $\mathbf{V}$ converges at $\mathbf{x}$
2. If div $\mathbf{V}(\mathbf{x}) > 0$, then $\mathbf{V}$ diverges at $\mathbf{x}$

Definition (curl $\mathbf{V}$). The **curl** of a vector field $\mathbf{V}$ on $\mathbb{R}^3$ is,

$$\operatorname{curl}\mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

which evaluates to,

$$\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) \cdot \mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) \cdot \mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \cdot \mathbf{k}$$

curl$(\mathbf{V})$ tells us how much the flow lines is $\mathbf{V}$ rotate. In particular, a gradient field has no center of rotation or else the level curves would intersect.

Proposition 6 (Curl of a Gradient). For any C^2 function f,

$$\operatorname{curl}\nabla f = \nabla \times (\nabla f) = 0$$

That is, the curl of any gradient is the zero vector.

Corollary. If $\operatorname{curl}\mathbf{V} = \nabla \times \mathbf{V} \neq 0$, then $\mathbf{V}$ is not a gradient field.

Proposition 7 (Divergence of a Curl). For any C^2 vector field $\mathbf{V}$,

$$\operatorname{div}\operatorname{curl}\mathbf{V} = \nabla \cdot (\nabla \times \mathbf{V}) = 0$$

That is, the divergence of any curl is zero.

Corollary. If $\operatorname{div}\mathbf{V} = \nabla \cdot \mathbf{V} \neq 0$, then $\mathbf{V}$ is not solenoidal.

Integrals over Paths and Surfaces

Summary

In this section, we will see the following variations of integrals:

Path Integral	$\int_{\mathbf{c}} f ds = \int_a^b f(\mathbf{c}(t)) \cdot \lVert\mathbf{c}'(t)\rVert \, dt$
Line Integral	$\int_{\mathbf{c}} \mathbf{F} ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$
Surface Integral (Scalar)	$\iint_S f dS = \iint_D f(\mathbf{\Phi}(u,v)) \cdot \lVert\mathbf{T}_u \times \mathbf{T}_v\rVert \, dudv$
Surface Integral (Vector)	$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, dudv$

For an interpretation of double, triple, and line integrals in terms of weighted sums, please see this **reference**.

Line Integrals of Vector Fields

Consider a parameterized curve $\mathbf{c}(t)$

$$\begin{aligned} \mathbf{c}(t) &: [a,b] \to U \subseteq \mathbb{R}^3 \\ t &\mapsto \mathbf{c}(t) \end{aligned}$$

which is assumed to be simple and oriented.

Definition (Path Integral). Given a curve $\mathbf{c} : [a,b] \to \mathbb{R}^3$ that is of class C^1, the **path integral** of $f : \mathbb{R}^3 \to \mathbb{R}$ along c is,

$$\int_{\mathbf{c}} f ds = \int_a^b f(\mathbf{c}(t)) \left\|\mathbf{c}'(t)\right\| dt$$

Remark. If $\mathbf{c}(t)$ is piecewise C^1 or $f(\mathbf{c}(t))$ is piecewise continuous, then we can break I into pieces over which $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuous. We then sum the integrals over the pieces.

The assumption that $\mathbf{c}(t)$ is simple tells us that it is a one-to-one on $[a,b]$.

Example 46: Oriented Simple Curves

1. Let $0 \le t \le 1$. The following curves have the same image,

$$\begin{aligned} \mathbf{c}_1(t) &= (t,t,t) \\ \mathbf{c}_2(t) &= (1-t, 1-t, 1-t) \end{aligned}$$

   but opposite orientations.

2. Let $0 \le t \le 1$. The following curves have the same image,

$$\begin{aligned} \mathbf{c}_1(t) &= (\cos t, \sin t) \\ \mathbf{c}_2(t) &= (\cos 2t, \sin 2t) \end{aligned}$$

   but $\mathbf{c}_2(t)$ is not simple.

We now consider the problem of integrating a vector field along a path. We can approximate the **work done** by the force field $\mathbf{F}$ on a particle moving along a path $\mathbf{c} : [a,b] \to \mathbb{R}^3$ as,

$$\int_a^b \mathrm{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

Definition (Line Integral). Given a curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ that is of class C^1, the **line integral** of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ along c is,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

Remark (Notation). Let $\mathbf{c}(t) = (x(t), y(t), z(t))$. Then,

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

is the integral,

$$\int_a^b (F_1 \cdot \mathbf{i} + F_2 \cdot \mathbf{j} + F_3 \cdot \mathbf{k}) \cdot (x'(t) \cdot \mathbf{i} + y'(t) \cdot \mathbf{j} + z'(t) \cdot \mathbf{k}) dt$$

which we can re-write by an abuse of notation as,

$$\int_a^b F_1 \cdot x'(t) + F_2 \cdot y'(t) dt + F_3 \cdot z'(t) dt$$

Example 47: $\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$

We will calculate the work of the force field

$$\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$$

along the line segment given by,

$$\mathbf{c}(t) = (t, t) \quad 0 \leq t \leq 1$$

By definition, this is,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

which evaluates to,

$$\int_0^1 \left(t^2 \cdot \mathbf{i} + t \cdot \mathbf{j}\right) \cdot (\mathbf{i} + \mathbf{j}) dt = \frac{5}{6}$$

Example 48: $\mathbf{F} = y \cdot \mathbf{i}$

We will calculate the work of the force field

$$\mathbf{F} = y \cdot \mathbf{i}$$

along the unit circle oriented counter-clockwise,

$$\mathbf{c}(t) = (\cos t, \sin t) \quad 0 \leq t \leq 2\pi$$

By definition, this is,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (\sin t \cdot \mathbf{i}) \cdot (-\sin t \cdot \mathbf{i} + \cos t \cdot \mathbf{j}) dt = -\pi$$

Example 49: $\mathbf{F} = y \cdot \mathbf{i}$

Consider the work of the same force field

$$\mathbf{F} = y \cdot \mathbf{i}$$

along the curve,

$$\mathbf{c}(t) = (\cos 2t, \sin 2t) \quad 0 \leq t \leq 2\pi$$

This is **not a simple curve** because the unit circle is covered twice by the image of $\mathbf{c}(t)$. Computing the line integral,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \sin 2t \cdot \mathbf{i} \cdot (-2\sin 2t \cdot \mathbf{i} + 2\cos 2t \cdot \mathbf{j}) dt = -2\pi$$

as opposed to $-\pi$.

Remark. The line integral can be thought of as the path integral of the tangential component $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)$ of $\mathbf{F}$ along $\mathbf{c}$.

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \right] \|\mathbf{c}'(t)\| \, dt \\ &= \int_a^b [\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)] \, \|\mathbf{c}'(t)\| \, dt. \end{aligned}$$

Definition (Reparameterization)**.** Let $\mathbf{h} : [a, b] \to [a', b']$ be a one-to-one C^1 real-valued function. If $\mathbf{c} : [a', b'] \to \mathbb{R}^3$ is piecewise C^1,

$$\mathbf{p} := (\mathbf{c} \circ h) : [a, b] \to \mathbb{R}^3$$

is a reparameterization of $\mathbf{c}$.

Theorem 17 (Change of Parameterization)**.** Let $\mathbf{F}$ be a vector field continuous on the C^1 path $\mathbf{c} : [a', b'] \to \mathbb{R}^3$. Given a reparameterization $\mathbf{p} : [a,b] \to \mathbb{R}^3$ of $\mathbf{p} := (\mathbf{c} \circ h) : [a,b] \to \mathbb{R}^3$, we have that,

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \begin{cases} +\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{h} \text{ increases monotonically} \\ -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{h} \text{ decreases monotonically} \end{cases}$$

Corollary. If $\mathbf{p}$ is orientation-preserving, then,

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{c} \mathbf{F} \cdot d\mathbf{s}$$

If $\mathbf{p}$ is orientation-reversing, then,

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{c} \mathbf{F} \cdot d\mathbf{s}$$

Example 50: $\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$

We will calculate the work of the force field

$$\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$$

along the line segment given by,

$$\mathbf{c}(t) = (t^2, t^2) \quad 0 \le t \le 1$$

By definition, this is,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

which evaluates to,

$$\int_0^1 \left(t^4\mathbf{i} + t^2\mathbf{j}\right) \cdot (2t\mathbf{i} + 2t\mathbf{j}) dt = \frac{5}{6}$$

Example 51: $\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$

We will calculate the work of the force field

$$\mathbf{F} = x^2 \cdot \mathbf{i} + y \cdot \mathbf{j}$$

along the line segment given by,

$$\mathbf{c}(t) = (1 - t, 1 - t) \quad 0 \le t \le 1$$

By definition, this is,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

which evaluates to,

$$\int_0^1 \Big((1-t)^2 \cdot \mathbf{i} + (1-t)\cdot \mathbf{j}\Big) \cdot (-\mathbf{i} - \mathbf{j})dt = -\frac{5}{6}$$

Remark. Unlike the line integral, the path integral is **not oriented**. In fact, path integrals are unchanged under re-parametrizations.

Recall that a vector field $\mathbf{F}$ is called a **gradient vector field** if $\mathbf{F} = \nabla f$ for some real-valued function f. In particular,

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Theorem 18 (Fundamental Theorem of Calculus)**.** Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and $\mathbf{c} : [a,b] \to \mathbb{R}^3$ is piecewise C^1. Then,

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Proof. Define a composite function $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = f(\mathbf{c}(t))$. Apply the chain rule to compute F':

$$F'(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$$

By the Fundamental Theorem of Calculus,

$$\int_a^b F'(t)dt = F(b) - F(a) = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Hence, the result follows from the fact that,

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

□

If we can recognize the integrand as a gradient, then the evaluation of the integral becomes much easier. This is summarized below:

Remark. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for $f : \mathbb{R}^3 \to \mathbb{R}$. Then,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Corollary. The value of the work of a gradient field is independent of the choice of path connecting the two endpoints. That is,

$$\int_{\mathbf{c}_1} \mathbf{F} d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} d\mathbf{s}$$

if $\mathbf{c}_1$ and $\mathbf{c}_2$ have the same endpoints.

Corollary. If $\mathbf{c}$ is closed, then $\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = 0$.

If $\mathbf{c}$ is a closed curve, then we write,

$$\oint_{\mathbf{c}} \mathbf{F} d\mathbf{s}$$

Remark. If $\mathbf{c}_1$ and $\mathbf{c}_2$ are two curves that differ only in orientation,

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$

It may be easier to parameterize the components $\mathbf{c}_i$ than the whole curve $\mathbf{c}$.

Remark. If $\mathbf{c}$ is an oriented curve that is made up of several oriented component curves $\mathbf{c}_1, \cdots, \mathbf{c}_n$, that is, $\mathbf{c} = \mathbf{c}_1 + \cdots \mathbf{c}_n$, then,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{\mathbf{c}_n} \mathbf{F} \cdot d\mathbf{s}$$

Example 52: Verification of Path Independence

Consider the force field,

$$\begin{aligned} \mathbf{F} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= \nabla \left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \right) \end{aligned}$$

along the curve,

$$\mathbf{c}(t) = \left(t, t^2, t\right) \quad 0 \leq t \leq 1$$

Both from applying the definitions or our theorems,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(1,1,1) - V(0,0,0) = \frac{1}{2}$$

Definition (Flux)**.** The **flux** of a vector field $\mathbf{F}$ across $\mathbf{c}$ is,

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{n}(t) dt$$

where $\mathbf{n}(t)$ is the normal vector.

Flux and work are independent of the choice of parameterization for $\mathbf{c}$, but they are not independent of the choice of orientation.

Parameterized Surfaces

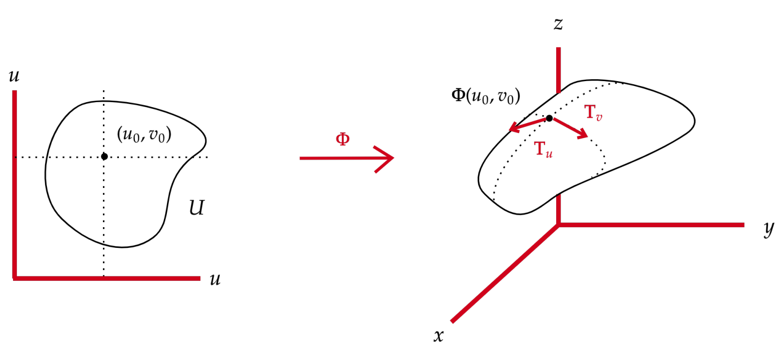
Definition (Parameterized Surface)**.** A **parametrization of a surface**

$$\mathbf{\Phi} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

is a map defined over a domain U in $\mathbb{R}^2$. The **surface** S corresponding to $\mathbf{\Phi}$ is its image $S = \mathbf{\Phi}$. We can write,

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$$

Remark. If $\mathbf{\Phi}$ is C^1, then S is called a **differentiable** surface. This condition is equivalent to saying that $x(u, v)$, $y(u, v)$, and $z(u, v)$ are C^1.



Suppose that $\mathbf{\Phi}$ is differentiable at $(u, v) \in \mathbb{R}^2$. The tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ to the curves $\Phi(t, u_0)$ and $\Phi(t, v_0)$ on the surface are,

$$\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$
$$\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

Remark. $\mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface at the point (u_0, v_0).

Definition (Regular Surface)**.** We say that a surface S is **regular at** $\Phi(u_0, v_0)$ is $\mathbf{T}_u \times \mathbf{T}_v \neq 0$ at (u_0, v_0). If this condition holds at all points on the surface, then S is called **regular**.

The condition that $\mathbf{T}_u \times \mathbf{T}_v(u,v) \neq 0$ suggests that the partials $\mathbf{\Phi}_u$ and $\mathbf{\Phi}_v$ are linearly independent everywhere, which ensures the existence of the tangent plane at every point of $\mathbf{\Phi} = S$.

Example 53: $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$

Given a C^1 function f, define the map $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$. We have that $\mathbf{T}_u = (1, 0, f_u)$ and $\mathbf{T}_v = (0, 1, f_v)$. Therefore,

$$\mathbf{T}_v \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & f_v \\ 0 & 1 & f_v \end{vmatrix} = -f_v \cdot \mathbf{i} - f_v \cdot \mathbf{j} + \mathbf{k} \neq \mathbf{0}$$

regardless of our choice of f.

Example 54: Upper Sheet of a Cone

The upper sheet of a cone is given by,

$$\mathbf{\Phi}(u,v) = (u \cos v, u \sin v, u)$$

where $u \geq 0$ and $0 < v < 2\pi$. Here,

$$\mathbf{T}_u = (\cos v, \sin v, 1)$$
$$\mathbf{T}_v = (-u \sin v, u \cos v, 0)$$

and consequently,

$$\mathbf{T}_v \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = -u \cos v \mathbf{i} - u \sin v \mathbf{j} + u \mathbf{k}$$

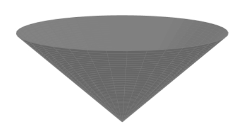
We have that $\mathbf{T}_u \times \mathbf{T}_v = \mathbf{0}$ if and only if $\|\mathbf{T}_u \times \mathbf{T}_v\| = \mathbf{0}$ and,

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \left(u^2 \cos^2 v + u^2 \sin^2 v + u^2\right)^{1/2} = 2\sqrt{u}$$

which is $\mathbf{0}$ if and only if $u = 0$.

The **upper sheet of the cone** is,

$$\mathbf{\Phi}(u,v) = (u \cos v, u \sin v, u)$$



Example 55: Helicoid

The helicoid is given by,

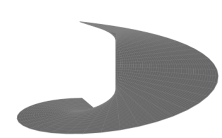
$$\mathbf{\Phi}(u,v) = (u \cos v, u \sin v, v)$$

where $u \geq 0$ and $0 < v < 2\pi$. Here,

$$\mathbf{T}_u = (\cos v, \sin v, 0)$$
$$\mathbf{T}_v = (-u \sin v, u \cos v, 1)$$

The **helicoid** is,

$$\mathbf{\Phi}(u,v) = (u \cos v, u \sin v, v)$$



and consequently,

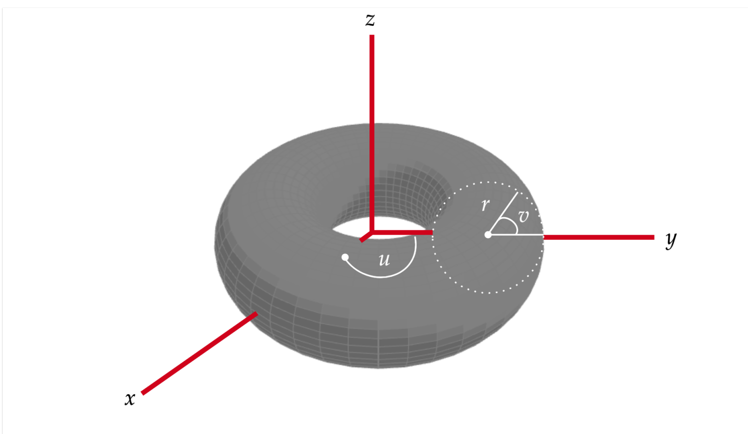
$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix}$$

Observe that this is equal to,

$$\text{sinv}\,\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k}$$

Hence, $\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{1+u^2} \neq 0$ for all u.

In the next example, we will see how to parameterize the **torus of revolution**. This is summarized in the diagram below:



Example 56: Torus of Revolution

The torus of revolution $\mathbf{\Phi}(u, v)$ is given by,

$$\begin{aligned} x(u,v) &= (a + r\cos v)\cos u \\ y(u,v) &= (a + r\cos v)\sin u \\ z(u,v) &= r\sin v \end{aligned}$$

where $0 < u, v < 2\pi$. Here,

$$\begin{aligned} \mathbf{T}_u &= (-\sin u(a + r\cos v), \cos u(a + r\cos v), 0) \\ \mathbf{T}_v &= (-r\sin v\cos u, -r\sin v\sin u, r\cos v) \end{aligned}$$

Computing the cross-product and simplifying gives,

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = r|(a + r\cos v)|$$

which is non-zero if $r \neq 0$ and $a > r$.

Example 57: Sphere

Using the **spherical coordinate transformation**,

$$\mathbf{\Phi}(u,v) = (a\sin v\cos u, a\sin v\sin u, a\cos v)$$

is a **sphere of radius** a where $0 < u < 2\pi$ and $0 < v < \pi$.

$$\mathbf{T}_u = (-a\sin v\sin u, a\sin v\cos u, 0)$$
$$\mathbf{T}_v = (a\cos v\cos u, a\cos v\sin u, -a\sin v)$$

Taking the cross-product and simplifying gives,

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = a^2\sin v$$

which is zero if $v = 0, \pi$.

Remark. As shown in the previous example, the condition that

$$\|\mathbf{T}_u \times \mathbf{T}_v\| \neq \mathbf{0}$$

is necessary but not sufficient for the existence of a tangent plane.

Area of a Surface

In this chapter, we will consider piecewise regular surfaces that are unions of images of parametrized surfaces $\mathbf{\Phi}_i : D_i \to \mathbb{R}^3$ for which:

- D_i is an elementary region in the plane
- Φ_i is C^1 and one-to-one, except possibly at the boundary
- S_i is regular except possibly at a finite number of points

Definition (Surface Area). The **surface area** $A(S)$ is,

$$A(S) = \iint_U \|\mathbf{T}_u \times \mathbf{T}_v\|\, dudv$$

where S is a parameterized surface.

If S is the union of surfaces S_i, then the area is the sum of the areas of S_i.

To find the surface area of a function, we are scaling by the area of the parallelogram spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$.

Example 58: Area of a Sphere

In this example, we will verify the standard formula for the

area of a sphere. Since $\|\mathbf{T}_u \times \mathbf{T}_v\| = a^2 \sin v$,

$$\begin{aligned} A(S) &= \int_{v=0}^{v=\pi} \int_{u=0}^{u=2\pi} \|\mathbf{T}_u \times \mathbf{T}_v\| \, du dv \\ &= 2\pi a^2 \int_0^\pi \sin v dv \\ &= 4\pi a^2 \end{aligned}$$

Exercise: Compute the area of a cylinder of radius a and height h.

$$\mathbf{\Phi}(u, v) = (a \cos v, a \sin v, u)$$

where $0 < v < 2\pi$ and $0 < u < h$.

Integrals of Scalar Functions Over Surfaces

In this section, we will define the integral of a **scalar function** f over a surface S. This generalizes the area of a surface, which corresponds to the integral over S of the scalar function $f(x, y, z) = 1$. Consider a surface S parameterized by a mapping $\mathbf{\Phi} : D \to S \subseteq \mathbb{R}^3$, where D is an elementary region. We write,

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

Definition (Integral of a Scalar Function Over a Surface)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function defined on a parameterized surface S.

$$\iint_S f(x, y, z) dS = \iint_S f dS = \iint_D f(\mathbf{\Phi}(u, v)) \, \|\mathbf{T}_u \times \mathbf{T}_v\| \, du dv$$

is the **integral of f over** S.

The surface integral is independent of the choice of the parameterization.

Remark. We can compute the **average value** of a function f as,

$$\frac{\iint_S f ds}{|A(S)|}$$

Example 59: Average over a Cone

We want to compute the average of the surface defined by,

$$f(x, y, z) = x + z^2$$

where D is the portion of the cone $x^2 + y^2 = z^2$ for which $1 \leq z \leq 4$. Parameterize the graph $z = f(x, y) = \sqrt{x^2 + y^2}$.

$$\Phi(u, v) = \left(u, v, \sqrt{u^2 + v^2}\right)$$

Taking the appropriate partials $\mathbf{T}_u$ and $\mathbf{T}_v$,

$$\mathbf{T}_u = \left(1, 0, \frac{u}{\sqrt{u^2 + v^2}}\right) \qquad \mathbf{T}_v = \left(0, 1, \frac{v}{\sqrt{u^2 + v^2}}\right)$$

gives $\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{2}$. Next,

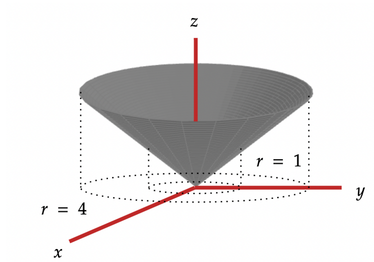
$$\begin{aligned}\iint_S f dS &= \iint_D f(\Phi(u,v)) \|\mathbf{T}_u \times \mathbf{T}_v\| \, dudv \\ &= \iint_D \left(u + u^2 + v^2\right) \sqrt{2} dudv\end{aligned}$$

We can use **polar coordinates** to compute this integral,

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=4} \left(r\cos\theta + r^2\right) r\sqrt{2} drd\theta = \sqrt{2} \cdot 15\pi$$

Hence, the average value of f is,

$$\frac{\iint_S f ds}{|A(S)|} = \frac{255}{30} = 8.5$$



Oriented Surfaces

An **oriented surface** is a two-sided surface with one side specified as positive ("outside") and one side specified as negative ("inside"). At each point $(x, y, z) \in S$, there are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfying $\mathbf{n}_1 = -\mathbf{n}_2$. Each of these normals can be associated with one side of the surface. Let $\mathbf{\Phi} : D \to \mathbb{R}^3$ be a parametrization of an oriented surface S. Suppose that S is regular at $\mathbf{\Phi}(u_0, v_0)$. Let $\mathbf{n}(\mathbf{\Phi}(u_0, v_0))$ be the unit normal to S at $\mathbf{\Phi}(u_0, v_0)$. Since,

$$\frac{(\mathbf{T}_{u_0} \times \mathbf{T}_{v_0})}{\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\|}$$

exists, it is defined and equal to $\pm\mathbf{n}(\Phi(u_0, v_0))$.

Remark. Φ is called **orientation-preserving** if we have the $+$ sign, and **orientation-reversing** if we have the $-$ sign.

The definition of an oriented surface given in this section assumes that a surface has two sides. This is in fact not necessary, e.g., the Möbius strip.

Remark. Any one-to-one parameterized surface for which $\mathbf{T}_v \times \mathbf{T}_u \neq 0$ can be considered as an oriented surface with a positive side determined by the direction of $\mathbf{T}_v \times \mathbf{T}_u$.

Theorem 19. Let S be an oriented surface and let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two regular orientation-preserving parametrizations, with $\mathbf{F}$ a continuous vector field defined on S. Then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}$$

If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation-reversing, then

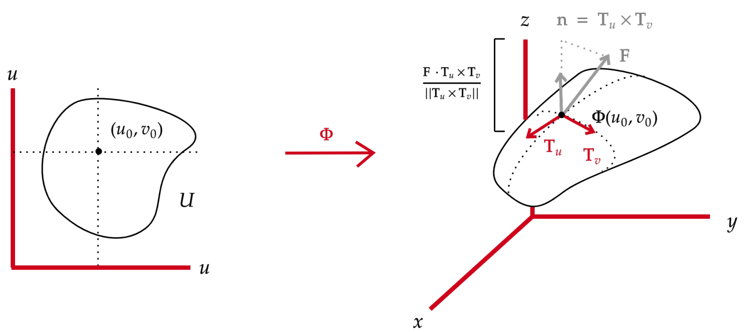
$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}$$

If f is a real-valued continuous function defined on S, and if $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are parametrizations of S, then

$$\iint_{\mathbf{\Phi}_1} f dS = \iint_{\mathbf{\Phi}_2} f dS$$

Surface Integrals of Vector Fields

We can define the integral of a vector field $\mathbf{F}$ over a surface S.



Definition (Surface Integral of Vector Fields). Let $\mathbf{F}$ be a vector field defined on a surface S, the image of a parameterized surface $\mathbf{\Phi}$.

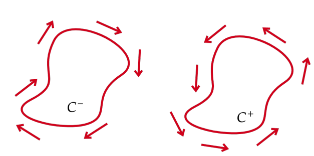
$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, dudv$$

is the **surface integral of** F **over** Φ.

Remark. This integral quantifies the **flux** of $\mathbf{F}$ across S.

Remark. The integral $\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S}$ is not dependent on the choice of parameterization for S, but it does depend on the orientation.

Integral Theorems and Vector Analysis



Green's and Stokes' Theorem

In this section, we will relate a line integral along a closed curve C in the plane to a double integral over the region enclosed by the curve.

Theorem 20 (Green's Theorem)**.** Let D be a simple region with an oriented piecewise continuous boundary C^+. Suppose that $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are of class C^1. Then,

$$\oint_{C^+} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Remark. If $\mathbf{F} = \nabla V$ is conservative, then $\nabla \times \mathbf{F} = 0$. Since,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot \mathbf{k}$$

Green's Theorem quantifies the amount by which $\mathbf{F}$ fails to be conservative by relating the two integrals:

$$\oint_{(\partial D)^+} \mathbf{F} \cdot d\mathbf{s} \quad \text{and} \quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The corollary of the previous remark is not true.

Remark. If $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is not necessarily conservative.

Proof. Consider the vector field $\mathbf{F}$ defined on $U = \mathbb{R}^2 - \{(0,0)\}$:

$$\mathbf{F} = \frac{-y}{x^2 + y^2} \cdot \mathbf{i} + \frac{x}{x^2 + y^2} \cdot \mathbf{j}$$

We will verify that $\nabla \times \mathbf{F} = 0$ everywhere on U.

$$\nabla \times \mathbf{F} = (Q_x - P_y) \cdot \mathbf{k}$$

Observe that $P_x = Q_y$ for all $x, y \in U$.

$$Q_x = \frac{-x^2 + y^2}{x^2 + y^2} \quad \text{and} \quad P_y = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

It follows that $(Q_x - P_y) \cdot \mathbf{k} = 0$. However, $\mathbf{F}$ was carefully chosen **not** to be conservative. Assume for a contradiction that $\mathbf{F}$ is conservative. The line integral must satisfy that,

$$\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$$

for any choice of closed curve $\mathbf{c}(t)$. Take the unit circle

$\mathbf{c}(t) = (\cos t, \sin t)$ where $0 \leq t \leq 2\pi$. We obtain,

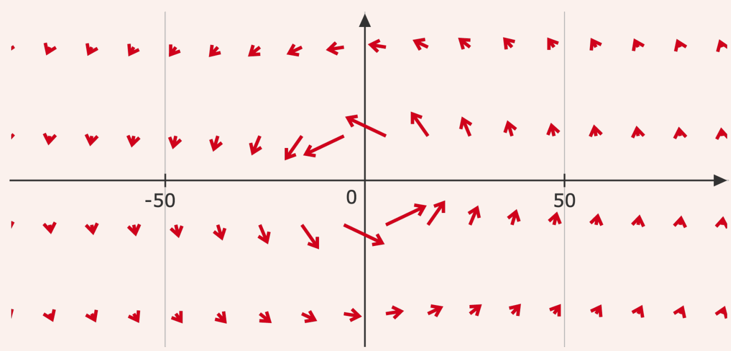
$$\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} dt \neq 0$$

□

There is more to learn from this counter-example.

Example 60: Visualizing the vector field $\mathbf{F}$

The vector field $\mathbf{F} = \frac{-y}{x^2+y^2} \cdot \mathbf{i} + \frac{x}{x^2+y^2} \cdot \mathbf{j}$ can be visualized as,



Remark. Let $\mathbf{c}(t)$ be an arbitrary closed curve. Consider

$$\mathbf{F} = \frac{-y}{x^2+y^2} \cdot \mathbf{i} + \frac{x}{x^2+y^2} \cdot \mathbf{j}$$

If $\mathbf{c}(t)$ contains $(0,0)$ in its interior, then,

$$\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 2\pi$$

Otherwise,

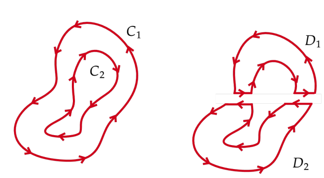
$$\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$$

Let D be a region in $\mathbb{R}^2$ containing the origin. There exists $r > 0$ such that $D_r(\mathbf{0}) \subseteq D$. The area of B_r is 2π, implying that the line integral evaluates to 2π since the remaining work is 0 by the previous calculation:

$$\iint_{D-D_r(\mathbf{0})} \underbrace{(Q_x - P_y)}_{=0} \, dxdy = 0$$

It is possible for the region D to contain holes, in which case there are multiple boundary curves. In this case, we write:

$$\oint_{\partial D^+} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D_1^+} \mathbf{F} \cdot d\mathbf{s} + \oint_{\partial D_2^+} \mathbf{F} \cdot d\mathbf{s}$$



Example 61: Verifying Green's Formula by an Example

We will verify Green's Formula for the vector field,

$$\mathbf{F} = \frac{1}{2}(-y + x)$$

where D is taken to be the interior of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parameterizing the curve gives that,

$$\mathbf{c}(t) = (a\cos t, b\sin t) \quad \text{where} \quad 0 \le t \le 2\pi$$

Taking the derivative of $\mathbf{c}(t)$,

$$\mathbf{c}'(t) = (-a\sin\theta, b\cos\theta)$$

Hence, $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{1}{2} \cdot ab$ because,

$$\mathbf{F}(\mathbf{c}(t)) = \left(\frac{1}{2} - b\sin\theta, \frac{1}{2}a\cos\theta\right)$$

Integrating from $t = 0$ to $t = 2\pi$ gives $ab \cdot \pi$. Thus,

$$\oint_{\partial D^+} \mathbf{F} \cdot ds = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = ab \cdot \pi$$

On the other hand,

$$P(x,y) = -\frac{1}{2} \cdot y \qquad Q(x,y) = \frac{1}{2} \cdot x$$

Computing the partial derivatives of P and Q,

$$P_y = -\frac{1}{2} \qquad Q_x = \frac{1}{2}$$

It follows that,

$$\iint_D (Q_x - P_y) dxdy = \iint_D \frac{1}{2} - \left(-\frac{1}{2}\right) dxdy = ab \cdot \pi$$

since this integral is simply the area of an ellipse.

Example 62: Verifying Green's Formula by an Example

We will verify Green's Formula on the line integral,

$$\oint_{\mathbf{c}} \left(2x^3 - y^3\right) dx + \left(x^3 + y^3\right) dy$$

where **c** is the unit circle in the x-y plane oriented counter-clockwise. Taking $\mathbf{c}(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$,

$$\oint_{\mathbf{c}} = \int_0^{2\pi} -2\cos^3 t \cdot \sin t + \sin^4 t + \cos^4 t + \sin^3 t \cdot \cos t dt$$

Applying the double angle formula, this evaluates to $\frac{3\pi}{2}$. Now,

$$Q_x = 3x^2 \quad \text{and } P_y = -3y^2$$

so $Q_x - P_y = 3(x^2 + y^2)$. In polar coordinates,

$$\begin{aligned} \iint_D (Q_x - P_y)\, dxdy &= \int_0^{2\pi} \int_0^1 3r^2 \cdot r dr d\theta \\ &= 2\pi \int_0^1 3r^2 dr \\ &= \frac{3\pi}{2} \end{aligned}$$

where r is the Jacobian.

A generalized formulation of Green's Theorem is that,

Theorem 21 (Stokes' Theorem)**.** Define the vector field,

$$\mathbf{F}(x, y) = P(x, y) \cdot \mathbf{i} + Q(x, y) \cdot \mathbf{j}$$

If **F** is continuously differentiable and defined on D, then,

$$\oint_{(\partial D)^+} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

where the **curl** of the vector field $\nabla \times \mathbf{F}$ represents its circulation.

Remark. Fix a vector field **F** as,

$$\mathbf{F} = P(x, y) \cdot \mathbf{i} + Q(x, y) \cdot \mathbf{j}$$

Stokes' Theorem can be seen as generalizing Green's Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (Q_x - P_y)\, dxdy$$

Remark. For any two surfaces S_1 and S_2,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}_1 = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}_2$$

because Stokes' Theorem tells us that both integrals are equal to,

$$\oint_{\mathbf{c}} \mathbf{F} d\mathbf{s}$$

where $(\partial S_1)^+ = \mathbf{c} = (\partial S_2)^+$

Proposition 8. If S is a closed surface, then,

$$\oiint (\underbrace{\nabla \times \mathbf{F}}_{\mathbf{G}}) \cdot d\mathbf{S} = \mathbf{0}$$

where $\mathbf{G}$ is called a **solenoidal vector field**.

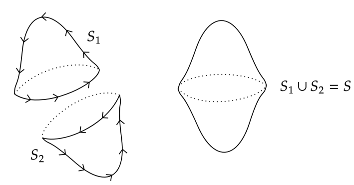
Proof. Write $S = S_1 \cup S_2$ and split the integral:

$$\oiint (\underbrace{\nabla \times \mathbf{F}}_{\mathbf{G}}) \cdot d\mathbf{S} = \iint_{S_1} + \iint_{S_2} = \oint \mathbf{F} \cdot d\mathbf{S}_1 + \oint \mathbf{F} \cdot d\mathbf{S}_2$$

Exploiting the orientations of each component,

$$\oint_{(\delta \mathbf{S}_1)^+} \mathbf{F} \cdot d\mathbf{S}_1 + \oint_{(\delta \mathbf{S}_2)^+} \mathbf{F} \cdot d\mathbf{S}_2 = \oint_{(\delta \mathbf{S}_1)^+} \mathbf{F} \cdot d\mathbf{S}_1 + \oint_{(\delta \mathbf{S}_1)^-} \mathbf{F} \cdot d\mathbf{S}_2$$

so we can cancel terms and obtain the desired integral. □



Corollary. The net flux of a solenoidal vector field is always 0.

Gauss' Divergence Theorem

In this section, we will quantify the extent to which a vector field $\mathbf{F}$ fails to be solenoidal. Recall that a solenoidal vector field,

$$\mathbf{F} = \nabla \cdot \mathbf{F} \neq 0$$

satisfies that $\nabla \cdot \mathbf{F} = 0$.

Theorem 22 (Gauss' Divergence Theorem). Let $\mathbf{F}$ be a C^1 vector field in $U \subseteq \mathbb{R}^3$. Given a bounded open subset V of U where ∂V is regular or piece-wise regular,

$$\oiint_{(\partial V)^+} \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F}) dx dy dz$$

where the left-hand side is the flux of $\mathbf{F}$ across $(\partial V)^+$ and the right-hand side is the integral divergence of $\mathbf{F}$.

Recall that the sign of $\nabla \times \mathbf{F}$ is a measure of the contraction or expansion of the flow lines of $\mathbf{F}$. Specifically,

$$\mathbf{F} < 0 \implies \text{Contraction}$$
$$\mathbf{F} > 0 \implies \text{Expansion}$$

Theorem 23 (Gauss' Law)**.** Consider the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \quad \text{where} \quad \mathbf{r} = x \cdot \mathbf{i} + y \cdot \mathbf{j} + z \cdot \mathbf{k}$$

Observe that the norm of $\mathbf{F}$ is,

$$\|\mathbf{F}\| = \left\| \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \right\| = \left\| \frac{1}{\|\mathbf{r}\|^3} \right\| \cdot \|\mathbf{r}\| = \frac{1}{\|\mathbf{r}\|^2} = \frac{1}{x^2 + y^2 + z^2}$$

defined on $U = \mathbb{R}^3 - \{\mathbf{0}\}$.

$$\oiint_{S_1} \mathbf{F} \cdot d\mathbf{s} = \begin{cases} 4\pi & \text{if } \mathbf{0} \notin S_2 \\ 0 & \text{if } \mathbf{0} \in S_2 \end{cases}$$

To prove Gauss' Law, fix a sphere at the origin and apply the Divergence Theorem to the region between the sphere and the surface.

Definition (Outgoing Flux)**.** The **outgoing flux** of a vector field $\mathbf{F}$ is,

$$\int_{(\partial D)^+} \mathbf{F} ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{n}(t) dt$$

where $\mathbf{n}(t) = (y'(t), -x'(t))$.

Remark. The outgoing flux of $\mathbf{F}$ across $(\partial D)^+$ satisfies,

$$\int_{(\partial D)^+} \mathbf{F} ds = \iint_D (\nabla \cdot \mathbf{F}) dx dy$$

Proof. Since $\mathbf{F} = P(x,y) \cdot \mathbf{i} + Q(x,y) \cdot j$,

$$\begin{aligned} \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{n}(t) dt &= \int_a^b (P\mathbf{i} + Q\mathbf{i}) \cdot (y'(t), x'(t))\, dt \\ &= \int_{(\partial D)^+} - \underbrace{P}_{\tilde{P}} dx + \underbrace{Q}_{\tilde{Q}} dy \\ &= \iint_D (P_x + Q_y) dx dy \\ &= \iint_D (\nabla \cdot \mathbf{F}) dx dy \end{aligned}$$

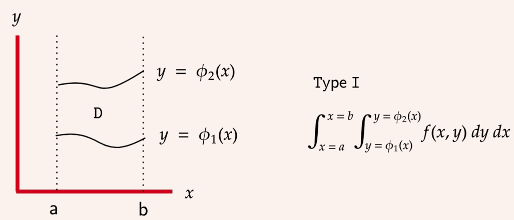
$\square$

Appendix

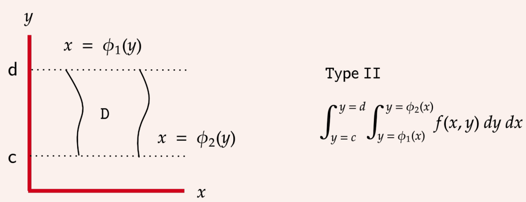
Double and Triple Integrals

The **three types** of **simple domains of integration** are,

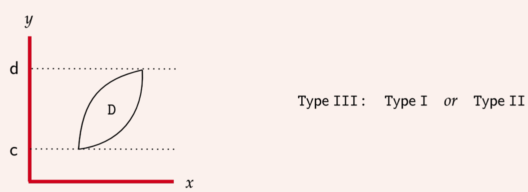
Definition (Type I). A **Type I** domain of integration is,



Definition (Type II). A **Type II** domain of integration is,



Definition (Type III). A **Type III** domain of integration is,

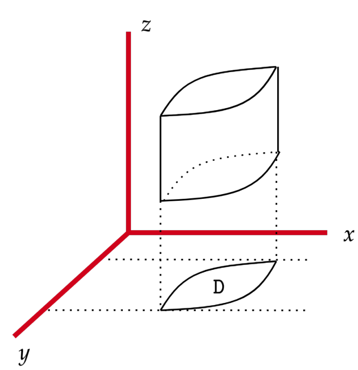


Remark. To compute a **triple integral**,

$$\iiint_D f(x, y, z) dxdydz$$

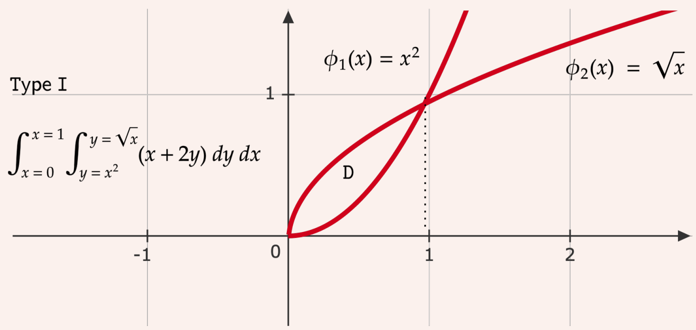
we decompose D into simple domains of integration,

$$\int_{x=a}^{x=b} \left[\int_{y=l_1(x)}^{y=l_2(x)} \left[\int_{z=\Psi_1(x,y)}^{z=\Psi_2(x,y)} f(x, y, z) dz \right] dy \right] dx$$



Example 63: $\iint_D (x+2y)dxdy$

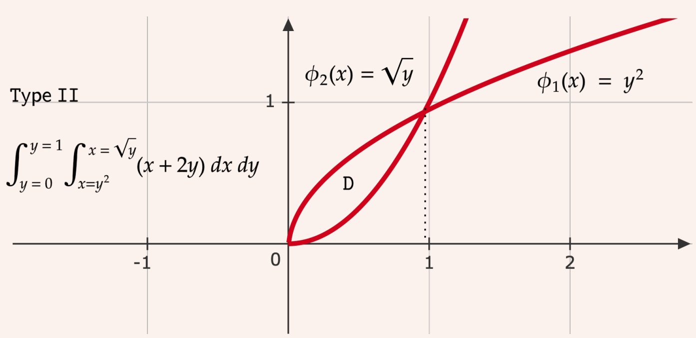
We will compute $\iint_D (x+2y)dxdy$ over a Type I domain.



In this case,

$$\int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=\sqrt{x}}(x+2y)dy\right]dx = \int_{x=0}^{x=1}\left[xy+y^2\right]_{y=x^2}^{y=\sqrt{x}}dx$$
$$= \int_{x=0}^{x=1} x^{3/2} - x^3 + x - x^4 dx$$
$$= 9/20$$

We can also compute $\iint_D (x+2y)dxdy$ over a Type II domain.



Example 64: Volume of a Sphere of Radius r

The formula for the volume of a sphere is

$$8\int_{x=0}^{x=r}\left[\int_{y=0}^{y=\sqrt{r^2-x^2}}\left[\int_{z=0}^{z=\sqrt{r^2-x^2-y^2}}dz\right]dy\right]dx = \frac{4}{3}\pi r^2$$

Change of Variables

The formula for **change of variables** for simple integrals is,

> **Theorem 24.** Under the following conditions,
>
> 1. f is continuous
> 2. $u \mapsto x(u)$ is continuously differentiable on $[a, b]$
>
> we can relate the two integrals,
>
> $$\int_a^b f(x(u)) \frac{dx}{du}(u) \cdot du = \int_{x(a)}^{x(b)} f(x) dx$$

Suppose that $u \mapsto x(u)$ is one-to-one on $[a, b] := I^*$. Let I be the interval whose endpoints are given by $x(a)$ and $x(b)$. That is,

$$I = \begin{cases} [x(a), x(b)] & x \text{ is increasing} \iff x_u(u) \geq 0 \text{ on } [a, b] \\ [x(b), x(a)] & x \text{ is decreasing} \iff x_u(u) \geq 0 \text{ on } [a, b] \end{cases}$$

If $x_u(u) \geq 0$ on $[a, b]$, then,

$$\int_a^b f(x(u)) \frac{dx}{du}(u) \cdot du = \int_{x(a)}^{x(b)} f(x) dx$$

Otherwise,

$$\int_a^b f(x(u)) \frac{dx}{du}(u) \cdot du = -\int_{x(a)}^{x(b)} f(x) dx$$

We combine these to obtain,

$$\int_{I^*} f(x(u)) \left| \frac{dx}{du}(u) \right| du = \int f(x) dx$$

where the absolute value covers both cases.

> **Theorem 25.** If $T : A^* \subseteq \mathbb{R}^2 \to A \subseteq \mathbb{R}^2$ is bijective and C^1, then,
>
> $$\iint_{A=T(A^*)} f(x, y) dx dy = \iint_{A^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$
>
> where
>
> $$\frac{\partial(x, y)}{\partial(u, v)} = \det \underbrace{\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{(\mathbf{D}(T))}$$
>
> is the determinant of the Jacobian matrix of T.

Remark. By properties of the determinant,

$$\frac{\partial(u,v)}{\partial(x,y)} = \mathbf{D}(T^{-1}) = (\mathbf{D}(T))^{-1} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

Remark. If $T : A^* \to A$ maps ∂A^* to ∂A in a one-to-one and onto fashion at $\det(\mathbf{D})(T) \neq 0$, then T is one-to-one and onto.

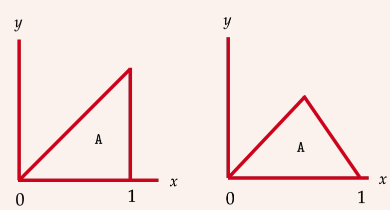
Example 65: $\iint (x+y)dxdy$

We will compute $\iint_A (x+y)dxdy$ where

$$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \text{ and } 0 \leq x \leq 1\}$$

Define the transformation

$$T : (u,v) \longmapsto \begin{cases} x(u,v) = u+v & \implies u = \dfrac{x+y}{2} \\ y(u,v) = u-v & \implies v = \dfrac{x-y}{2} \end{cases}$$



We first calculate the determinant of the Jacobian,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which gives the integral,

$$\iint_{A^*} 2u \cdot 2dudv = \int_{v=0}^{v=1} \int_{u=v}^{u=1-v} 2u \cdot 2dudv = \frac{1}{2}$$

where $x+y = 2u$ by our choice of $x = u+v$ and $y = u-v$.

Example 66: $\iint_A \left(1+x^2+y^2\right)^{3/2} dxdy$

We will compute $\iint_A \left(1+x^2+y^2\right)^{3/2} dxdy$ where A is the unit disk in the $x-y$ plane. Define the transformation,

$$T : (r,\theta) \longmapsto (x = r\cos\theta, y = r\sin\theta)$$

We begin by writing A in polar coordinates,

$$\{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi\}$$

The determinant of the Jacobian matrix is,

$$(\mathbf{D})(T) = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\sigma \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

which implies that,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r$$

It follows that the desired integral is,

$$\begin{aligned} \iint_{A^*} \left(1+r^2\right)^{3/2} r dr d\theta &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (1+r^2)^{3/2} r dr d\theta \\ &= 2\pi \int_1^2 u^{3/2} \frac{1}{2} du \\ &= \frac{2\pi}{5}(4\sqrt{2}-1) \end{aligned}$$

Theorem 26. If $T : A^* \subseteq \mathbb{R}^3 \to A \subseteq \mathbb{R}^3$ is bijective and C^1, then,

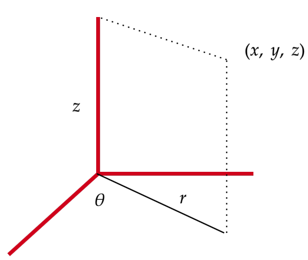
$$\iiint_{A=T(A^*)} f(x,y,z) dx dy dz$$

is equal to the integral,

$$= \iiint_{A^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du dv dw$$

Spherical and Cylindrical Coordinates

Cylindrical coordinates involve the transformation,



$$\begin{aligned} 0 &\leq r \leq \infty \\ 0 &\leq \theta \leq 2\pi \\ -\infty &< z < \infty \end{aligned}$$

$$\begin{aligned} x(r,\theta,z) &= r\cos\theta \\ y(r,\theta,z) &= r\sin\theta \\ z(r,\theta,z) &= z \end{aligned}$$

with the determinant,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{pmatrix} x_r & x_\theta & x_z \\ y_r & y_q & y_z \\ z_r & z_\theta & z_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

Example 67: $\iiint_A \left(x^2+y^2+z^2\right) dxdydz$

We will use cylindrical coordinates to compute the integral

$$\iiint_A \left(x^2+y^2+z^2\right) dxdydz$$

along the z axis, where,

$$A = \{(x,y,z) \mid x^2+y^2 \leq 2 \text{ and } -2 \leq z \leq 3\}$$

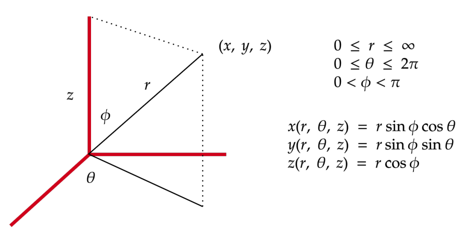
Applying the change of variables formula,

$$\iiint_{A^*} (r^2\cos^2\theta + r^2\sin^2\theta + z^2) r dr d\theta dz$$

We obtain that,

$$\int_{\theta=0}^{\theta=2\pi}\int_{z=-2}^{z=3}\int_{r=0}^{r=\sqrt{2}} (r^2+z^2) r dr dz d\theta = \pi \cdot \frac{100}{3}$$

Spherical coordinates involve the transofmration,



with the determinant,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \det\begin{pmatrix} x_n & x_\theta & x_\phi \\ y_n & y_\gamma & y_\phi \\ z_n & z_\theta & z_\phi \end{pmatrix}$$

which evaluates to,

$$\begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & -\sin\phi\cos\phi & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix} = -r^2\sin\phi$$

Example 68: Two-Step Integration

Let $a > 0$, $b > 0$, and $c > 0$. We will compute the integral,

$$\iiint_A \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{c^2}{d^2}\right) dxdydz$$

where,

$$A = \left\{(x, y, z) \in \mathbb{R} \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

We begin by defining the transformation,

$$x = au \quad y = bv \quad z = cw$$

so that,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2u^2}{a^2} + \frac{b^2v^2}{a^2} + \frac{c^2w^2}{c^2}$$

Thus, $A^* = \{(u, v, w) \mid u^2 + v^2 + w^2 \leq 1\}$ is a unit solid sphere. In particular,

$$\frac{\partial(x, y, z)}{\partial(u, w, w)} = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$$

Putting everything together,

$$\iiint_{A^*} (u^2 + v^2 + w^2) abc \cdot dudvdw$$

evaluates to,

$$\int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^4 \sin\phi drd\theta d\phi = abc \cdot \frac{4\pi}{5}$$