Origami-Constructible Numbers

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1 Origami-Constructible Numbers

1.1 Definition

We consider origami in the context of complex numbers and algebra. In an origami construction, we start with a sheet of paper that we can consider to be infinitely large. The sheet of paper also has two points marked on it: 0 and 1. The sheet represents the complex plane, with the real axis going through the points 0 and 1 and the imaginary axis perpendicular to the real axis, intersecting it at the point 0. In this way, we can talk about points and complex numbers interchangeably.

Origami constructions consist simply of a series of folds in the paper. When the paper is folded and unfolded, it leaves a crease which, in our constructions, acts as a line. In our origami constructions, a point only exists if it lies at the intersection of two creases. The types of folds we can do, all of which are straight lines, are defined by the axioms in Section 1.2. A complex number x is *origami-constructible* if, starting with a sheet of paper with 0 and 1 marked, we can make a series of folds such that two of the lines intersect at a point p that corresponds to x's position on the complex plane.

1.2 The Axioms of Origami

Huzita's axioms are a well-known set of folding axioms in origami. According to Hatori [3], Robert Lang has proven that these axioms are complete, *i.e.*there is no origami fold that cannot be done with these axioms. Hatori also shows that the axioms could be more concise; all six axioms can be reproduced using the single axiom O6. Huzita's axioms are as follows:

O1. Given points p_1 and p_2, we can fold a line that goes through both of them.

O2. Given points p_1 and p_2, we can fold p_1 onto p_2 (*i.e.* find the perpendicular bisector of segment p_1p_2).

O3. Given two lines l_1 and l_2, we can fold l_1 onto l_2 (*i.e.* bisect the angle between them).

O4. Given a point p and a line l, we can fold a line perpendicular to l that goes through p.

O5. Given two points p_1 and p_2 and a line l, we can fold p_1 onto l with a line that goes through p_2.

O6. Given two points p_1 and p_2 and two lines l_1 and l_2, we can fold p_1 onto l_1 and p_2 onto l_2 with a single line.

We can think of each axiom as a function that takes points and lines as parameters and evaluates to a line, namely the fold made in the application of the axiom). For simplicity, we can define a seventh function Isect, which takes two lines as parameters and evaluates to their point of intersection. One should note that Isect is undefined when applied to parallel lines.

1.3 Constructing the Axes

When we begin the process of origami construction, we have only the complex plane (our sheet of paper), and two points: 0 and 1. Applying O1 to 0 and 1 gives us the real axis. Applying O4 to 0 and the real axis gives us the imaginary axis, but we do not have any reference points on it. We can obtain the point i quite easily. First we apply O4 to 1 and the real axis to obtain l_1, which is parallel to the imaginary axis and goes through 1. We then evaluate $O5(0, 1, l_1)$, which gives us a line l_2 that passes through 1 and i. l_2 intersects the imaginary axis at the point i.

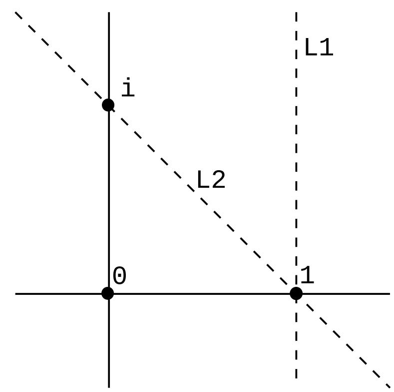

Figure 1: Marking the axes and the point i

2 Field Operations

The origami-constructible numbers form a subfield of $\mathbb{C}$. They are contained in $\mathbb{C}$ and we will show that, for any origami-constructible numbers α and β, $\alpha - \beta$ and $\alpha\beta$ are origami-constructible and α^{-1} is origami-constructible if $\alpha \neq 0$.

2.1 Elementary Operations

To assist us in our origami calculations, we will define two more elementary operations. These are essentially macros that can be created by composing the functions from Huzita's axioms.

E1. Given a point p and a line l, fold a line parallel to l that goes through p.

E1 consists of two applications of O4. It can be defined as $\mathrm{E1}(p, l) = \mathrm{O4}(p, \mathrm{O4}(p, l))$. The first application of O4 gives us a line l_1 that is perpendicular to l and passes through p. The second application gives us a line l_2 that is perpendicular to l_1 (and therefore parallel to l) and passes through p.

E2. Given a point p and a line l, reflect p across l .

We first apply O4 to p and l to get a line l_1 that is perpendicular to l and passes through p. We then pick some point p_1 on l that is not on l_1 — such a point can be made by taking the intersection of l with a line that goes through p and either 0, 1, or i. Now we apply O5 to p, p_1, and l_1 to fold p over l and we keep our plane folded. While folded, we can mark the line going through p and p_1. This marks the point p_2, which is the reflection of p across l. We should note that E2 as a function returns a point, whereas most of our functions return lines.

2.2 Addition

The addition operation can be performed with two applications of O1 and two applications of E1. We start out with three points: p_1, p_2, and 0. We apply functions to obtain a point p_3 that is equal to $p_1 + p_2$. Essentially what we do here is find the fourth corner of the parallelogram with corners 0, p_1, and p_2. The method of addition is given in Figure 2:

$$
\begin{aligned}
l_1 &\leftarrow \mathrm{O1}(0, p_1)\\
l_2 &\leftarrow \mathrm{O1}(0, p_2)\\
l_3 &\leftarrow \mathrm{E1}(p_2, l_1)\\
l_4 &\leftarrow \mathrm{E1}(p_1, l_2)\\
p_3 &\leftarrow \mathrm{Isect}(l_3, l_4)
\end{aligned}
$$

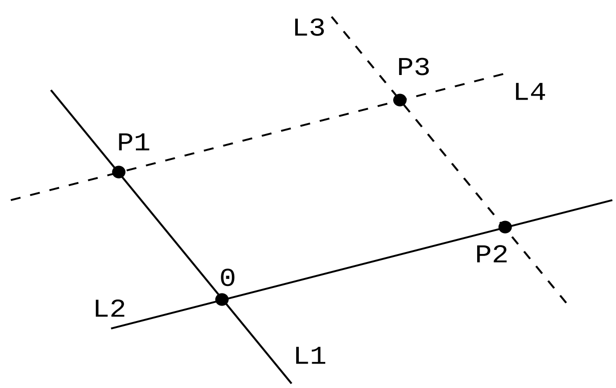

Figure 2: Addition of complex numbers in origami

When our two addends p_1 and p_2 are scalar multiples of each other, we run into difficulty with this method because the parallelogram lies on a single line and is degenerate. When this is the case we use another method for addition. Assume, without loss of generality, that p_2 has greater magnitude than p_1. First, we apply O1 to mark the line l_1 passing through 0, p_1, and p_2. Next, we fold p_2 onto p_1 using O2 and keep the paper folded over (in the case where $p_1 = p_2$, we instead make this fold by applying O4 to p_1 and l_1). Now our point p_3 is lying on top of 0. With the paper still folded, we apply O4 to 0 and l_1. This will mark our sum p_3.

In origami, negating a number is equivalent to reflecting it across 0. To do this to a point p, we simply evaluate $\mathrm{E2}(p, \mathrm{O4}(p, \mathrm{O1}(0, p)))$.

2.3 Multiplication

The first thing we need to note about multiplication is that we can implement it with two smaller operations: multiplication by a real number and multiplication by i. Since $(a, b)(c, d) = ac + (ad + bc)i - bd$, this (along with our addition operation) suffices. We use the properties of similar triangles to multiply a number p_1 by a real number r. The method is shown in Figure 3:

$$\begin{array}{rcl} l_1 & \leftarrow & \mathrm{O1}(0, p_1) \\ l_2 & \leftarrow & \mathrm{O1}(1, p_1) \\ l_3 & \leftarrow & \mathrm{E1}(r, l_2) \\ p_2 & \leftarrow & \mathrm{Isect}(l_1, l_3) \end{array}$$

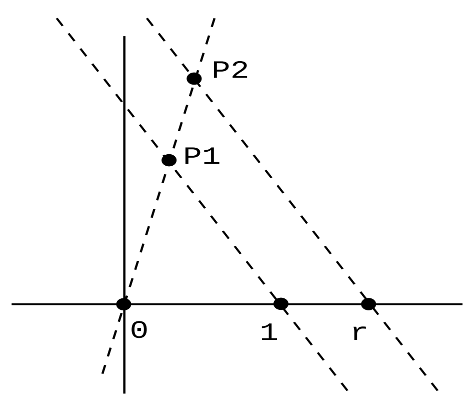

Figure 3: Multiplication by a real number in origami

We should note that this method will not work if p_1 is a real number. If this is the case, though, we can simply add i to p_1, multiply by r, and project this product onto the real axis with an application of O4.

To multiply a number by i we need to rotate it $\pi/2$ radians counterclockwise around 0. It is not difficult to verify that this can be accomplished with the following applications.

$$\begin{aligned} l_1 &\leftarrow \mathrm{O1}(0, p_1) \\ l_2 &\leftarrow \mathrm{O4}(0, l_1) \\ l_3 &\leftarrow \mathrm{O2}(l_1, l_3) \\ p_2 &\leftarrow \mathrm{E2}(p_1, l_3) \end{aligned}$$

Now that we can add, negate, multiply by reals, and multiply by i, we can combine these operations to multiply any two complex numbers.

2.4 Inversion

We know that the inverse of a complex number (a, b) is equal to $(a - bi)/(a^2 + b^2)$. To perform this calculation, we simply need to add division by real numbers to our bag of tricks. Again, this can be done using similar triangles. The process of dividing p_1 by r is nearly identical to real multiplication, and is shown in Figure 4:

$$\begin{aligned} l_1 &\leftarrow \mathrm{O1}(0, p_1) \\ l_2 &\leftarrow \mathrm{O1}(r, p_1) \\ l_3 &\leftarrow \mathrm{E1}(1, l_2) \\ p_2 &\leftarrow \mathrm{Isect}(l_1, l_3) \end{aligned}$$

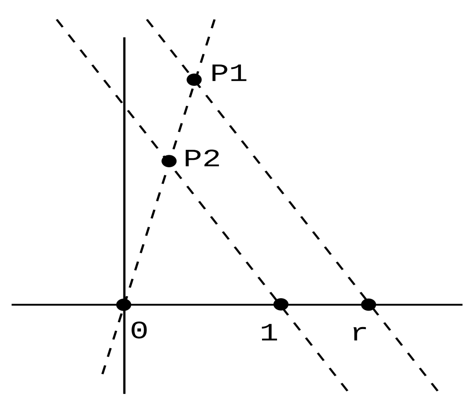

Figure 4: Division by a real number in origami

We run into problems when p_1 is a real number, but we can solve this just as we do when multiplying. With our added ability to divide by real numbers, we can now take the inverse of any nonzero complex number on our plane.

3 Square Roots

To take the square root of a complex number, we consider its polar representation. We then take the square root of its magnitude and cut its angle in half to obtain the complex square root. We must therefore provide two new operations: taking the square root of a positive real number and bisecting an angle. Bisecting an angle in origami is trivial — we simply apply O2 to the two lines that define the angle.

Taking the square root of a real number r using origami requires the use of circle geometry. The equation

$$\left(\frac{x+1}{2}\right)^2 = \left(\frac{x-1}{2}\right)^2 + y^2$$

gives us $y = \sqrt{x}$. With this knowledge, we consider a circle centred at $(0, (r-1)/2)$ with radius $(r+1)/2$ and note that it will intersect the real axis at the point $p_3 = (\sqrt{r}, 0)$. We do not need to draw the entire circle — only its intersection with the positive real axis. The method for constructing p_3, the square root of r, is shown in Figure 5:

$$\begin{array}{rcl} p_1 & \leftarrow & ir \\ p_2 & \leftarrow & (p_2 - i)/2 \\ l_3 & \leftarrow & \mathrm{O5}(-i, p_2, \text{real axis}) \\ p_3 & \leftarrow & \mathrm{E2}(-i, l_3) \end{array}$$

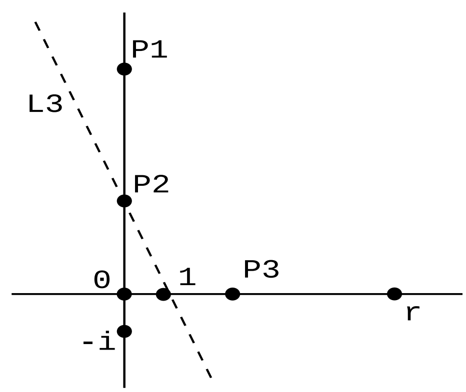

Figure 5: Taking the square root of a real number in origami

To take the square root of a complex number p_1, we first bisect the angle between the positive real axis and p_1 to obtain the angle bisector l_1, then rotate p_1 onto the positive real axis using O5 and E2 to get the positive real magnitude of p_1, which we will call r. We now take the square root of this magnitude, call it p_2. Finally, we rotate p_2 back to l_1 using O5 and E2 to obtain p_3, which is the complex square root of our original point p_1.

4 Cube Roots

Hatori Koshiro [3] gives a method for solving cubic equations in $\mathbb{R}[x]$ of the form $x^3 + ax^2 + bx + c$ using origami. In his construction, we construct the points $p_1 = (a, 1)$ and $p_2 = (c, b)$. Considering the real axis as the x-axis and the imaginary axis to be the y-axis, we also construct the two lines l_1 and l_2, defined by $y = -1$ and $x = -c$, respectively. We then apply O6 to p_1, p_2, l_1, and l_2 to obtain a new line l_3. This new line is defined by the equation $y = mx + z$ (since we know it is not vertical), where m is a solution to the original cubic equation.

This fact can be seen more easily by considering two parabolas A and B, where A is the locus of points equidistant from p_1 and l_1 and B is the locus of points equidistant from p_2 and l_2 (see Figure 6). By our definition of l_3, we know that l_3 must be tangent to both A and B. A is defined by the equation $(x - a)^2 = 4y$ and B is defined by the equation $(y - b)^2 = 4cx$. Considering the two points of tangency of l_3 to A and B, let us call them (x_1, y_1) and (x_2, y_2) respectively. From our equation for A we know that l_3 has the equation $(x_1 - a)(x - x_1) = 2(y - y_1)$. We can now see that, in our equation $y = mx + z$ for l_3, $z = -m^2 - am$. From our equation for B we know that l_3 has the equation $(y_2 - b)(y - y_2) = 2c(x - x_2)$. From this we can obtain $m = 2c/(y_2 - b)$ and $z = y_2 - 2cx_2/(y_2 - b)$. From these equations, we get $z = b + c/m$, so we can finally see that $m^3 + am^2 + bm + c = 0$.

We can find the cube root of a positive real number r by solving $x^3 - r = 0$ in this manner. To take the cube root of a complex number p_1, we need to cube root its magnitude then trisect its angle. Like when we find the square root of a complex number, we can rotate p_1 onto the positive real axis, take its cube root, then rotate it back 1/3 of its original angle. This also requires angle trisection, which is more complicated than bisection.

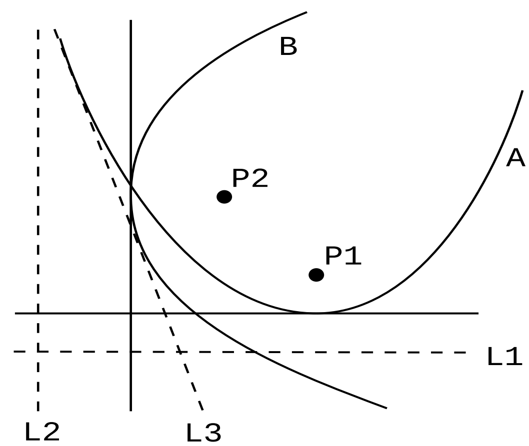

Figure 6: Solving a cubic in $\mathbb{R}[x]$ in origami

Angle trisection can be solved as a cubic polynomial. We can note the sine identity $\sin(3x) = 3\sin x - 4\sin^3 x$. Given an angle θ we can find $\theta/3$ by solving $x^3 - 3x/4 + (\sin\theta)/4 = 0$. The sine function and its inverse can be easily calculated using rotations and projections in origami.

5 Origami Field Theory

We showed in Section 2 that origami-constructible numbers form a subfield $\mathbb{O}$ of $\mathbb{C}$, so we can say that $\mathbb{Q} \subseteq \mathbb{O} \subseteq \mathbb{C}$. Sections 3 and 4 tell us that we can solve quadratic and cubic polynomials. This means that, for any subfield k of $\mathbb{O}$, if an extension K/k has degree 2 or 3, then K is also a subfield of $\mathbb{O}$. The consequence of this is as follows. We know that a complex number α is origami-constructible if there exists a tower of fields

$$\mathbb{Q} \subset K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_n = \mathbb{Q}(\alpha),$$

where every extension K_{i+1}/K_i has degree 2 or 3 [2]. If α is origami-constructible, its conjugate $\bar{\alpha}$ can be constructed from α by reflecting it across the imaginary axis by applying the operation E2.

A central question in origami construction is whether or not a regular n-gon can be constructed. If such an n-gon can be constructed centred at 0 with a point on 1, then the vertices of the n-gon are exactly the n^{th} roots of unity. Likewise, if a primitive n^{th} root of unity is origami-constructible, we can generate all n^{th} roots of unity via multiplication, then connect them to form a regular n-gon.

The primitive n^{th} root of unity is origami-constructible if and only if n is of the form $n = 2^a 3^b p_1 \dots p_n$, where each p_i is a prime of the form $p_i = 1 + 2^c 3^d$. We can prove this by induction. If the primitive p^{th} root of unity ζ_p is origami constructible, we can construct the primitive $(2p)^{th}$ and $(3p)^{th}$ roots of unity by solving $x^2 - \zeta_p = 0$ and $x^3 - \zeta_p = 0$ respectively. Consider also the field extension $\mathbb{Q}(\zeta_{pq})/\mathbb{Q}(\zeta_p)$, where $q = 1 + 2^c 3^d$ is prime. The Galois group of this cyclotomic extension is $\mathbb{I}_q^\times = \mathbb{I}_{2^c 3^d}$. We therefore know that there is a tower of fields between $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_{pq})$ in which each extension has degree 2 or 3, so ζ_{pq} must be origami constructible. Thus, for any n of the desired form, the primitive n^{th} root of unity is origami-constructible. If, on the other hand, n does not have the desired form, then for any tower of fields $\mathbb{Q} \subset K_0 \subset K_1 \subset \dots \subset K_n = \mathbb{Q}(\zeta_n)$ at least one of the extensions must have degree greater than 3, which means that ζ_n is not origami-constructible.

The smallest n for which a regular n-gon is not origami-constructible is 11. To construct the 11^{th} root of unity we would need to be able to take roots of higher order. Obviously we could construct ζ_{11} if we could take eleventh roots, but this is not actually necessary. Using Galois theory, we can see that the field extension $\mathbb{Q}(\zeta_{11})/\mathbb{Q}$ has Galois group $\mathbb{I}_{11}^\times = \mathbb{I}_{10}$. There is a tower of groups $\{0\} < \mathbb{I}_2 < \mathbb{I}_{10}$ with $[\mathbb{I}_2 : \{0\}] = 2$ and $[\mathbb{I}_{10} : \mathbb{I}_2] = 5$. We therefore know that there is a corresponding tower of fields $\mathbb{Q} \subset K \subset \mathbb{Q}(\zeta_{11})$ such that $|K/\mathbb{Q}| = 5$ and $|\mathbb{Q}(\zeta_{11})/K| = 2$. Therefore we could construct the primitive 11^{th} root of unity in origami if we could solve general quadratic equations (which we can) and general quintic equations (which we cannot, even with the use of general radicals). However, it is known [4] that the quintic equations that arise in the construction of ζ_{11} have solvable cyclic groups. Therefore the additional ability to determine fifth roots would allow us to construct the primitive 11^{th} root of unity.

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