COMP251: Red-black trees

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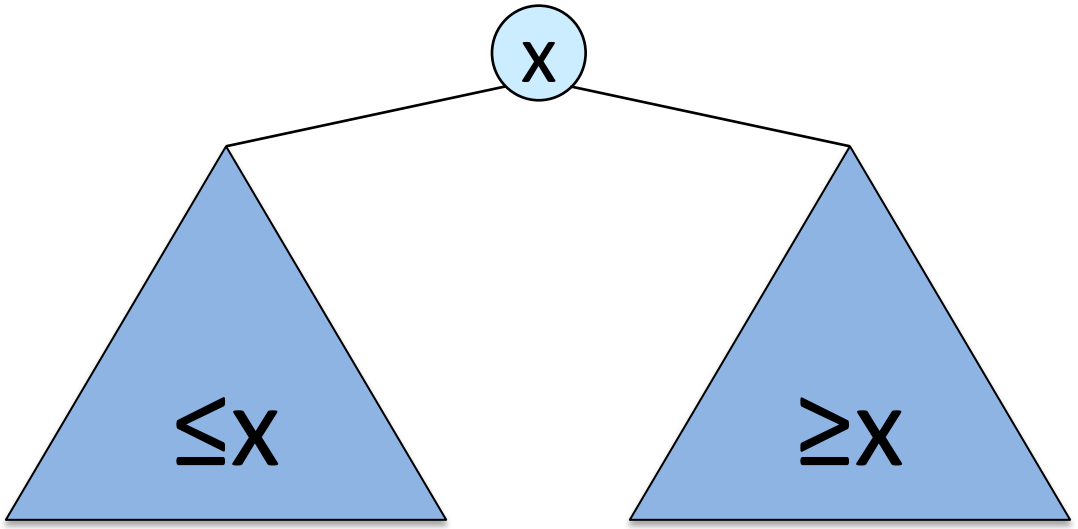
Based on (Cormen *et al.*, 2002)

Based on slides from D. Plaisted (UNC)

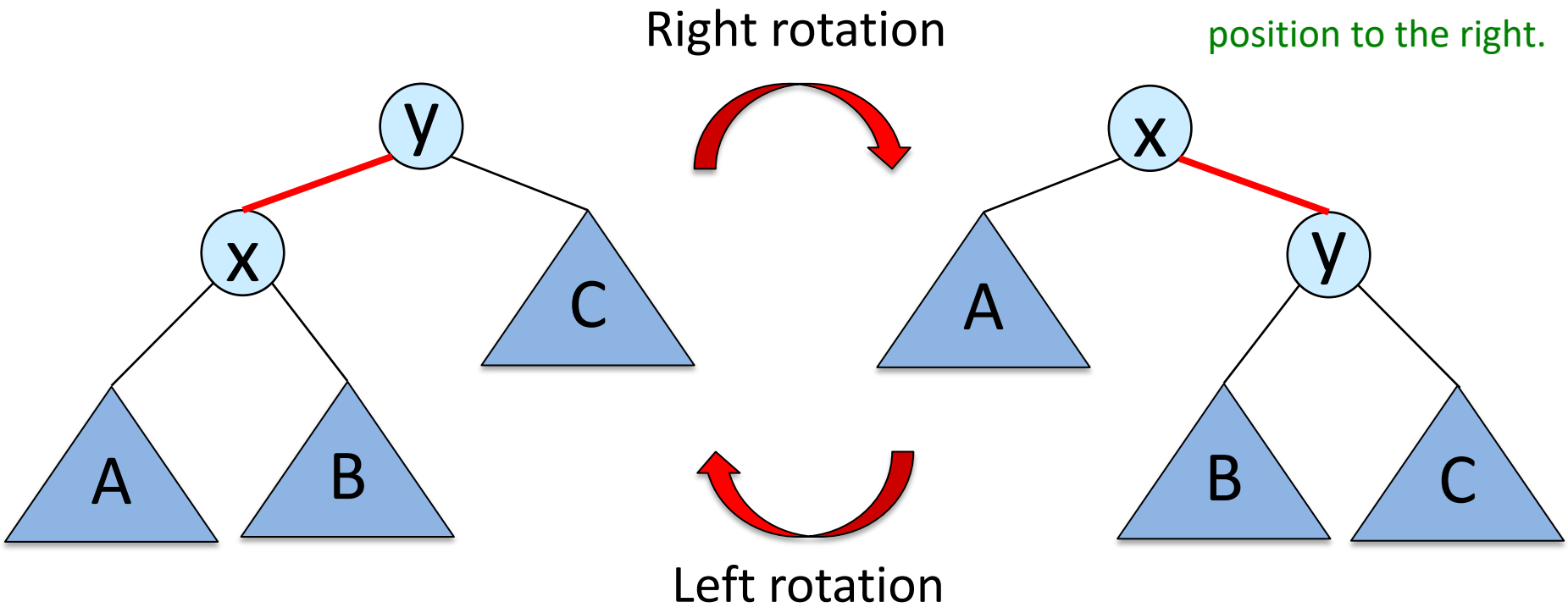
Recap: Balanced Binary Search Trees



- T is a rooted binary tree
- Key of a node x ≥ keys in its left subtree.
- Key of a node x ≤ keys in its right subtree.
- Use to store keys
- The running time of search/Insert/Delete operations depends on the height of the subtrees
- ⇒ Keep the height of subtrees as minimal as possible

Recap: Rotations

A right « rotation » moves the root one position to the right.



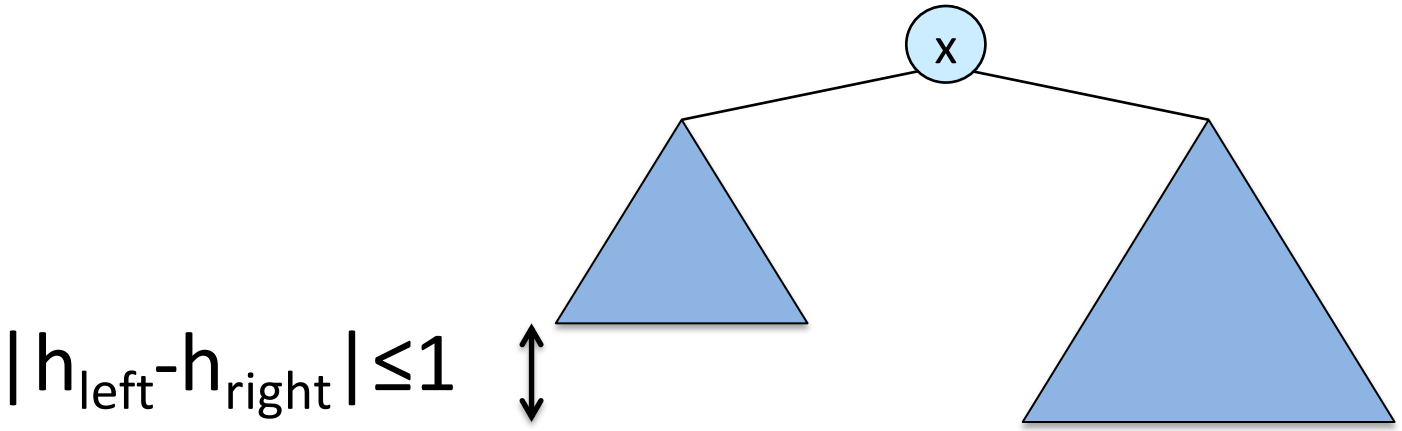
Rotations change the tree structure & **preserve the BST property**.

Proof: elements in B are ≥ x and ≤ y…

In both cases, everything in A < x < everything in B < y < everything in C

Recap: AVL trees

Definition: BST such that the heights of the two child subtrees of any node differ by at most one.



- Invented by G. Adelson-Velsky and E.M. Landis in 1962.
- AVL trees are self-balanced binary search trees.
- Insert, Delete & Search take O(log n) in average and worst cases.
- To satisfy the definition, the height of an empty subtree is -1

Red-black trees: Overview

- Red-black trees are a variation of binary search trees to ensure that the tree is **balanced**.
  - Height is $O(\lg n)$, where n is the number of nodes.
- Operations take $O(\lg n)$ time in the worst case.
- Invented by R. Bayer (1972).
- Modern definition by L.J. Guibas & R. Sedgewick (1978).

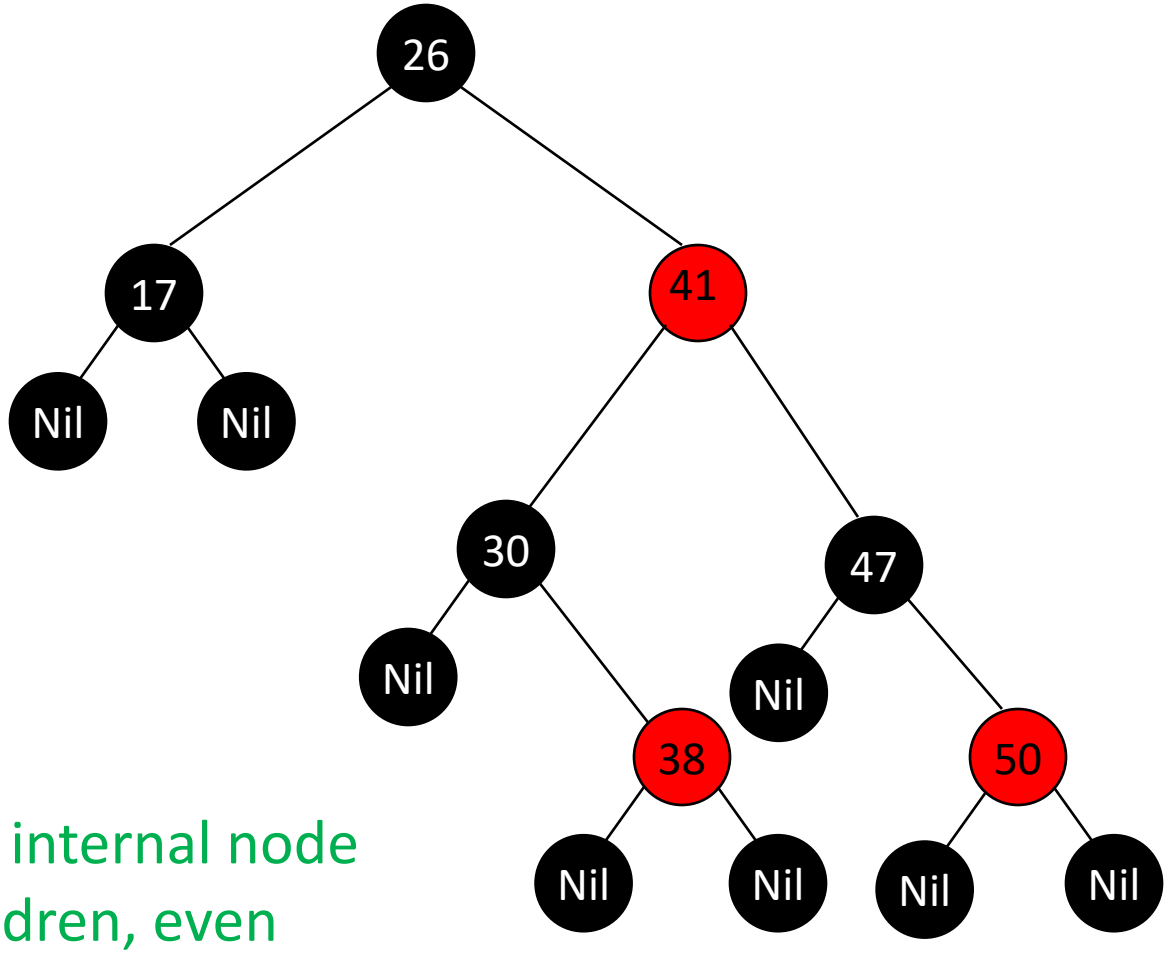
Red-black Tree

- Binary search tree + 1 bit per node: the attribute color, which is either **red** or **black**.
- All other attributes of BSTs are inherited:
  - *key*, *left*, *right*, and *parent*.
- All empty trees (leaves) are colored black.
  - Note: We can use a single sentinel, *nil*, for all the leaves of red-black tree *T*, with *color*[*nil*] = black. The root's parent is also *nil*[*T*].

Red-black (RB) Properties

1. Every node is either red or black.
2. The root is black.
3. All leaves (*nil*) are black.
4. If a node is red, then its children are black (i.e., no 2 consecutive red nodes).
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes (i.e., same black height).

Red-black Tree – Example

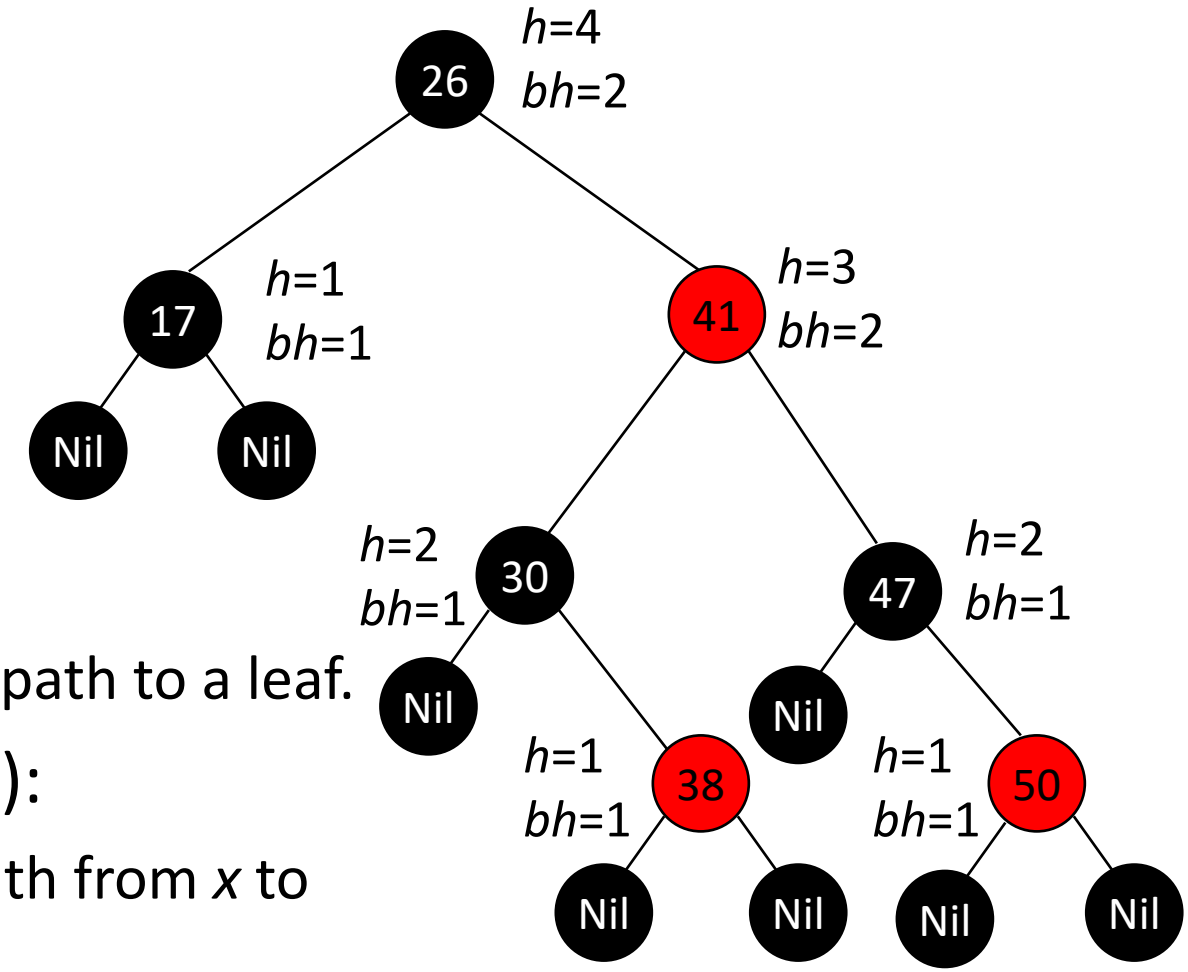


Note: every internal node has two children, even though *nil leaves are not usually shown*.

Height of a Red-black Tree

- Height of a node:
  - $h(x)$ = number of edges in the longest path to a leaf.
- Black-height of a node x, $bh(x)$:
  - $bh(x)$ = number of black nodes (including $nil[T]$) on a path from x to a leaf, not counting x.
- Black-height of a red-black tree is the black-height of its root.
  - By RB Property 5, black height is well defined.

Height of a Red-black Tree



- Height h(x):

  #edges in a longest path to a leaf.

- Black-height bh(x):

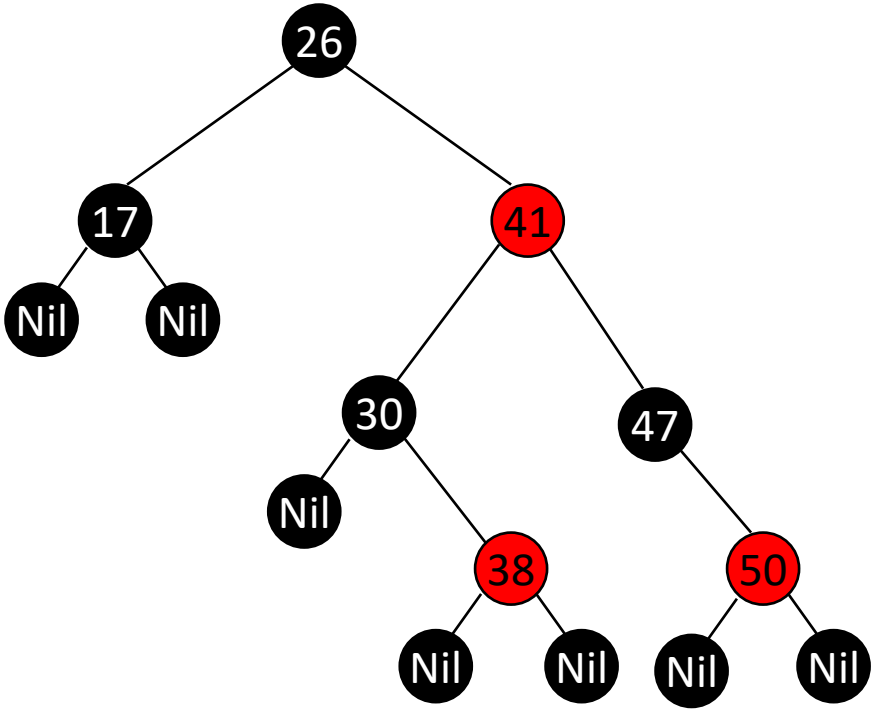
  # black nodes on path from *x* to leaf, *not counting x*.

- **Property: bh(x) ≤ h(x) ≤ 2 bh(x)**

Bound on RB Tree Height

Lemma 1: Any node x with height h(x) has a black-height bh(x) ≥ h(x)/2.

Proof: By RB property 4, ≤ h / 2 nodes on the path from the node to a leaf are red. Hence ≥ h/2 are black. ■



Bound on RB Tree Height

Lemma 2: The subtree rooted at any node x contains $\geq 2^{bh(x)} - 1$ internal nodes.

Proof: By induction on height of x.

- **Base Case:** Height $h(x) = 0 \Rightarrow x$ is a leaf $\Rightarrow \text{bh}(x) = 0$.
  Subtree has $\geq 2^0 - 1 = 0$ nodes.
- **Induction Step:**
  - Height $h(x) > 0 \Rightarrow$ Each child of x has black-height of either $bh(x)$ (child is red) or $bh(x) - 1$ (child is black).
  - By ind. hyp., each child has $\geq 2^{\text{bh}(x)-1} - 1$ internal nodes.
  - Subtree rooted at x has $\geq 2 \cdot (2^{\text{bh}(x)-1} - 1) + 1$
    $= 2^{\text{bh}(x)} - 1$ internal nodes. ■

Bound on RB Tree Height

Lemma 1: Any node x with height h(x) has a black-height bh(x) ≥ h(x)/2.

Lemma 2: The subtree rooted at any node x has $\geq 2^{bh(x)}-1$ internal nodes.

Lemma 3: A red-black tree with n internal nodes has height at most $2 \lg(n+1)$.

Proof:

- By lemma 2, $n \geq 2^{bh} - 1$,
- By lemma 1, $bh \geq h/2$, thus $n \geq 2^{h/2} - 1$.
- $\Rightarrow\ h \leq 2 \lg(n + 1)$.

Thus, a RB tree is balanced!

Insertion in RB Trees

- Insertion must preserve all red-black properties.
- Should an inserted node be colored Red? Black?
- Basic steps:
  1. Use BST Tree-Insert to insert a node *x* into *T.*
     - Procedure **RB-Insert(*x*)**.
  2. Color the node *x* in red.
  3. Use (1) node re-coloring, and (2) rotations to restore RB tree property.
     - Procedure **RB-Insert-Fixup**.

After step 2, some RB properties may be violated. Which ones?

Insertion

RB-Insert(*T*, *z*)

```
1.    y ← nil[T]
2.    x ← root[T]
3.    while x ≠ nil[T]
4.       do y ← x
5.          if key[z] < key[x]
6.             then x ← left[x]
7.             else x ← right[x]
8.    p[z] ← y
9.    if y = nil[T]
10.      then root[T] ← z
11.      else if key[z] < key[y]
12.         then  left[y] ← z
13.         else right[y] ← z
```

RB-Insert(*T*, *z*) Contd.

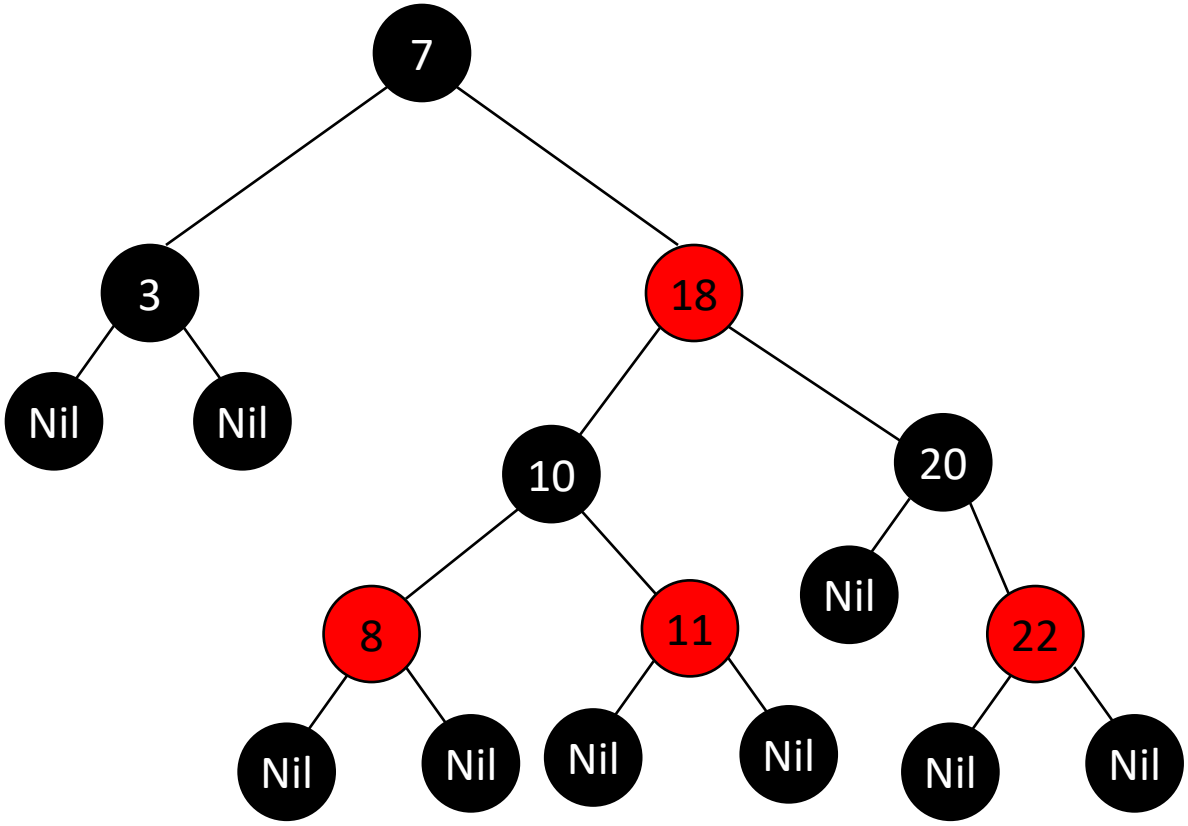
```
14.   left[z] ← nil[T]
15.   right[z] ← nil[T]
16.   color[z] ← RED
17.   RB-Insert-Fixup (T, z)
```

Principles:

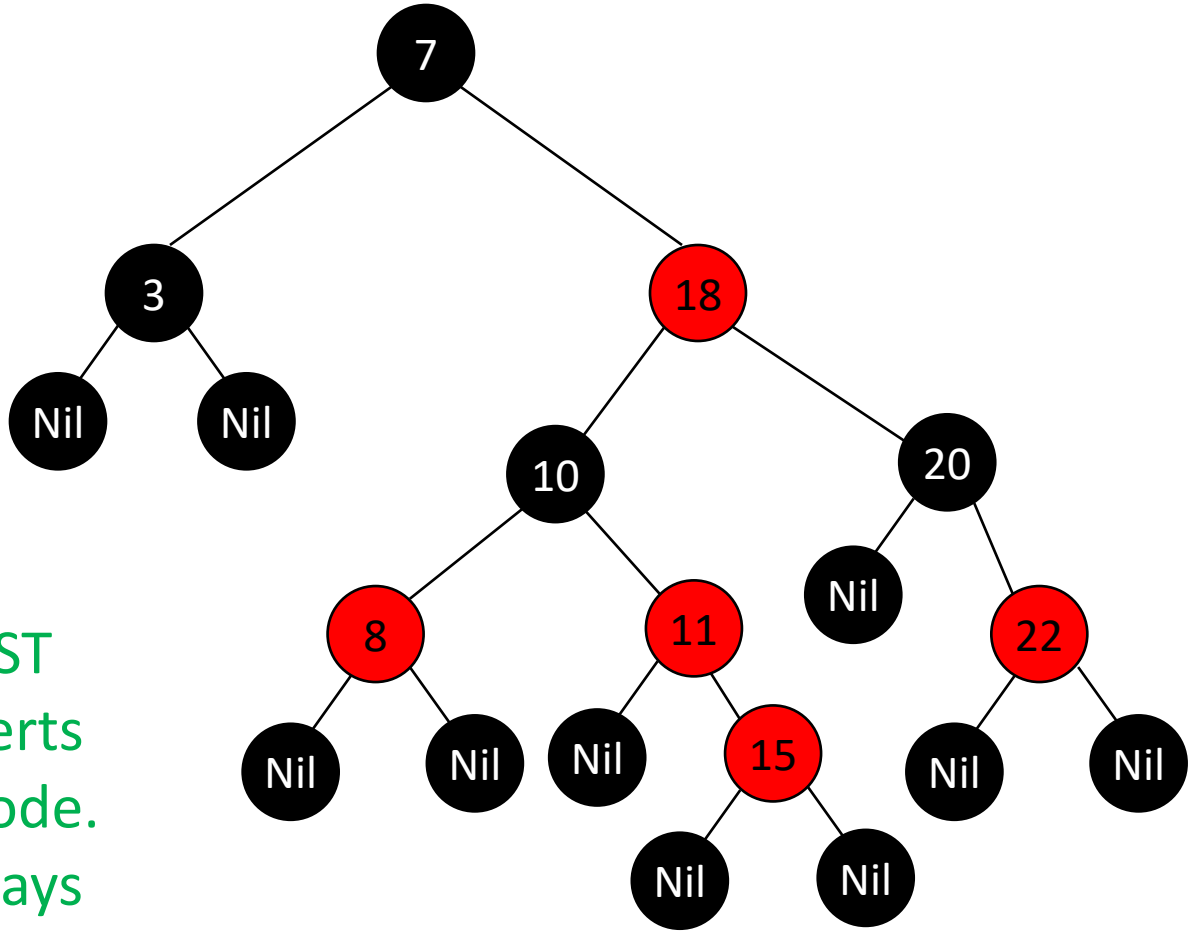
- Regular BST insert (line 1-15)
- Color assignment (line 16)
- Fixup (line 17).

What is happening in the fixup is obviously the more sophisticated procedure.

Insert RB Tree – Example



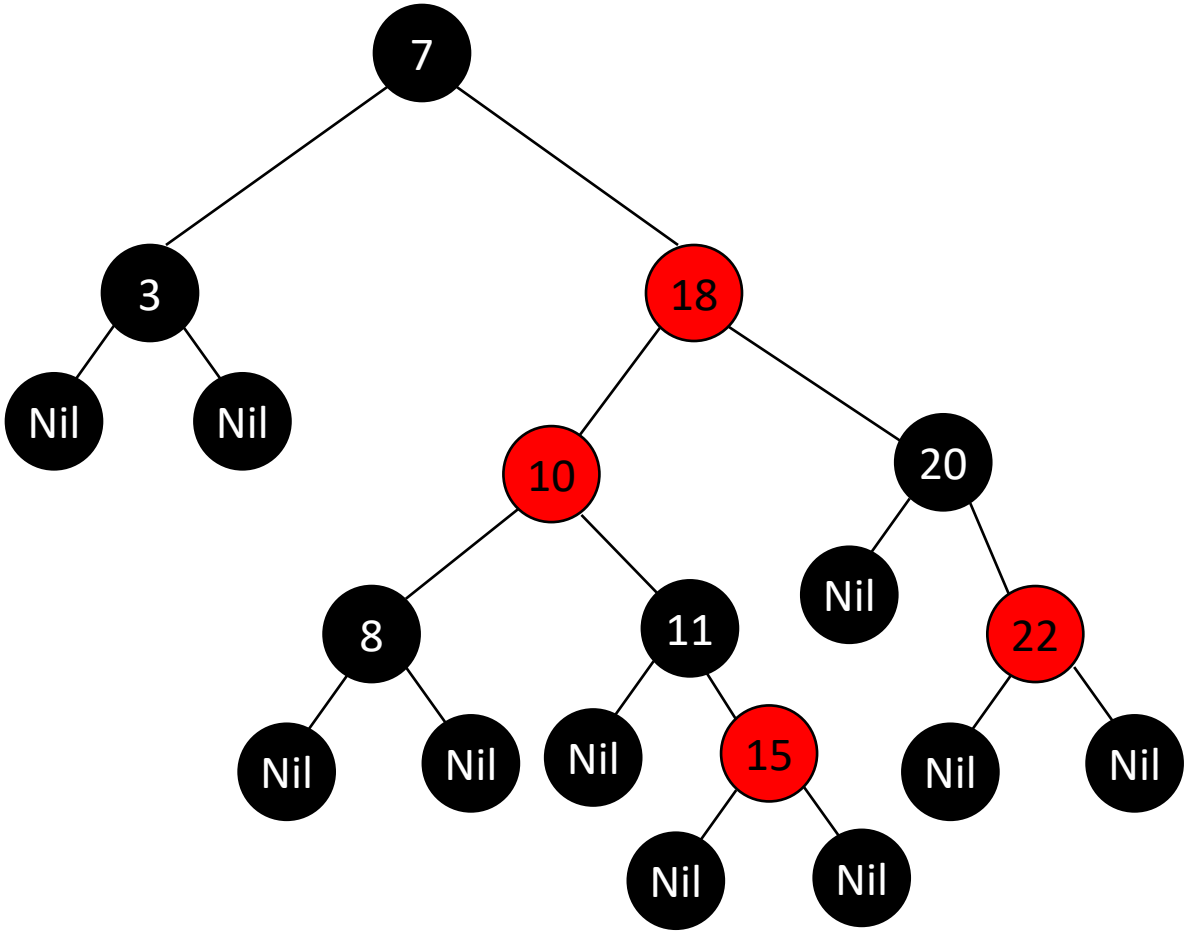
Insert RB Tree – Example



In fact, the BST insertion inserts as internal node. We must always keep all leaves as Nil (no key).

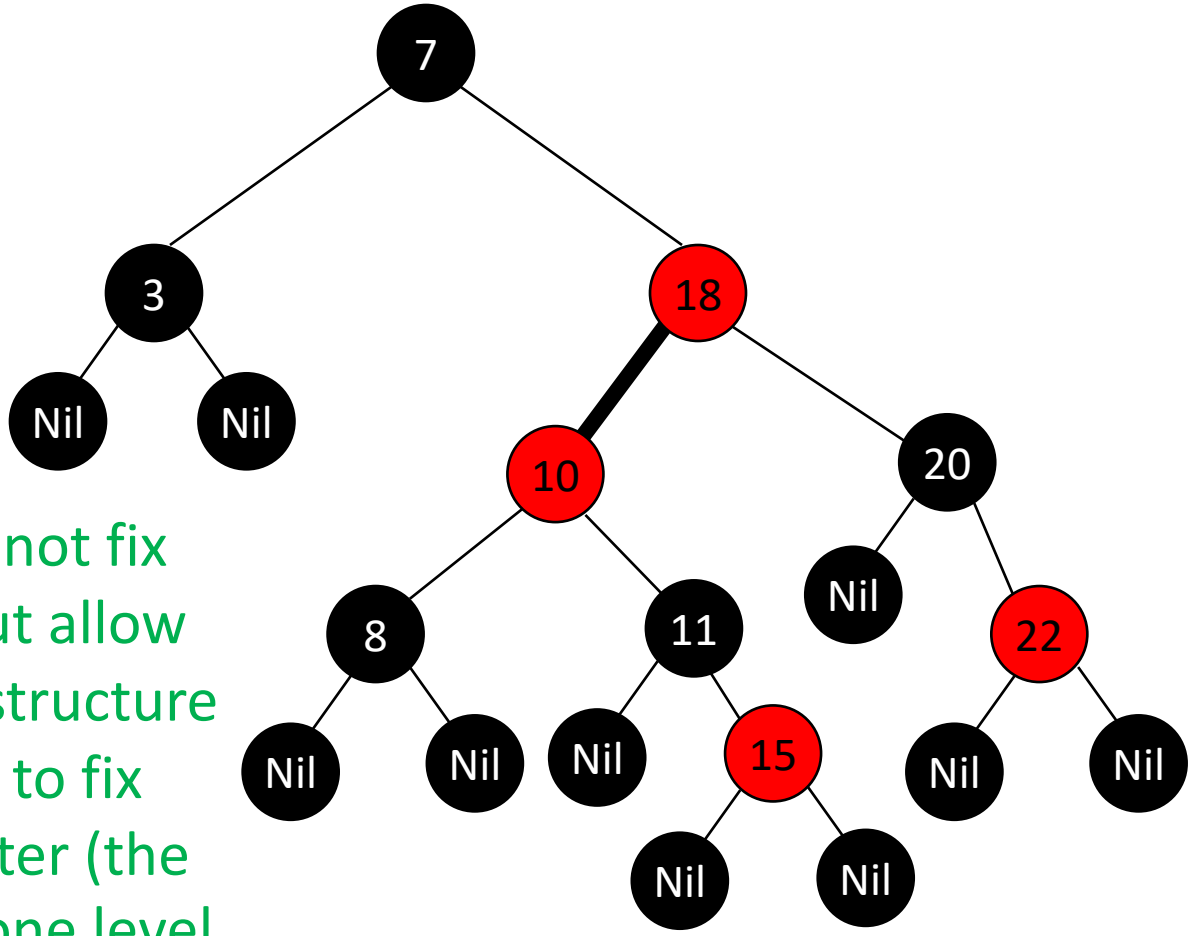
Insert(T,15)

Insert RB Tree – Example



Recolor 10, 8 &11

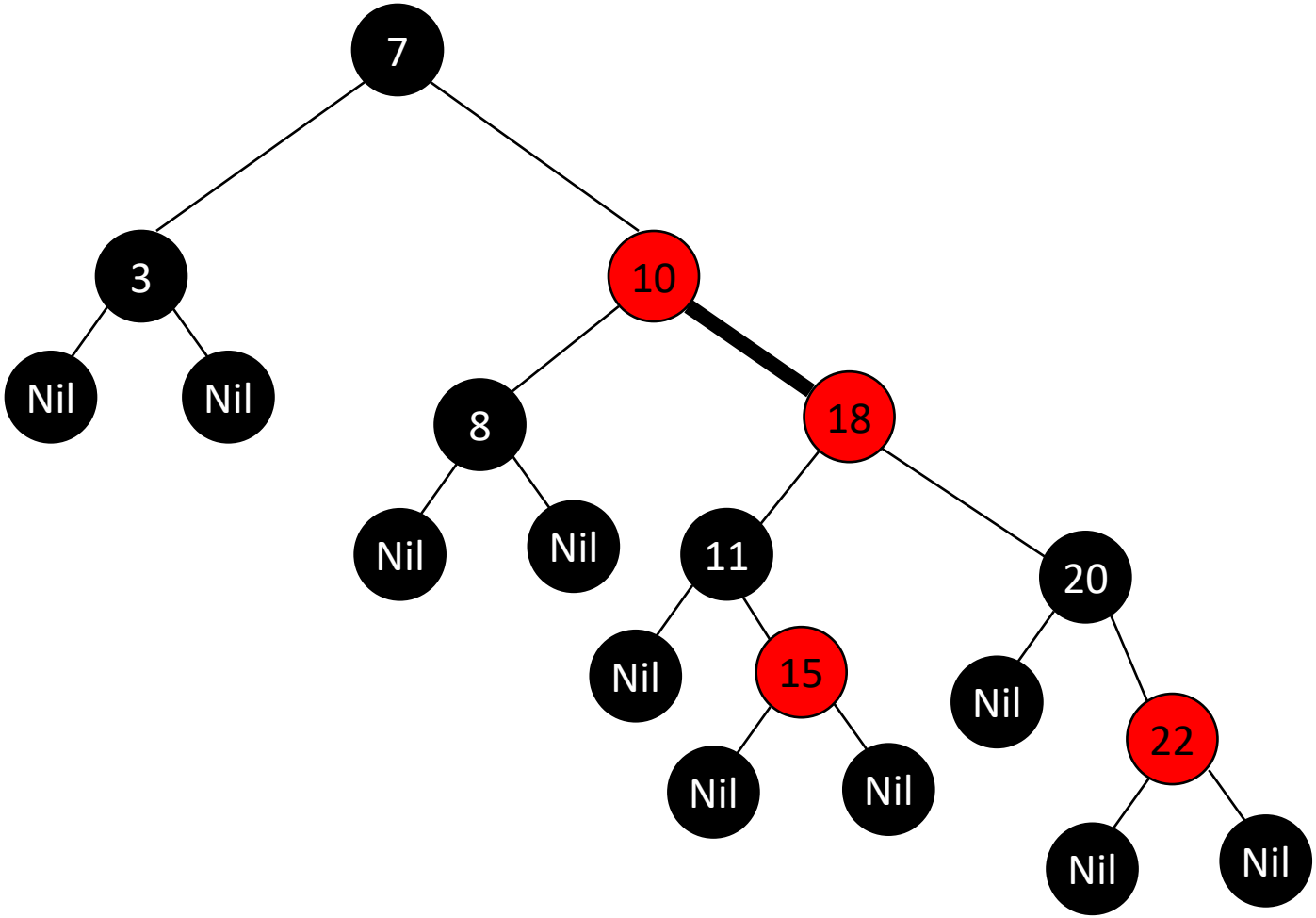
Insert RB Tree – Example



Rotations will not fix the conflict but allow to adjust the structure of the RB tree to fix the conflict after (the conflict goes one level upper).

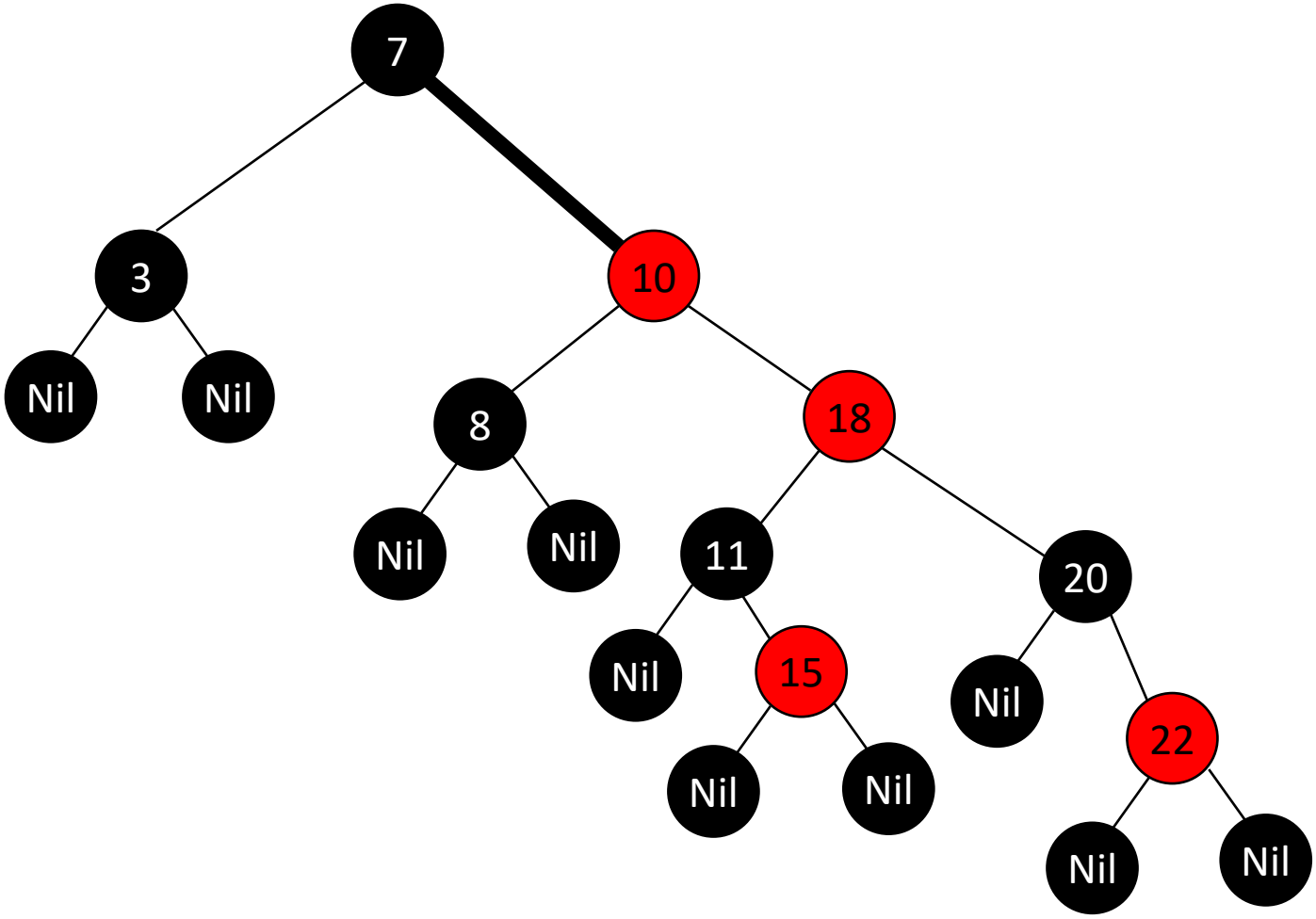
Right rotate at 18

Insert RB Tree – Example



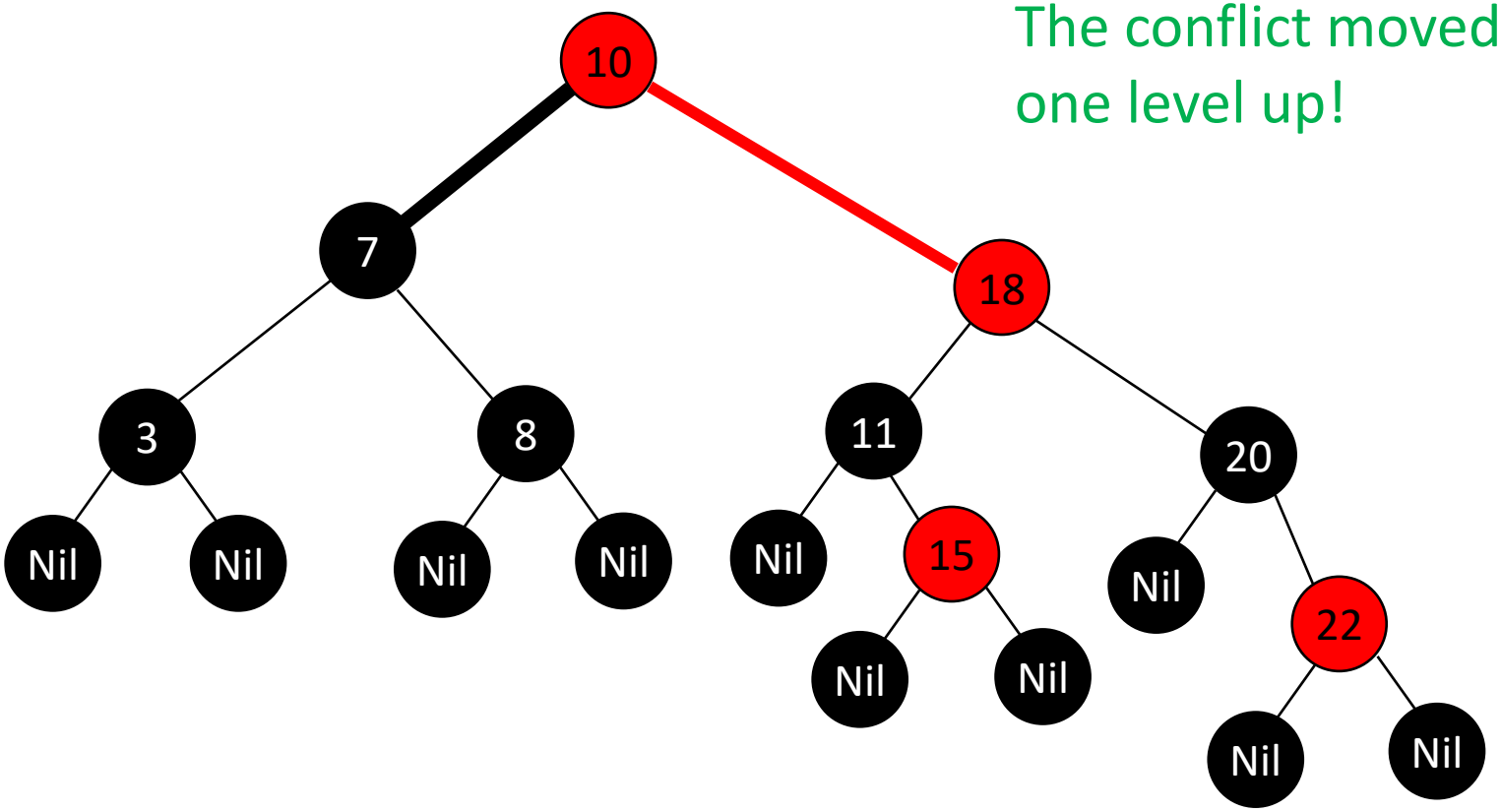
Parent & child with conflict are now aligned with the root.

Insert RB Tree – Example



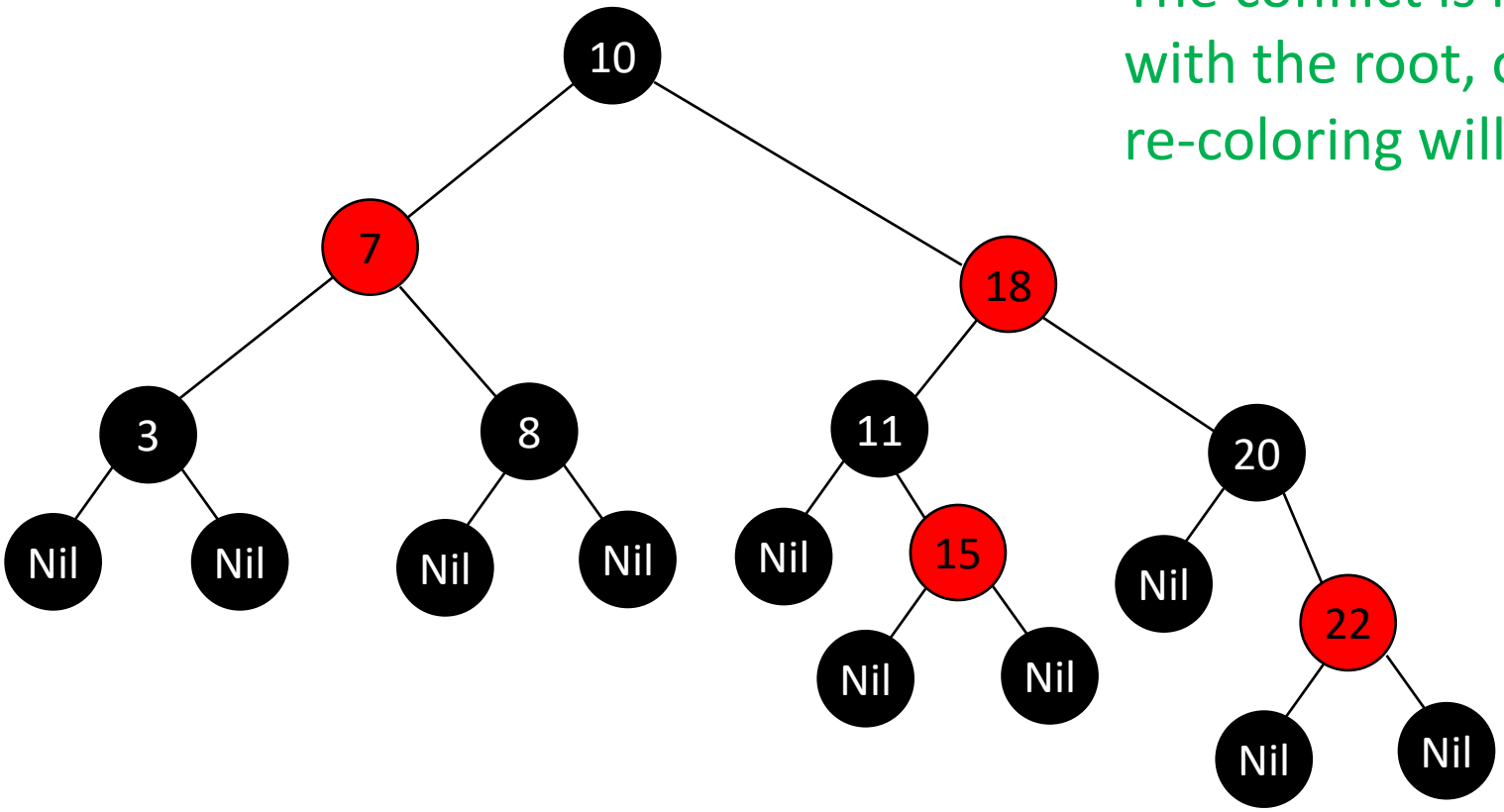
Left rotate at 7

Insert RB Tree – Example



Insert RB Tree – Example

The conflict is now with the root, only re-coloring will help.

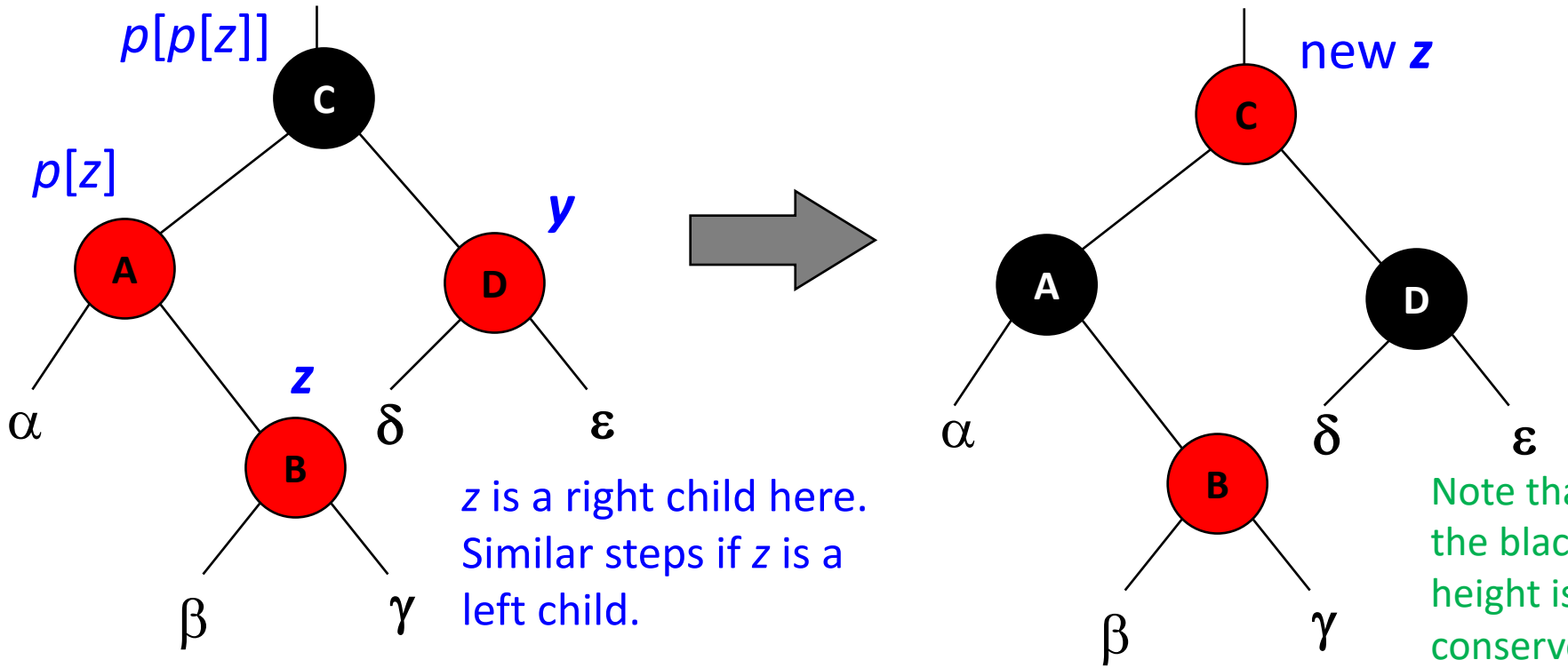


Recolor 10 & 7 (root must be black!)

General principles

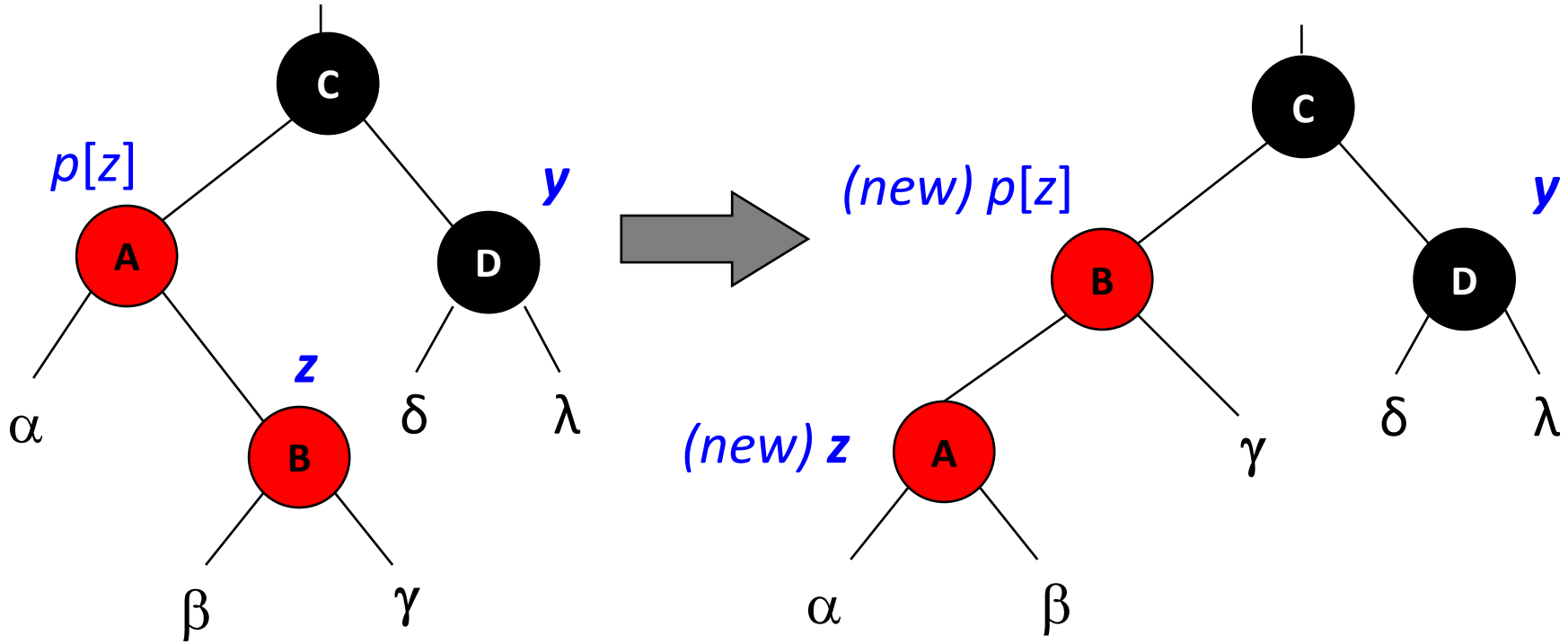
1. Proceed with a BST insertion and color the new node in red.
2. The only possible conflict is with the newly inserted node and its parent (i.e., both can be red). Fix the conflict as follow.
3. Let z be newly inserted node and y its uncle (i.e., the sibling of the parent of z). Until the conflict is resolved:
   - **Case 1:** y is red $\Rightarrow$ recolor the nodes to bring the conflict one level upper
   - **Case 2:** y is black, and conflict "aligned" with grand parent $\Rightarrow$ make one rotation to align the nodes in conflict
   - **Case 3:** y is black, and conflict "aligned" with grand parent $\Rightarrow$ make a rotation to bring the conflict one level upper
4. If needed, color the root in black

Case 1 – uncle *y* is red



- $p[p[z]]$ (z's grandparent) must be black, since z and $p[z]$ are both red and there are no other violations of property 4.
- Make $p[z]$ and y black $\Rightarrow$ now z and $p[z]$ are not both red. But property 5 might now be violated.
- Make $p[p[z]]$ red $\Rightarrow$ restores property 5.
- The next iteration has $p[p[z]]$ as the new z (i.e., z moves up 2 levels).

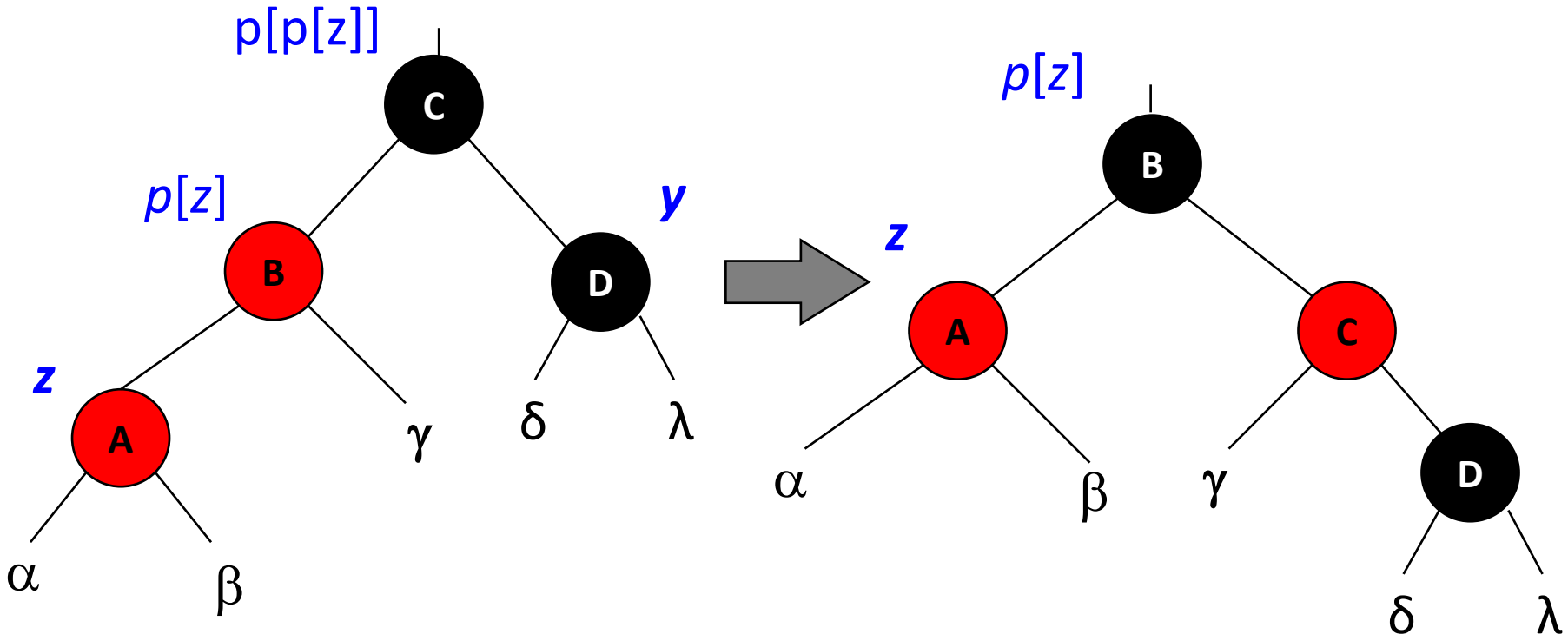
Case 2 – *y* is black, *z* is a right child



- Left rotate around *p*[*z*], *p*[*z*] and *z* switch roles ⇒ now *z* is a left child, and both *z* and *p*[*z*] are red.
- Takes us immediately to case 3.

This is an intermediate step that brings us to case 3.

Case 3 – *y* is black, *z* is a left child



- Make *p*[*z*] black and *p*[*p*[*z*]] red.
- Then right rotate right on *p*[*p*[*z*]] (in order to maintain property 4).
- No longer have 2 reds in a row.
- *p*[*z*] is now black $\Rightarrow$ no more iterations.

The rotation enables us to restore the black-height property lost with re-coloring.

Insertion – Fixup

RB-Insert-Fixup (*T*, *z*)

```
1.    while color[p[z]] = RED
2.        do if p[z] = left[p[p[z]]]
3.            then y ← right[p[p[z]]]
4.                 if color[y] = RED
5.                    then color[p[z]] ← BLACK  // Case 1
6.                         color[y] ← BLACK      // Case 1
7.                         color[p[p[z]]] ← RED  // Case 1
8.                         z ← p[p[z]]           // Case 1
```

Insertion – Fixup

RB-Insert-Fixup(*T, z*) (Contd.)

9. **else if** $z = right[p[z]]$ // color[y] $\neq$ RED

10. **then** $z \leftarrow p[z]$ // Case 2

11. LEFT-ROTATE(*T, z*) // Case 2

12. $color[p[z]] \leftarrow$ BLACK // Case 3

13. $color[p[p[z]]] \leftarrow$ RED // Case 3

14. RIGHT-ROTATE(*T,* $p[p[z]]$) // Case 3

15. **else** (if $p[z] = right[p[p[z]]]$)(same as **10-14**

16. with "right" and "left" exchanged)

17. $color[root[T]] \leftarrow$ BLACK

Algorithm Analysis

- $O(\lg n)$ time to get through RB-Insert up to the call of RB-Insert-Fixup.
- Within RB-Insert-Fixup:
  - Each iteration takes $O(1)$ time.
  - Each iteration but the last moves z up 2 levels.
  - $O(\lg n)$ levels $\Rightarrow$ $O(\lg n)$ time.
  - Thus, insertion in a red-black tree takes $O(\lg n)$ time.
  - Note: there are at most 2 rotations overall.

Correctness

Loop invariant:

- At the start of each iteration of the **while** loop,
  - z is red.
  - There is at most one red-black violation:
    - Property 2: z is a red root, or
    - Property 4: z and $p[z]$ are both red.

Correctness – Contd.

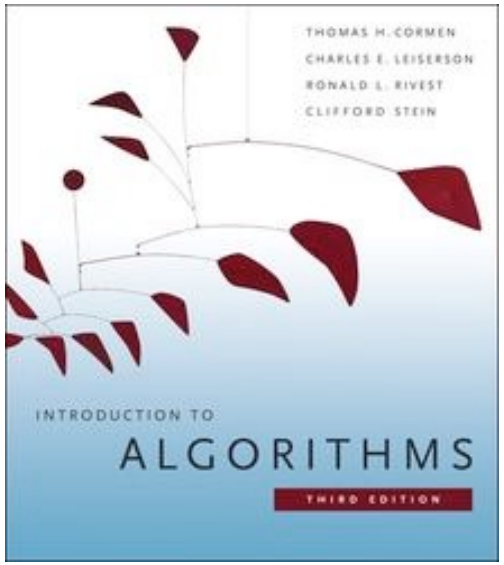
- **Initialization:** ✓
- **Termination:** The loop terminates only if $p[z]$ is black. Hence, property 4 is OK.
  The last line ensures property 2 always holds.
- **Maintenance:** We drop out when z is the root (since then $p[z]$ is sentinel $nil[T]$, which is black). When we start the loop body, the only violation is of property 4.
  - There are 6 cases, 3 of which are symmetric to the other 3. We consider cases in which $p[z]$ is a left child.
  - See cases 1, 2, and 3 described above.

AVL vs. Red-Black Trees

- AVL trees are more strictly balanced ⇒ faster search
- Red Black Trees have less constraints and insert/remove operations require less rotations ⇒ faster insertion and removal
- AVL trees store balance factors or heights with each node
- Red Black Tree requires only 1 bit of information per node

Further Readings

[CLRS2009] Cormen, Leiserson, Rivest, & Stein, *Introduction to Algorithms*. (available as E-book)



See Chapter 13 for the complete proofs & deletion