COMP251: Heaps & Heapsort

Giulia Alberini & Jérôme Waldispühl
School of Computer Science
McGill University

From (Cormen et al., 2002)

Based on slides from D. Plaisted (UNC)

Priority Queue

Assume a set of comparable elements or “keys”.

Like a queue, but now we have a more general definition of which element to remove next, namely the one with highest priority.

e.g. hospital emergency room

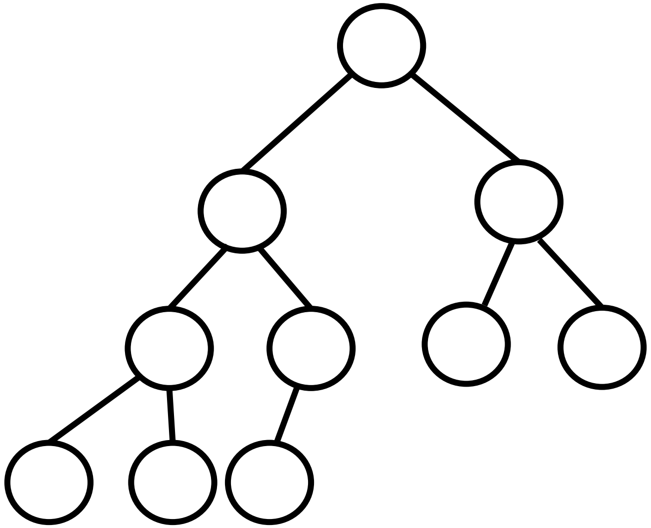
Priority Queue ADT

- add(key)
- removeMin()

“highest” priority = “number 1” priority

- peek()
- contains(element)
- remove(element)

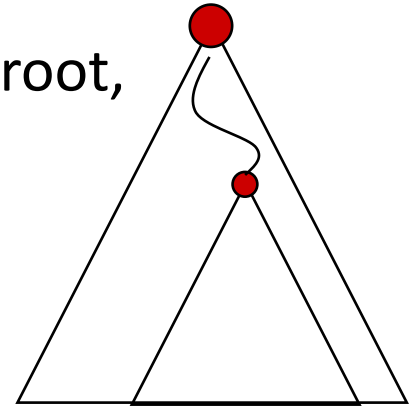
Complete Binary Tree (definition)



Binary tree of height h such that every level less than h is full, and all nodes at level h are as far to the left as possible

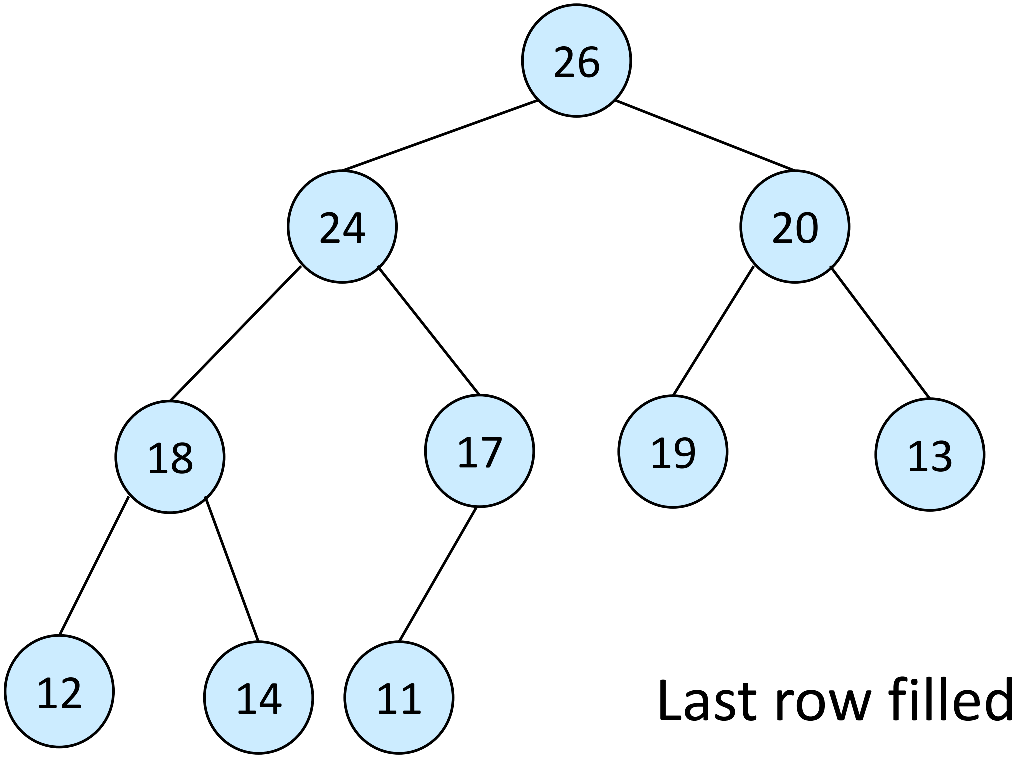
Heap data structure

- Tree-based data structure (here, binary tree, but we can also use k-ary trees)
- Max-Heap
  - Largest element is stored at the root.
  - for all nodes *i*, excluding the root, *A*[PARENT(*i*)] ≥ *A*[*i*].
- Min-Heap
  - Smallest element is stored at the root.
  - for all nodes *i*, excluding the root, excluding the root, *A*[PARENT(*i*)] ≤ *A*[*i*].
- Tree is filled top-down from left to right → Complete tree



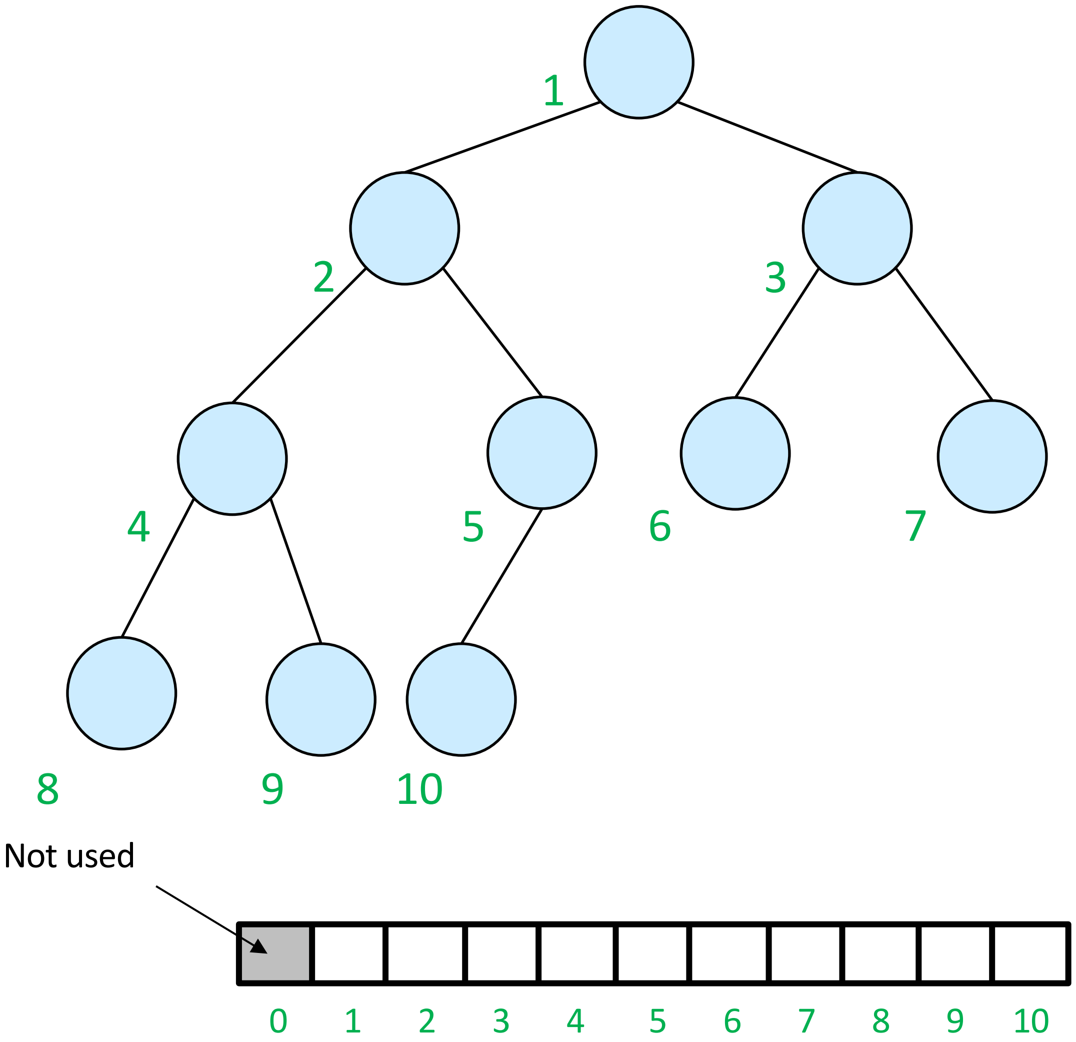
Heaps – Example

Max-heap as a binary tree.

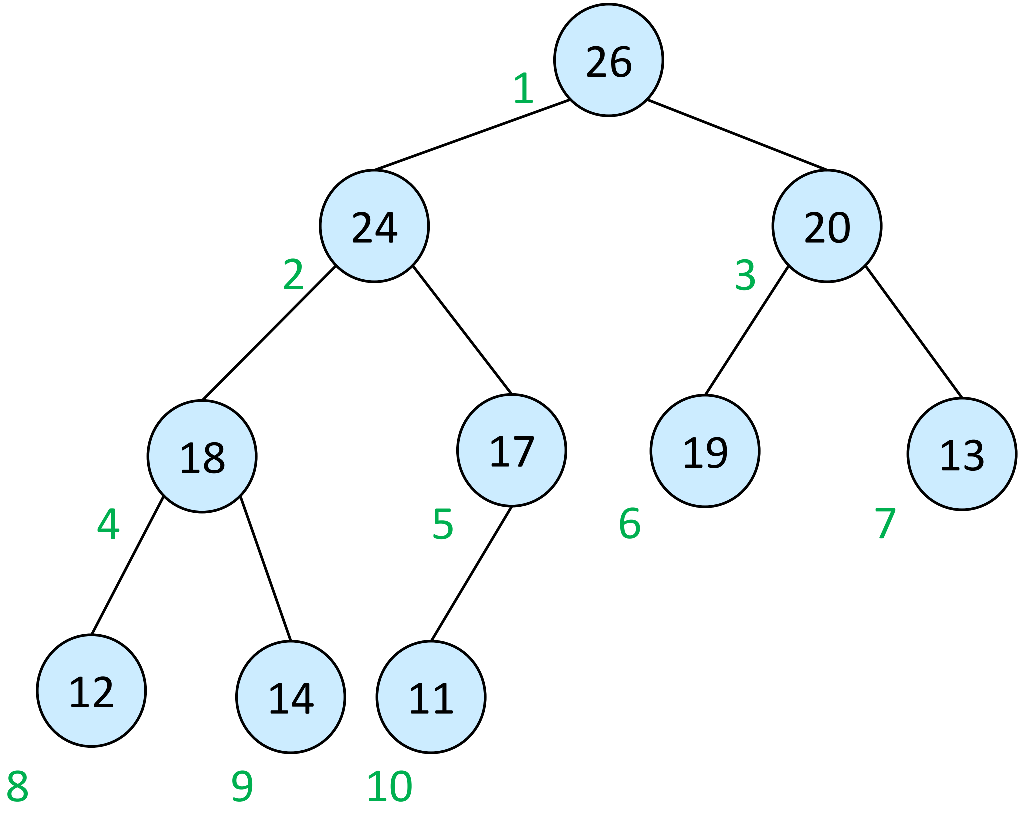


Last row filled from left to right.

Heap (array implementation)



Heap (array implementation)

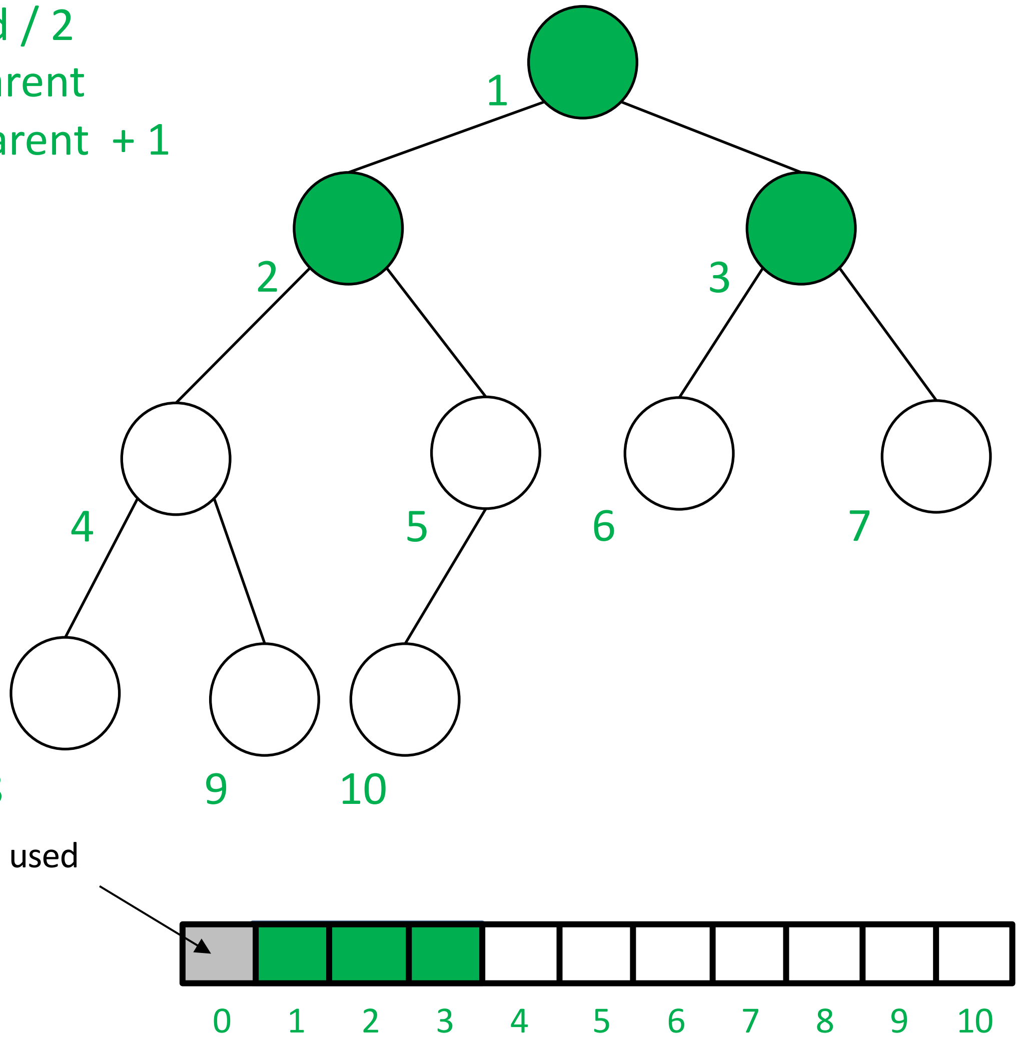


Not used

	26	24	20	18	17	19	13	12	14	11
0	1	2	3	4	5	6	7	8	9	10

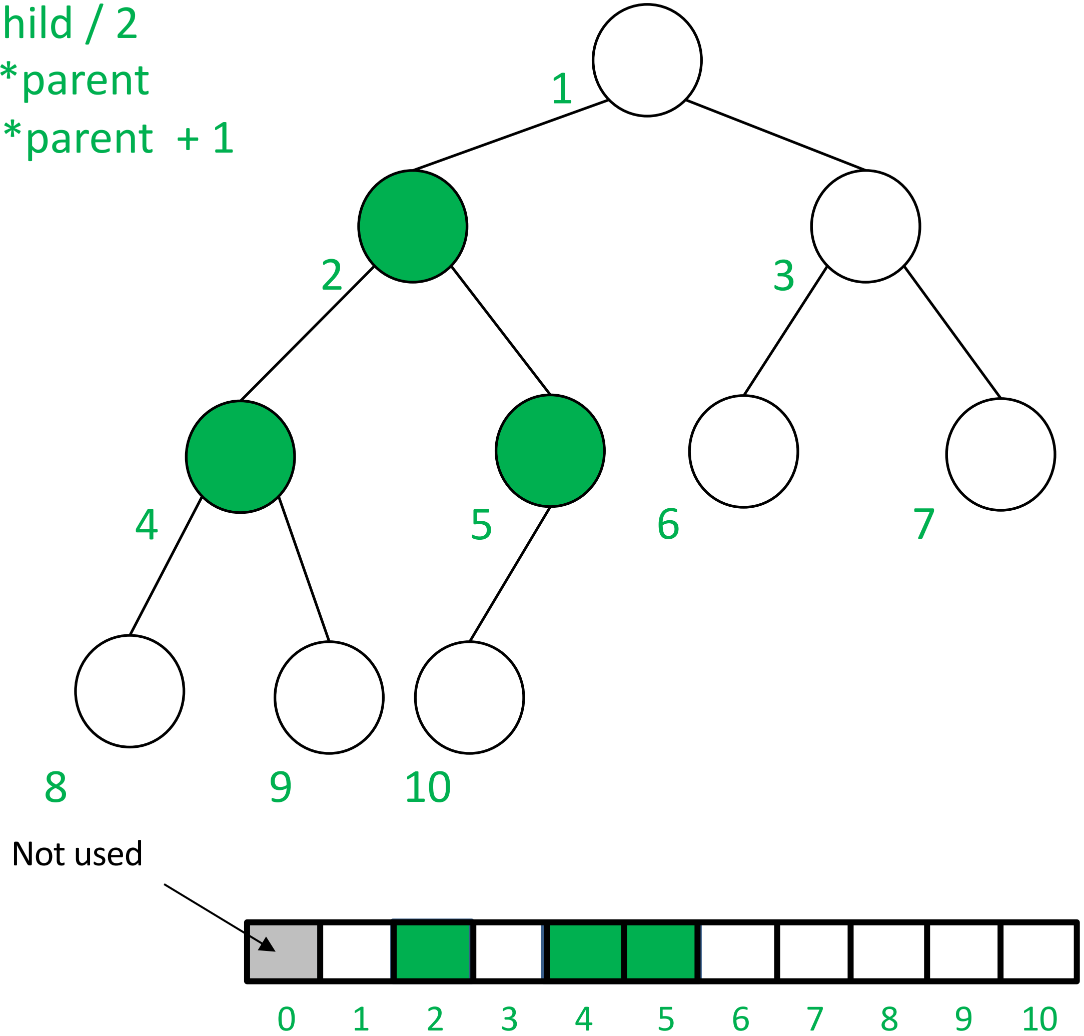
Heap index relations

parent = child / 2
left = 2*parent
right = 2*parent + 1



Heap index relations

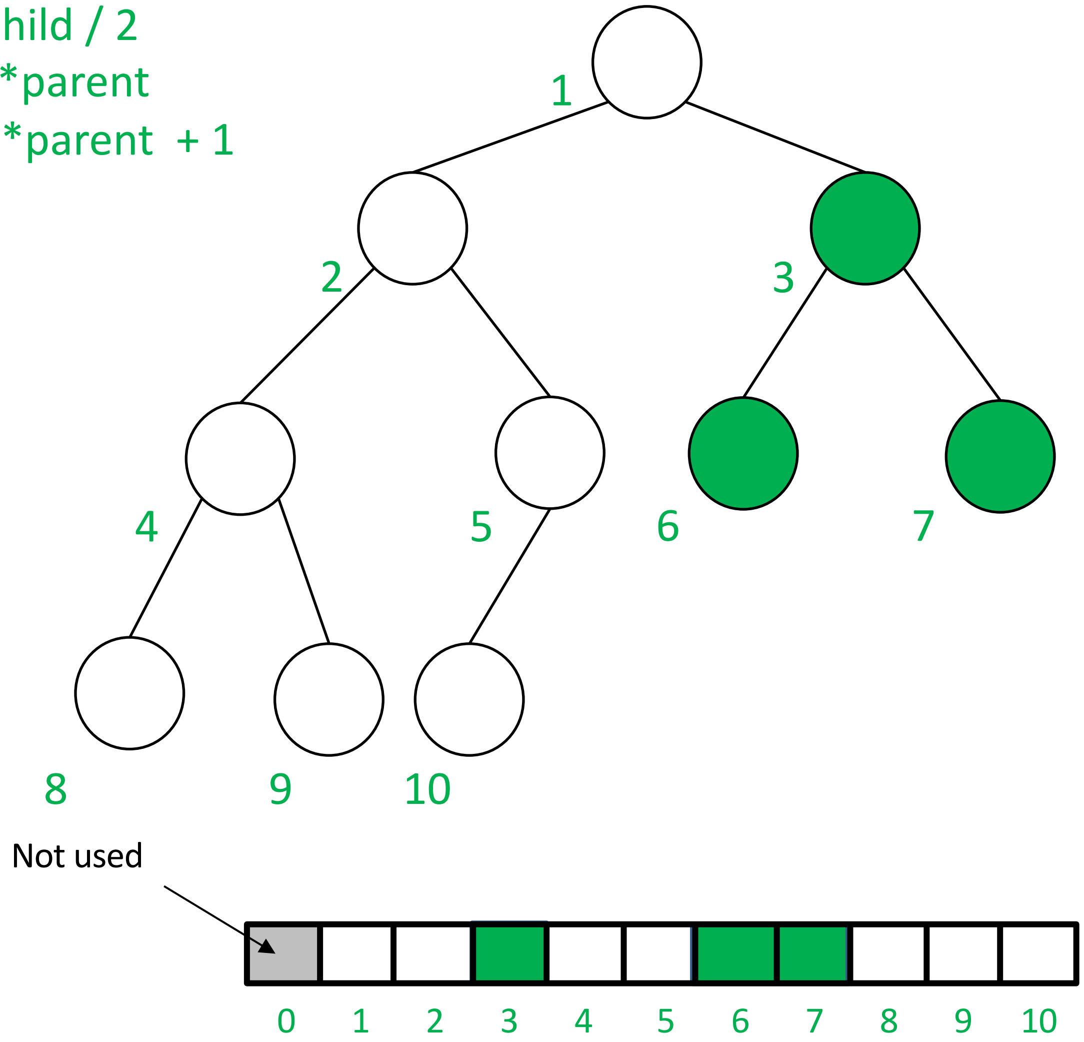
parent = child / 2
left = 2*parent
right = 2*parent + 1



Not used

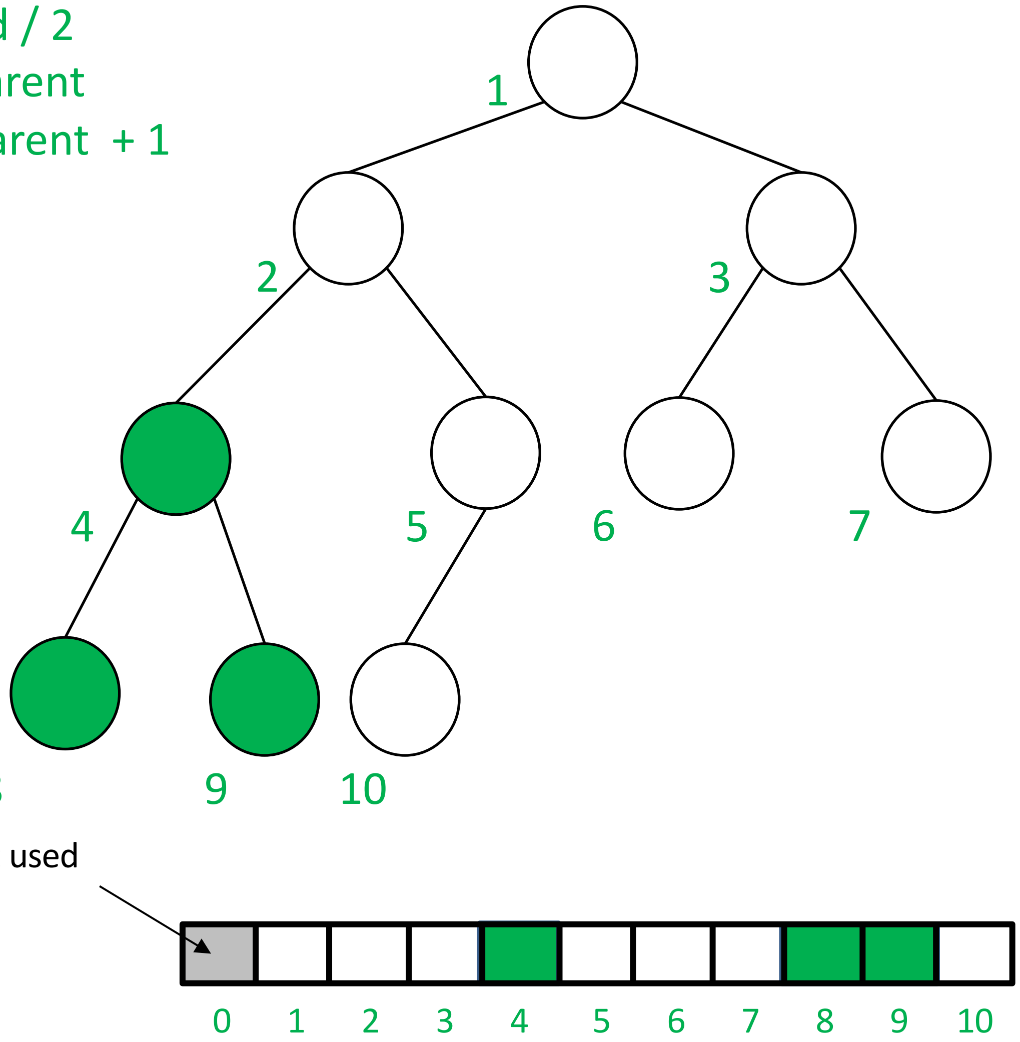
Heap index relations

parent = child / 2
left = 2*parent
right = 2*parent + 1



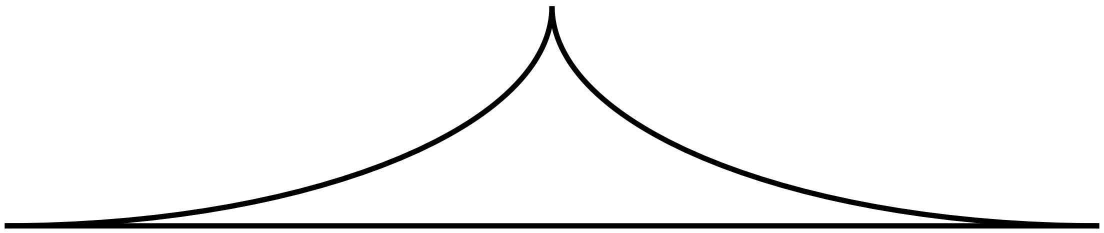
Heap index relations

parent = child / 2
left = 2*parent
right = 2*parent + 1



Height

- *Height of a node in a tree*: the number of edges on the longest simple path down from the node to a leaf.
- *Height of a heap = height of the root =* Θ(lg n).
- Most Basic operations on a heap run in *O*(lg *n*) time
- Shape of a heap



Sorting with Heaps

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  1. Convert the given array of size *n* to a max-heap (*BuildMaxHeap*)
  2. Swap the first and last elements of the array.
     - Now, the largest element is in the last position – where it belongs.
     - That leaves *n* – 1 elements to be placed in their appropriate locations.
     - However, the array of first *n* – 1 elements is no longer a max-heap.
     - Float the element at the root down one of its subtrees so that the array remains a max-heap (*MaxHeapify*)
     - Repeat step 2 until the array is sorted.

Heapsort

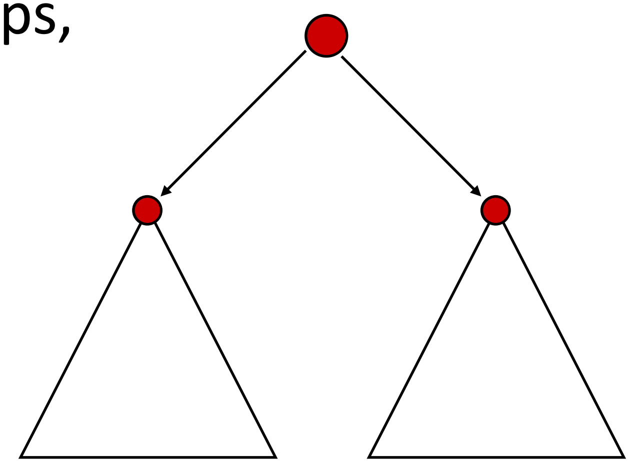
- Combines the better attributes of merge sort and insertion sort.
    - Like merge sort, worst-case running time is $O(n \lg n)$.
    - Like insertion sort, sorts in place.

- Introduces an algorithm design technique
    - Create data structure (*heap*) to manage information during the execution of an algorithm.

- The *heap* has other applications beside sorting.
    - Priority Queues

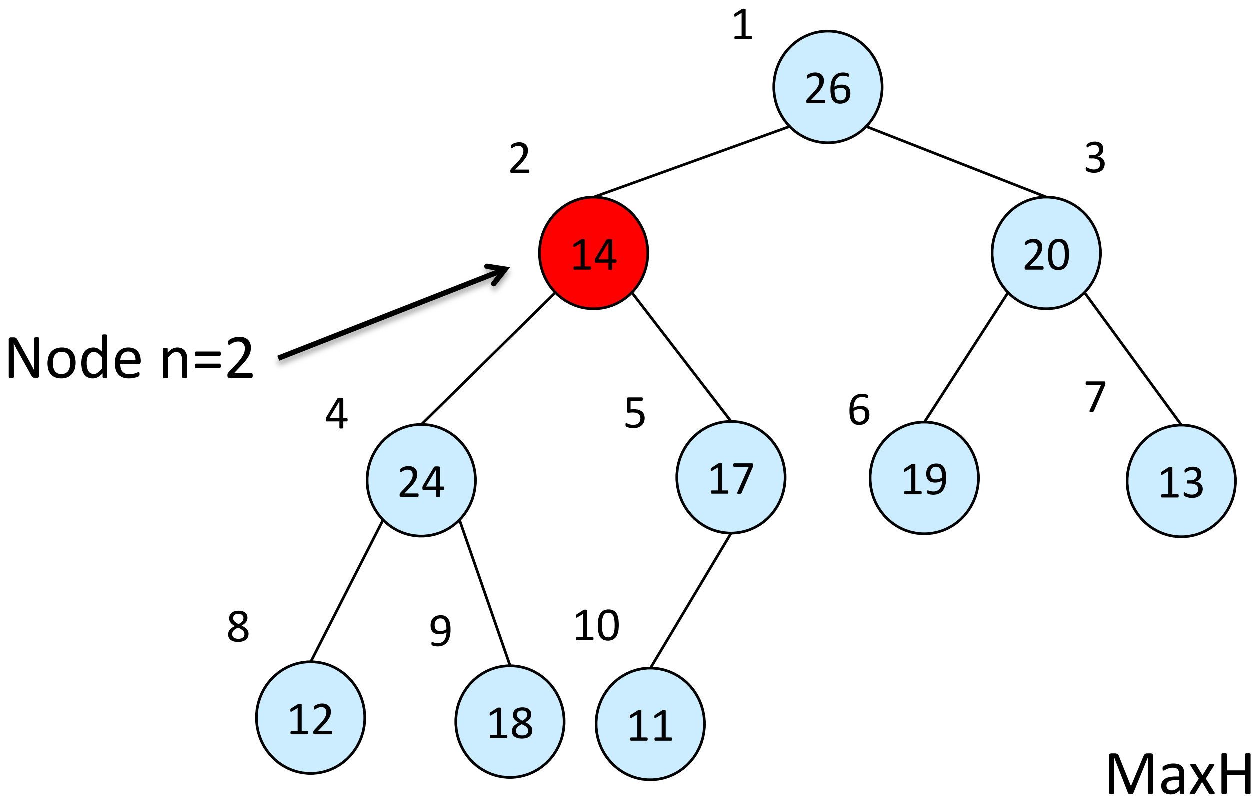
Maintaining the heap property

- Suppose two sub-trees are max-heaps, but the root violates the max-heap property.



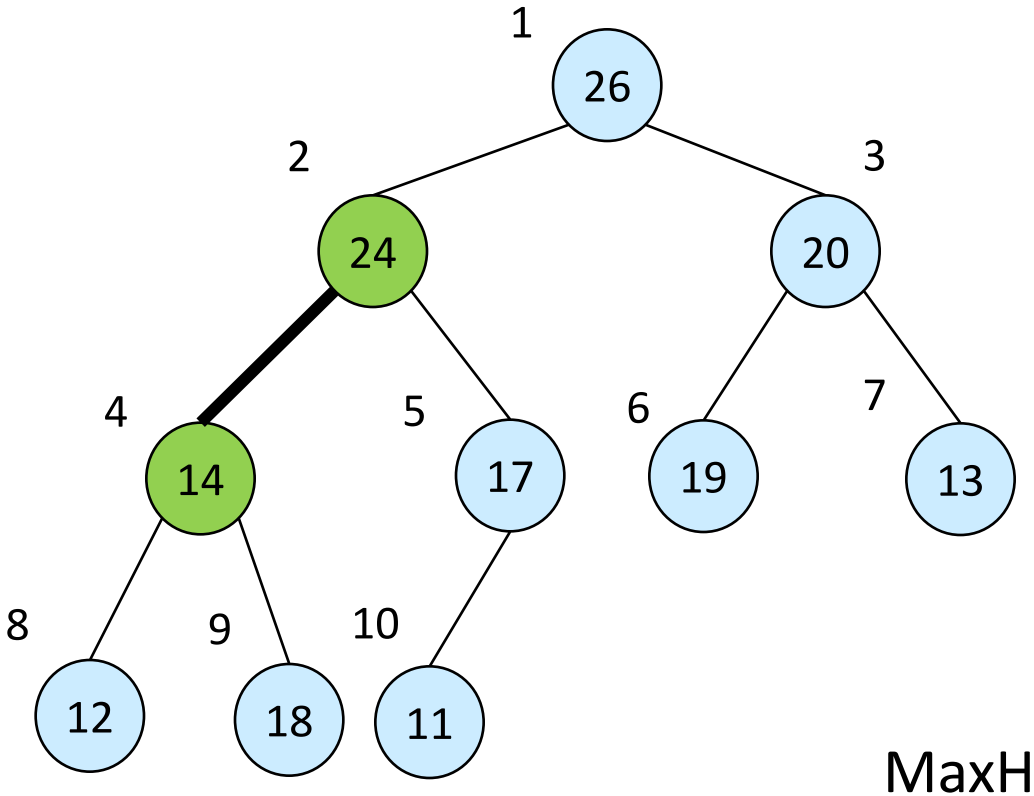
- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - The resulting tree may have a sub-tree that is not a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example



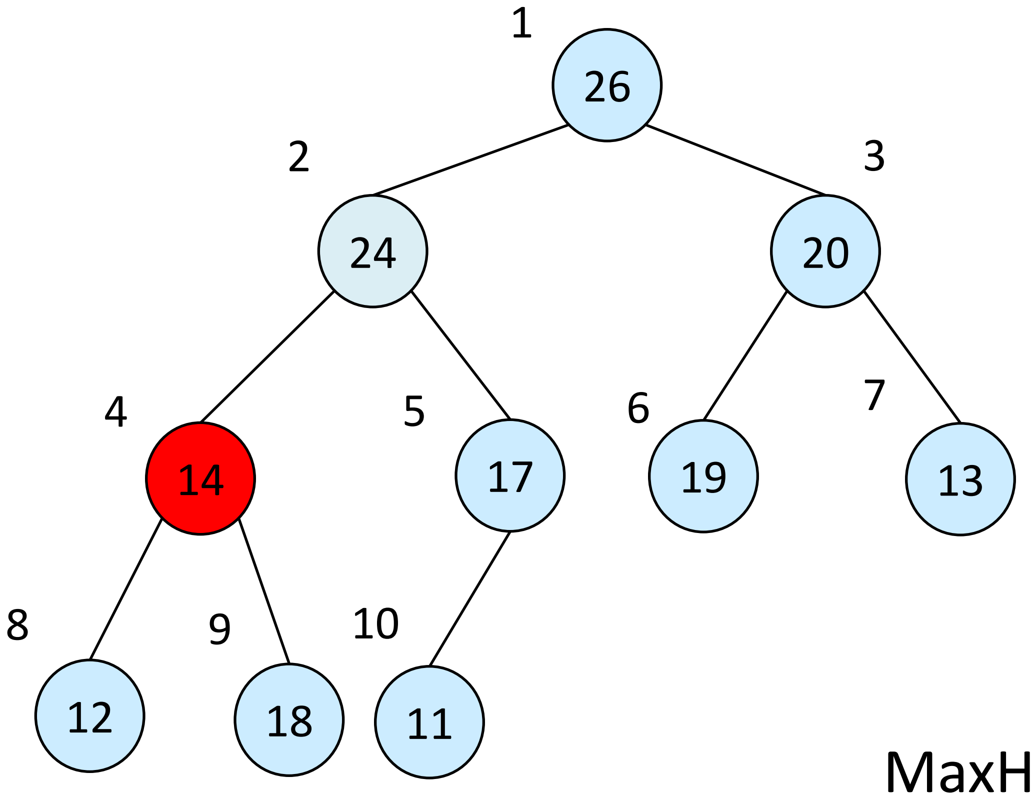
MaxHeapify(*A*, 2)

MaxHeapify – Example



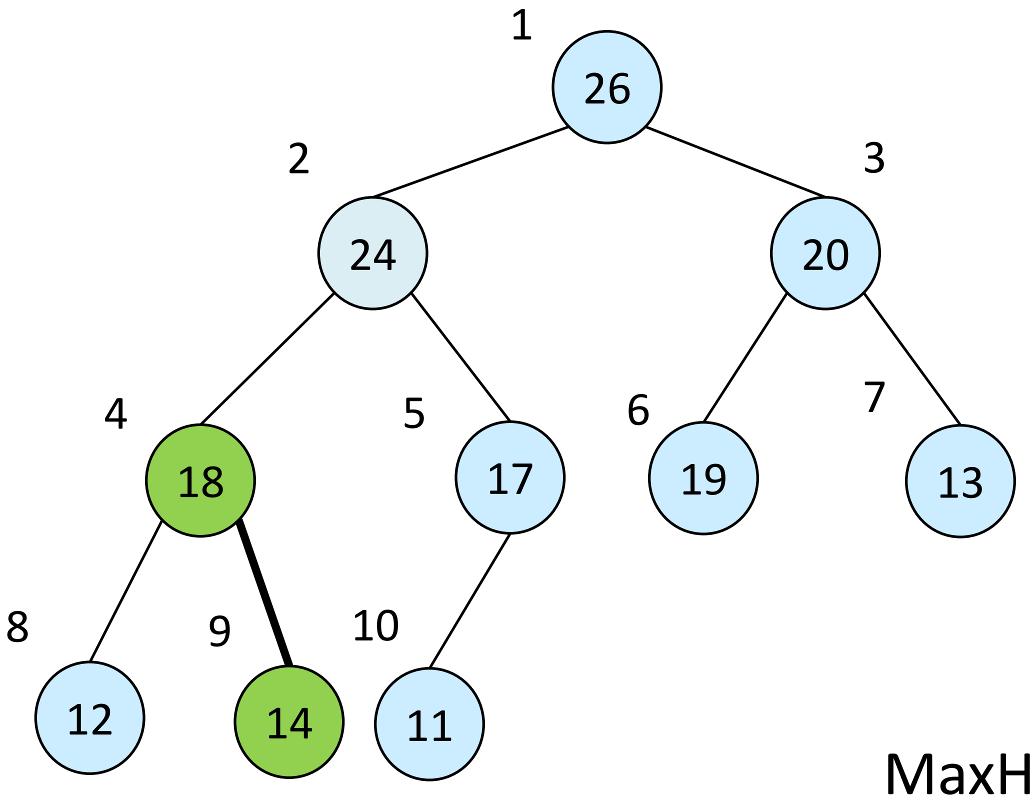
MaxHeapify(A, 2)

MaxHeapify – Example



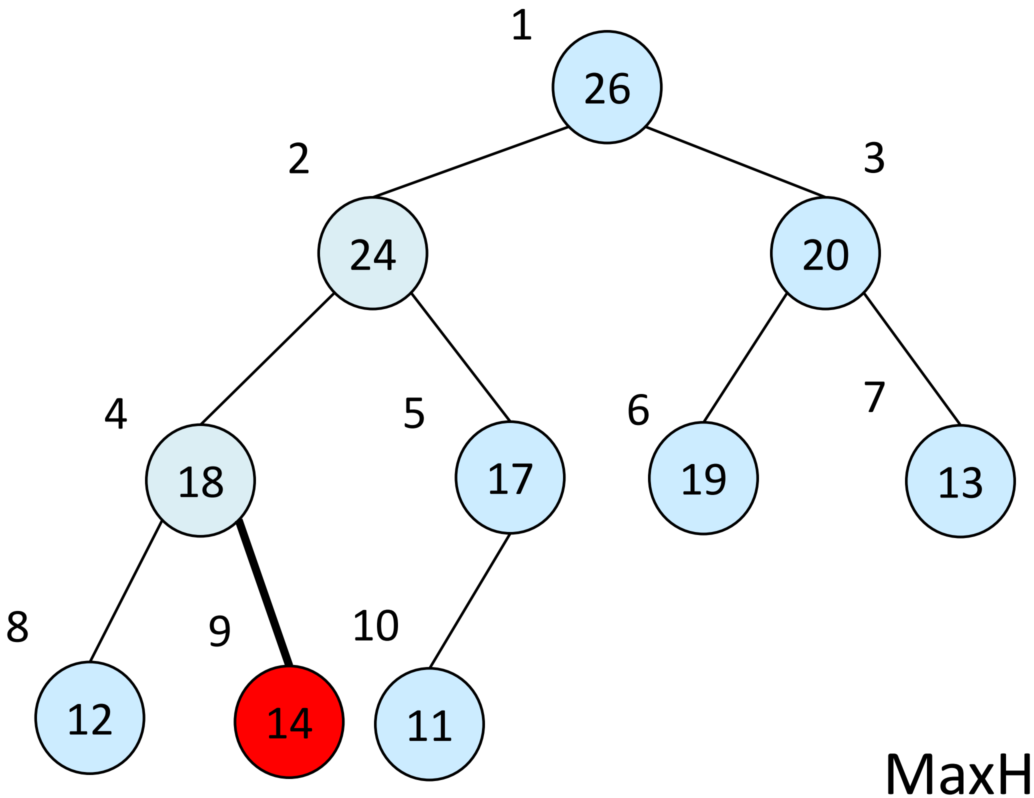
MaxHeapify(*A*, 2)
MaxHeapify(*A*, 4)

MaxHeapify – Example



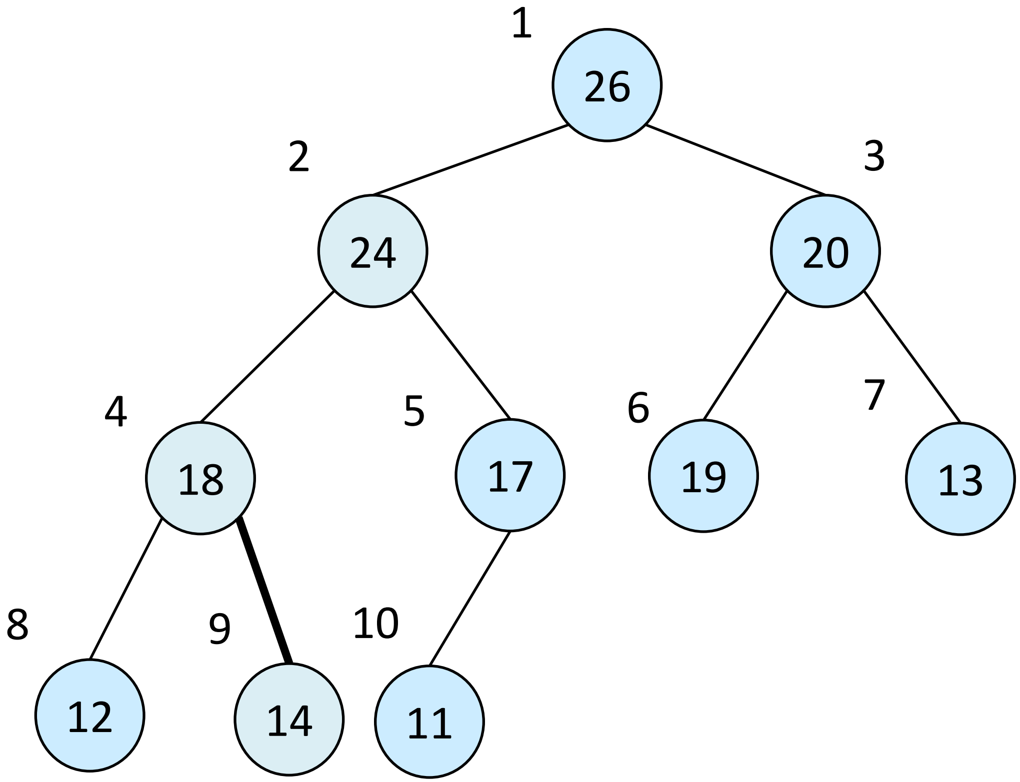
MaxHeapify(*A*, 2)
MaxHeapify(*A*, 4)

MaxHeapify – Example



MaxHeapify(*A*, 2)
MaxHeapify(*A*, 4)
MaxHeapify(*A*, 9)

MaxHeapify – Example



- Root : $A[1]$
- Left[i] : $A[2i]$
- Right[i] : $A[2i+1]$
- Parent[i] : $A[\lfloor i/2 \rfloor]$

MaxHeapify(A, 2)
MaxHeapify(A, 4)
MaxHeapify(A, 9)

Procedure MaxHeapify

Assumption: Left(*i*) and Right(*i*) are max-heaps.
n is the size of the heap.

```
MaxHeapify(A, i)
    l ← leftNode(i)        #use heap properties to
    r ← rightNode(i)       find the children of the root
    n ← HeapSize(i)        in the array
```

Procedure MaxHeapify

Assumption: Left(*i*) and Right(*i*) are max-heaps.
n is the size of the heap.

```
MaxHeapify(A, i)
    $l \leftarrow$ leftNode($i$)
    $r \leftarrow$ rightNode($i$)
    $n \leftarrow$ HeapSize($i$)
    if $l \leq n$ and $A[l] > A[i]$
        then $largest \leftarrow l$
        else $largest \leftarrow i$
    if $r \leq n$ and $A[r] > A[largest]$
        then $largest \leftarrow r$
```

#use heap properties to find the children of the root in the array

Compare the value of the root and its left and right children

Procedure MaxHeapify

Assumption: Left(*i*) and Right(*i*) are max-heaps.
n is the size of the heap.

```
MaxHeapify(A, i)
    $l \leftarrow$ leftNode($i$)
    $r \leftarrow$ rightNode($i$)
    $n \leftarrow$ HeapSize($i$)
    if $l \le n$ and $A[l] > A[i]$
        then $largest \leftarrow l$
        else $largest \leftarrow i$
    if $r \le n$ and $A[r] > A[largest]$
        then $largest \leftarrow r$
    if $largest \ne i$
        then exchange $A[i] \leftrightarrow A[largest]$
            MaxHeapify($A$, $largest$)
```

#use heap properties to find the children of the root in the array

Compare the value of the root and its left and right children

If the root is not the largest, then we swap and maxHeapify child

Procedure MaxHeapify

Assumption: Left(*i*) and Right(*i*) are max-heaps.
n is the size of the heap.

```
MaxHeapify(A, i)
    $l \leftarrow$ leftNode($i$)
    $r \leftarrow$ rightNode($i$)
    $n \leftarrow$ HeapSize($i$)
    if $l \leq n$ and $A[l] > A[i]$
        then $largest \leftarrow l$
        else $largest \leftarrow i$
    if $r \leq n$ and $A[r] > A[largest]$
        then $largest \leftarrow r$
    if $largest \neq i$
        then exchange $A[i] \leftrightarrow A[largest]$
            MaxHeapify($A$, $largest$)
```

#use heap properties to find the children of the root in the array

Compare the value of the root and its left and right children

Time to determine if there is a conflict and find the largest children is $\Theta(1)$

If the root is not the largest, then we swap and maxHeapify child

Procedure MaxHeapify

Assumption: Left(*i*) and Right(*i*) are max-heaps.
n is the size of the heap.

```
MaxHeapify(A, i)
    $l \leftarrow$ leftNode($i$)
    $r \leftarrow$ rightNode($i$)
    $n \leftarrow$ HeapSize($i$)
    if $l \le n$ and $A[l] > A[i]$
        then $largest \leftarrow l$
        else $largest \leftarrow i$
    if $r \le n$ and $A[r] > A[largest]$
        then $largest \leftarrow r$
    if $largest \ne i$
        then exchange $A[i] \leftrightarrow A[largest]$
            MaxHeapify($A$, $largest$)
```

#use heap properties to find the children of the root in the array

Compare the value of the root and its left and right children

Time to determine if there is a conflict and find the largest children is $\Theta(1)$

Time to fix the subtree rooted at one of *i*'s children is O(size of subtree)

If the root is not the largest, then we swap and maxHeapify child

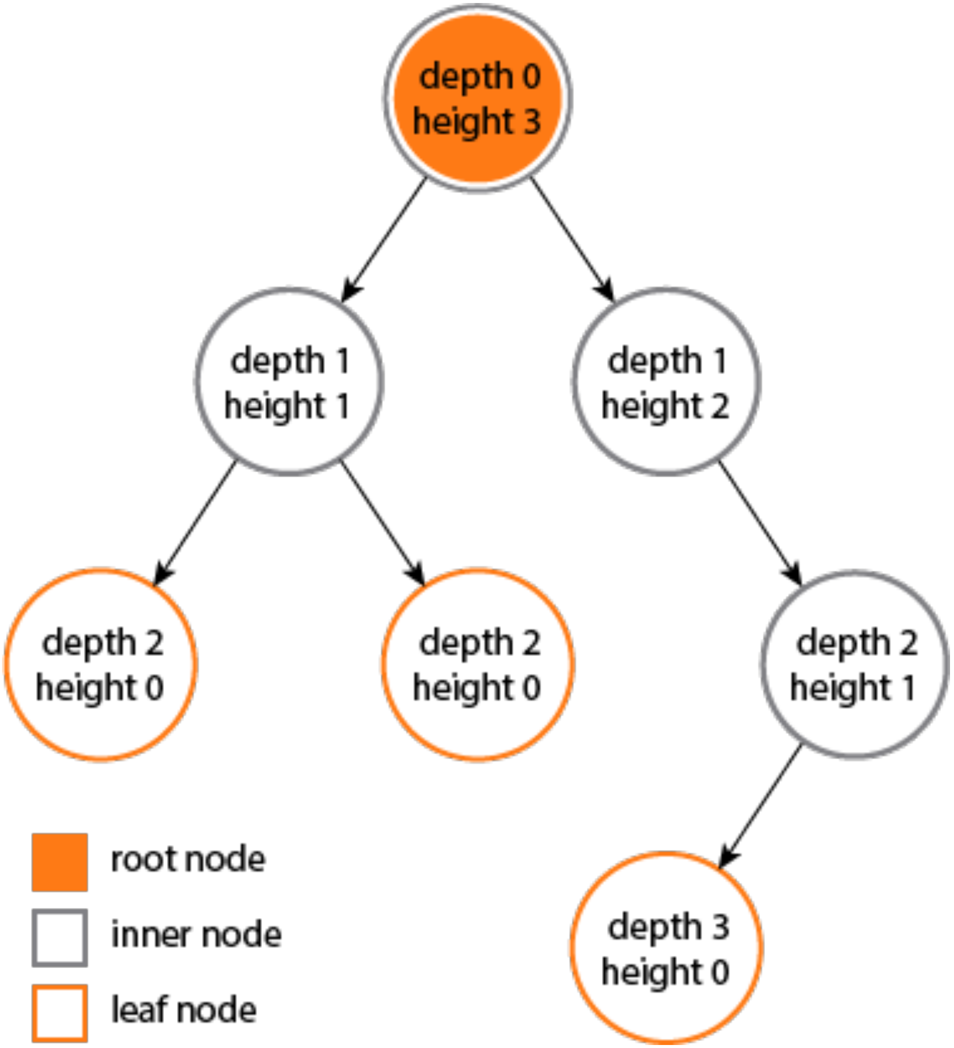
Worst case running time of MaxHeapify(*A*, *0*)

- *Size of a tree = number of nodes in this tree*
- *T(n): time used for an input of size n (a tree with n nodes)*
- $T(n) = T(\textit{size of the largest subtree}) + \Theta(1)$
- *Size of the largest subtree* $\leq 2n/3$ (worst case occurs when the last row of tree is exactly half full)

$\Rightarrow T(n) \leq T(2n/3) + \Theta(1) \Rightarrow$ **$T(n) = O(\lg n)$**

Alternately, MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied

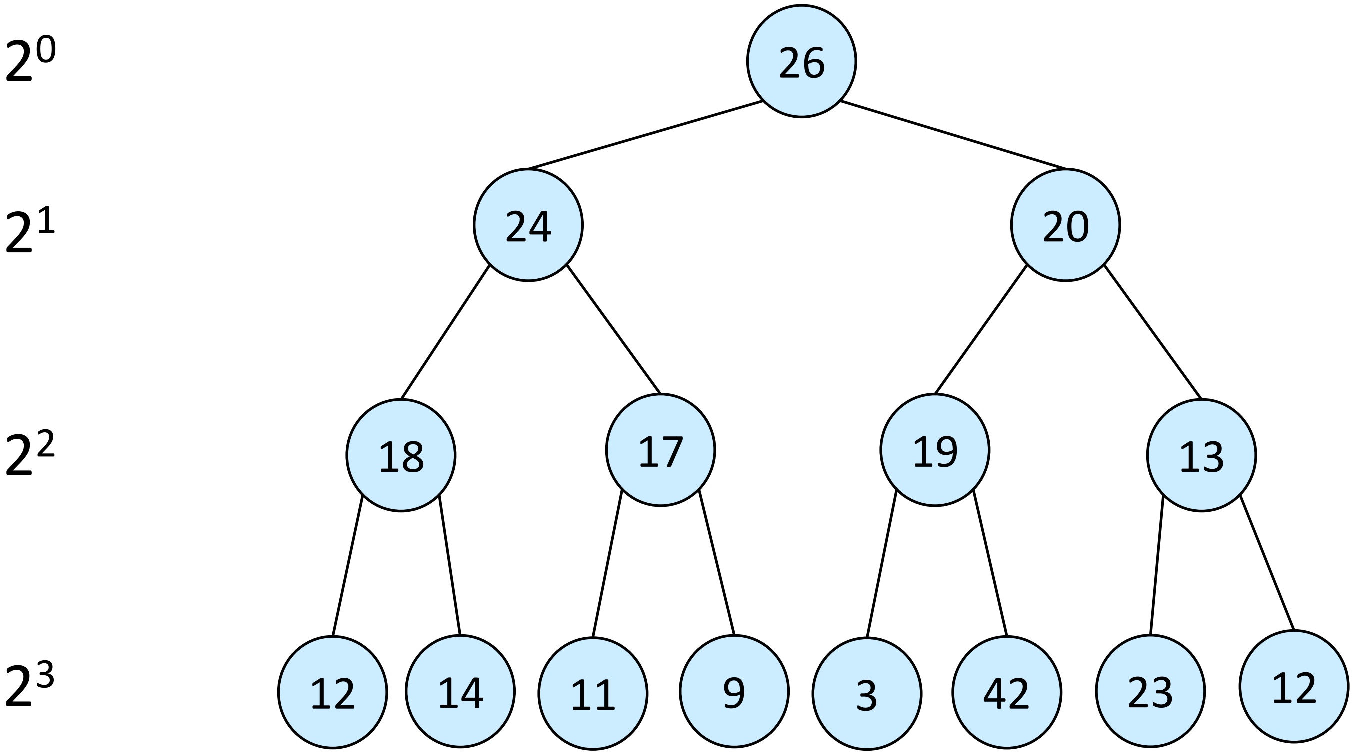
Height vs. Depth



https://stackoverflow.com/questions/29326512/what-is-the-difference-between-the-height-of-a-tree-and-depth-of-a-tree

Maximum capacity of a heap

Max # nodes / level



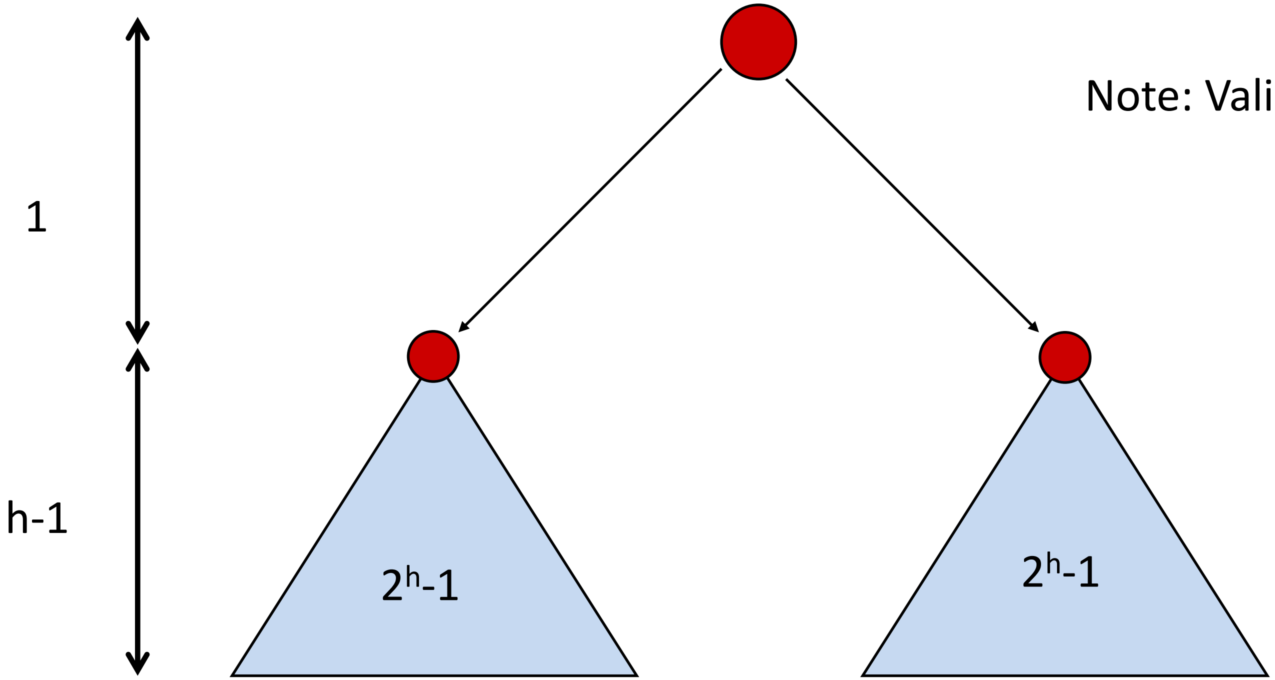
Maximum capacity of a binary tree of height h = $2^{h+1} - 1$

Heap of height h+1 has at least (2^{h+1}-1) + 1 nodes

1 node in last row

$$\Longrightarrow n_h \geq 2^h \Longrightarrow \log_2 n_h \geq h \Longrightarrow h = O(\log n)$$

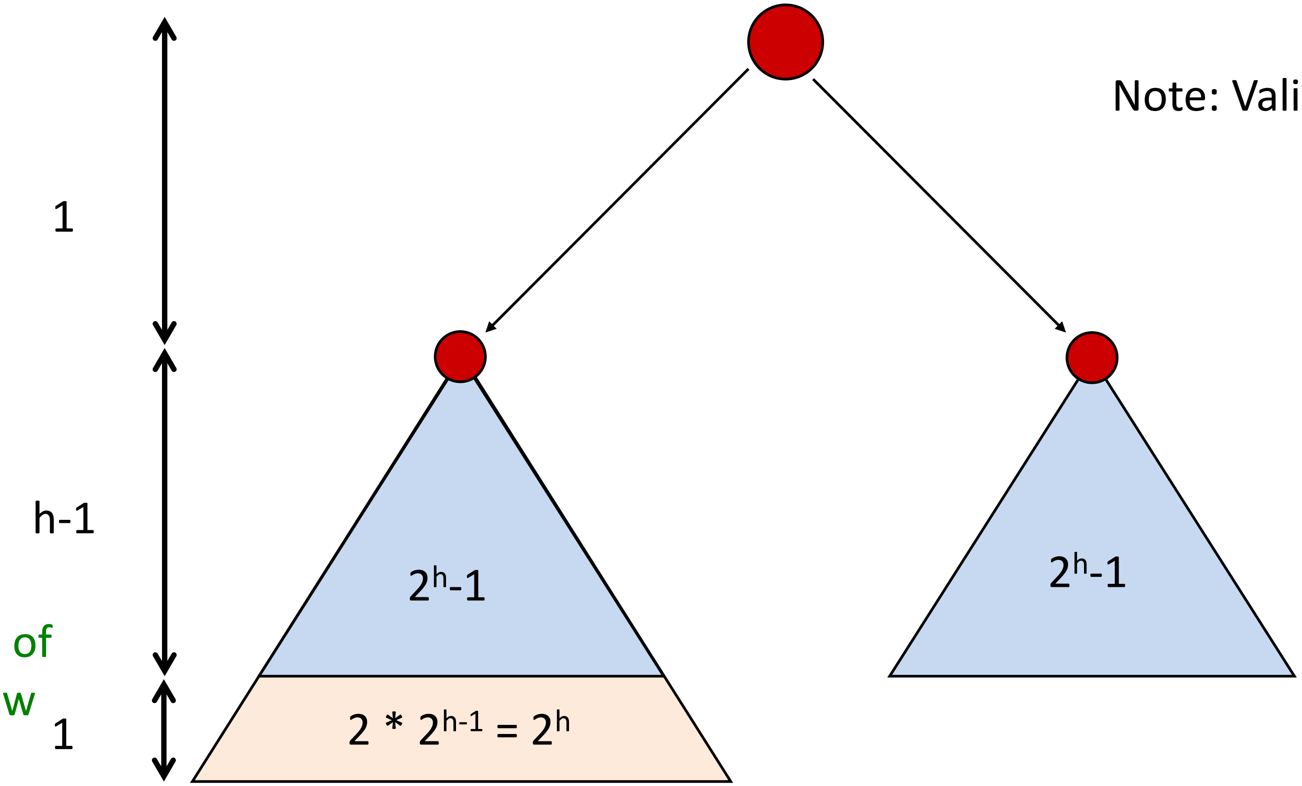
Worst case running time of MaxHeapify



Note: Valid iff $h \geq 1$

Worst case running time of MaxHeapify

Note: Valid iff $h \geq 1$



1

h-1

1

2^h-1

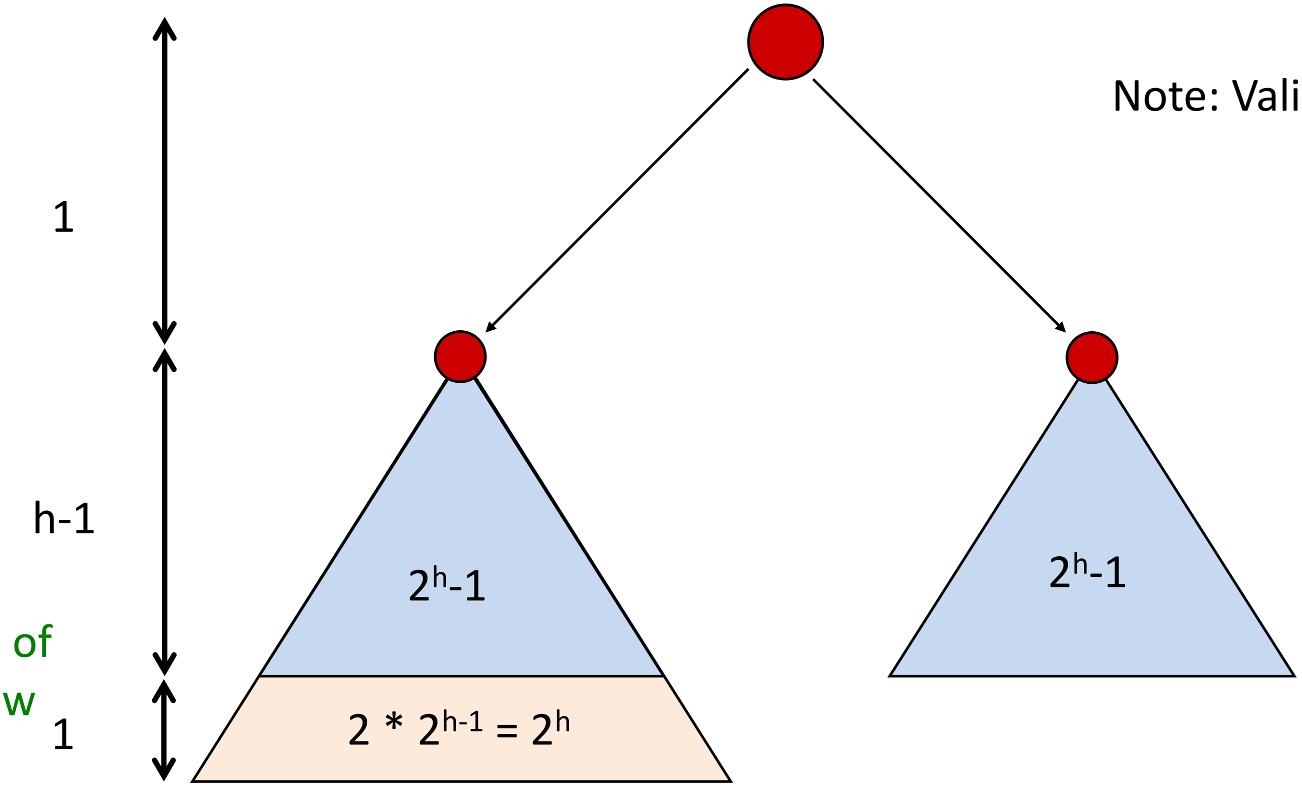
2^h-1

$2 * 2^{h-1} = 2^h$

The last row is double the size of the previous row

Worst case running time of MaxHeapify

Note: Valid iff $h \geq 1$

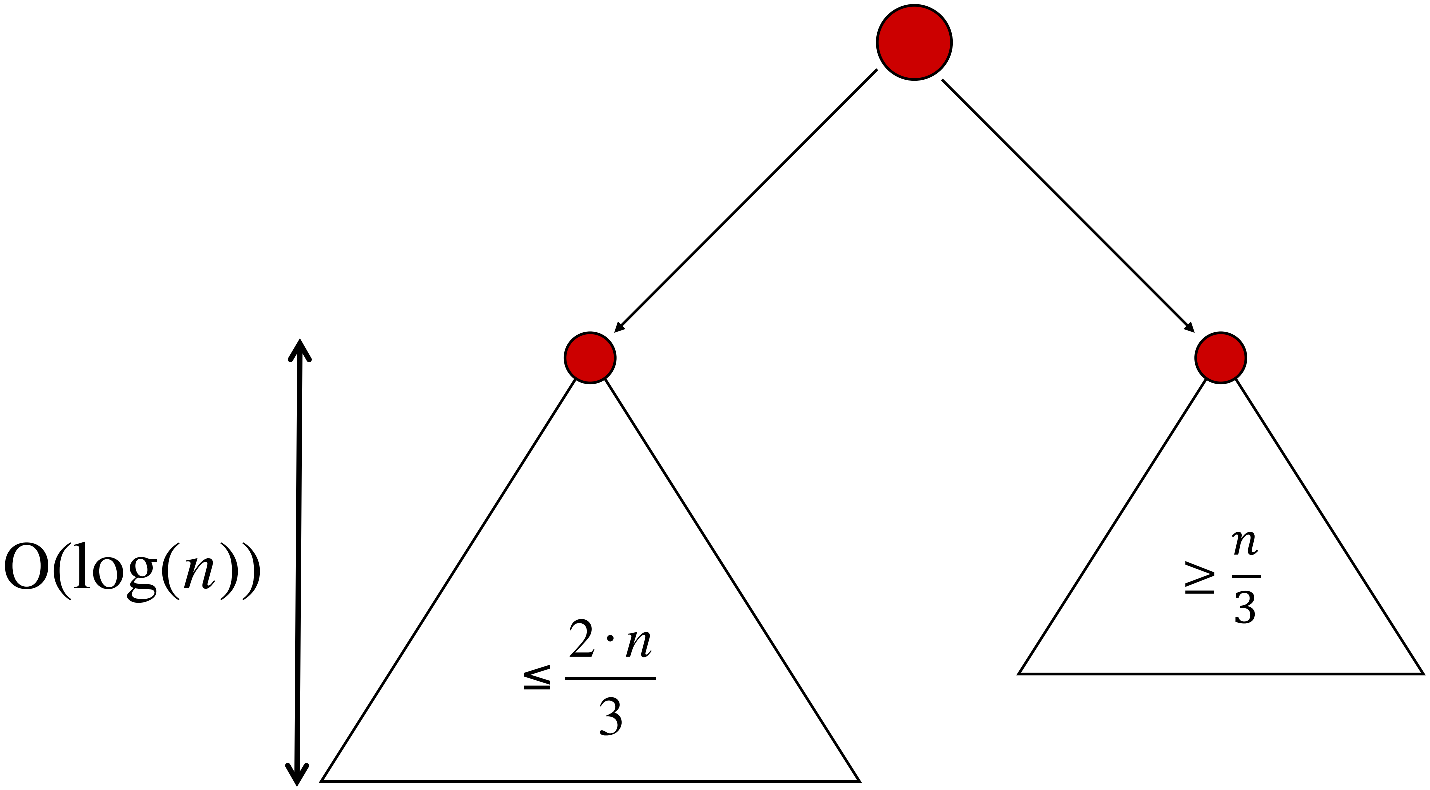


The last row is double the size of the previous row

Total in heap (n): $n = 3 \cdot 2^h - 1$

Total left subtree $n_{left} \leq 2^{h+1} - 1 = \frac{3}{3} \cdot 2 \cdot (2^h - \frac{1}{2}) = \frac{2}{3} \cdot (3 \cdot 2^h - \frac{3}{2}) \leq \frac{2}{3} \cdot n$

Worst case running time of MaxHeapify



$O(\log(n))$

$\leq \frac{2 \cdot n}{3}$

$\geq \frac{n}{3}$

Building a heap

- Use *BuildMaxHeap* to convert an array *A* into a max-heap.
- Call MaxHeapify on each element in a bottom-up manner.

```
BuildMaxHeap(A)
n ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
        do MaxHeapify(A, i, n)
```

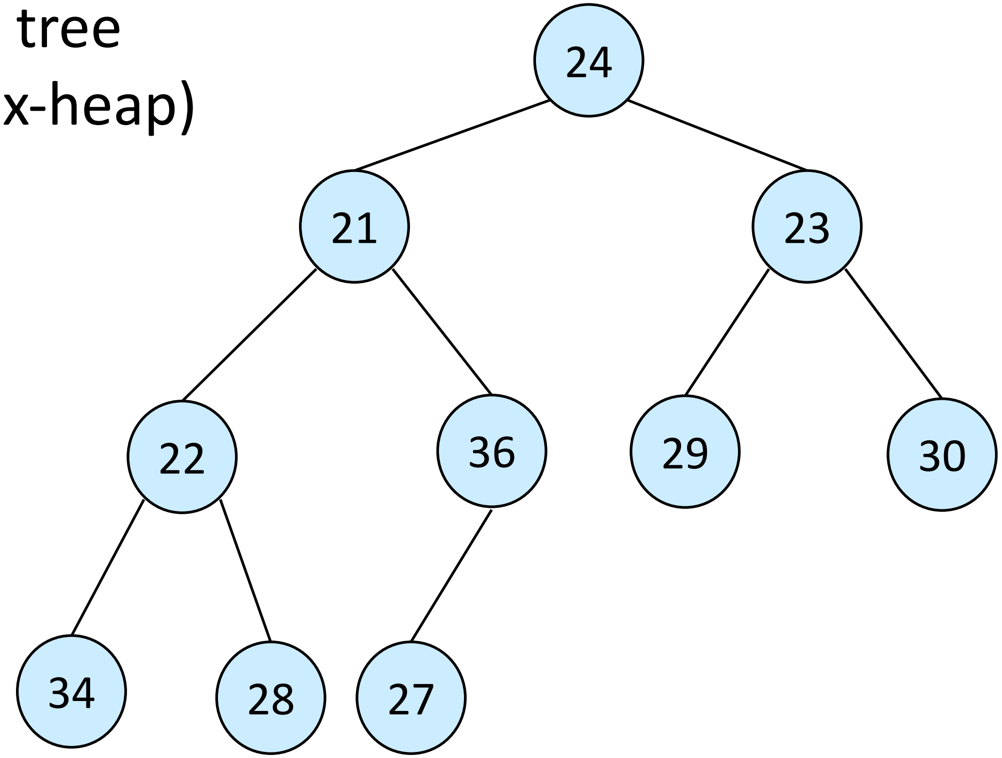
Length(a)/2 is the midpoint. At the right, everything is a child.

BuildMaxHeap – Example

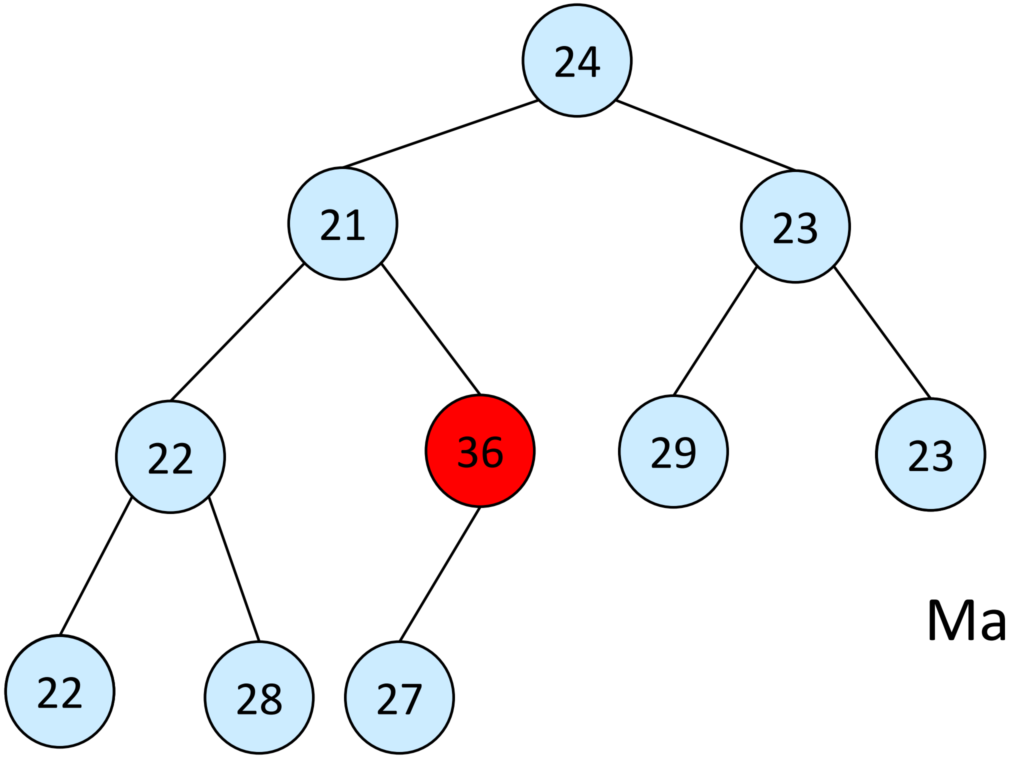
Input Array:

24	21	23	22	36	29	30	34	28	27

Starting tree
(<u>not</u> max-heap)

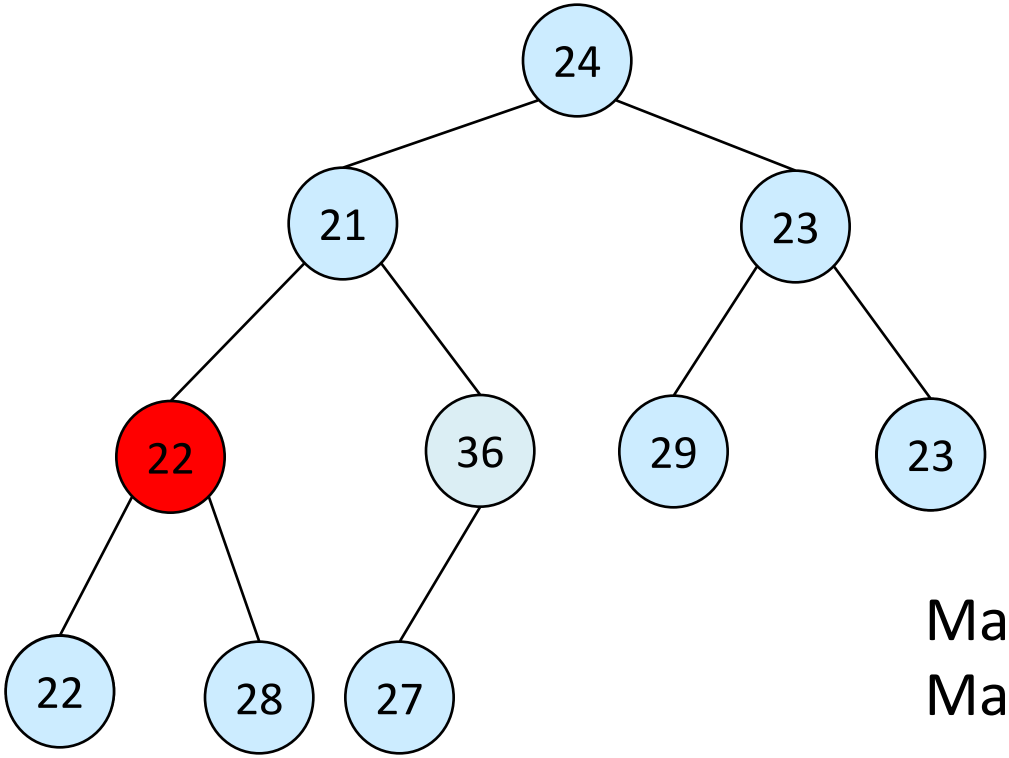


BuildMaxHeap – Example



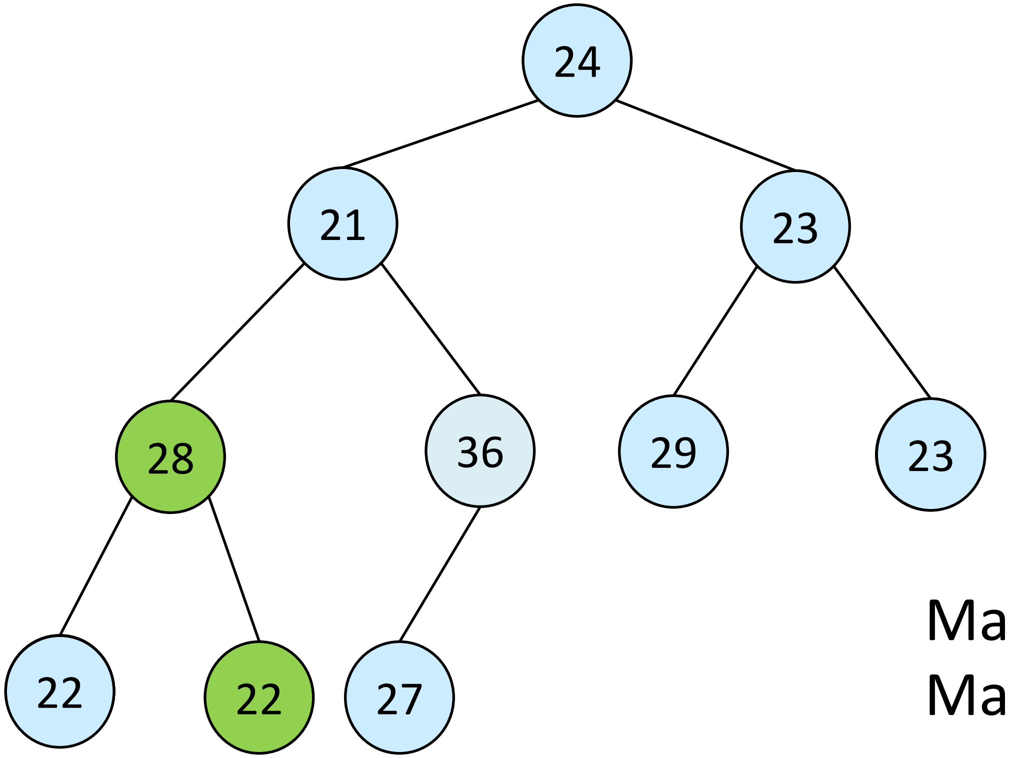
MaxHeapify($\lfloor 10/2 \rfloor = 5$)

BuildMaxHeap – Example



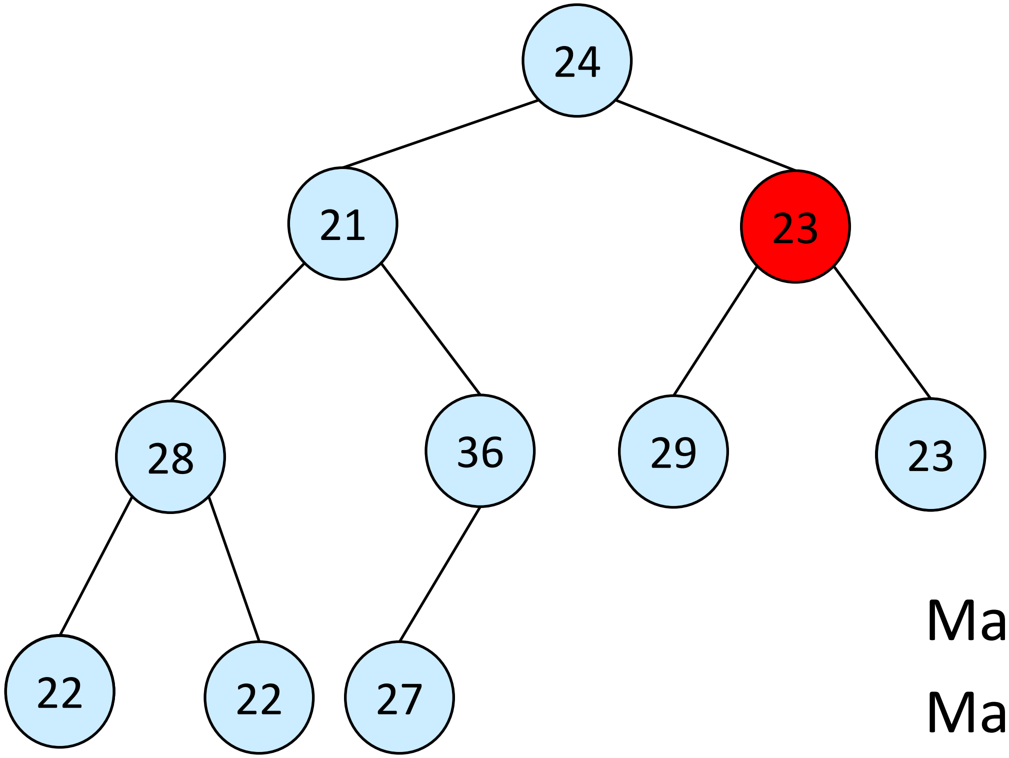
MaxHeapify(⌊10/2⌋ = 5)
MaxHeapify(4)

BuildMaxHeap – Example



MaxHeapify($\lfloor 10/2 \rfloor = 5$)
MaxHeapify(4)

BuildMaxHeap – Example

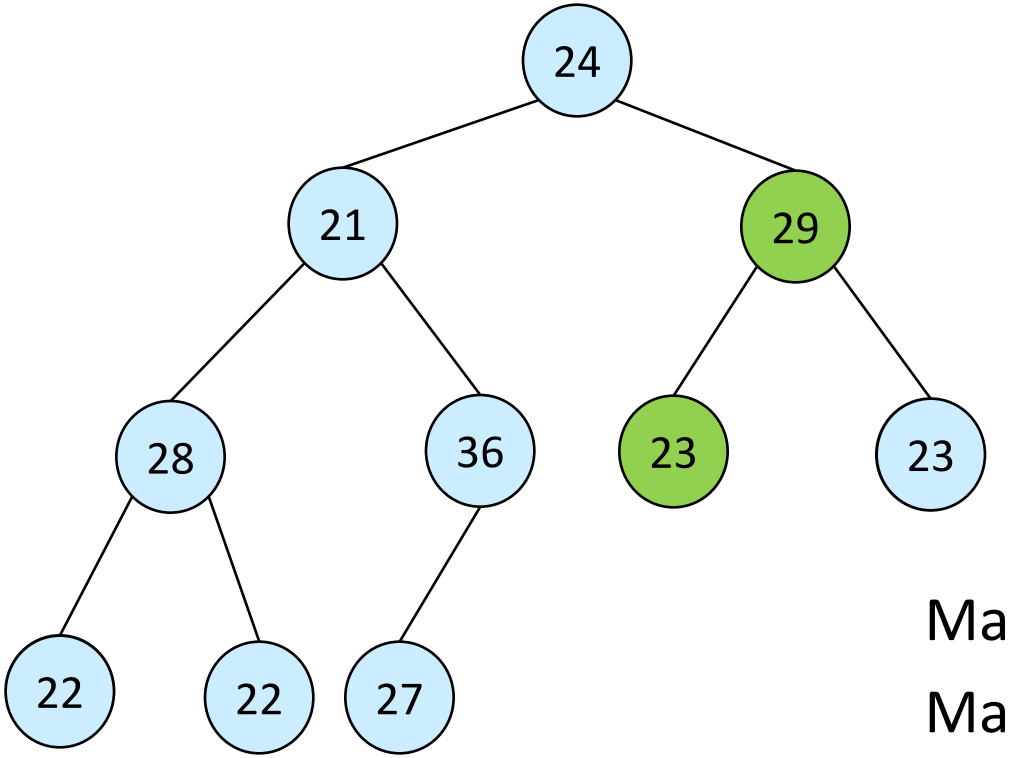


MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

BuildMaxHeap – Example

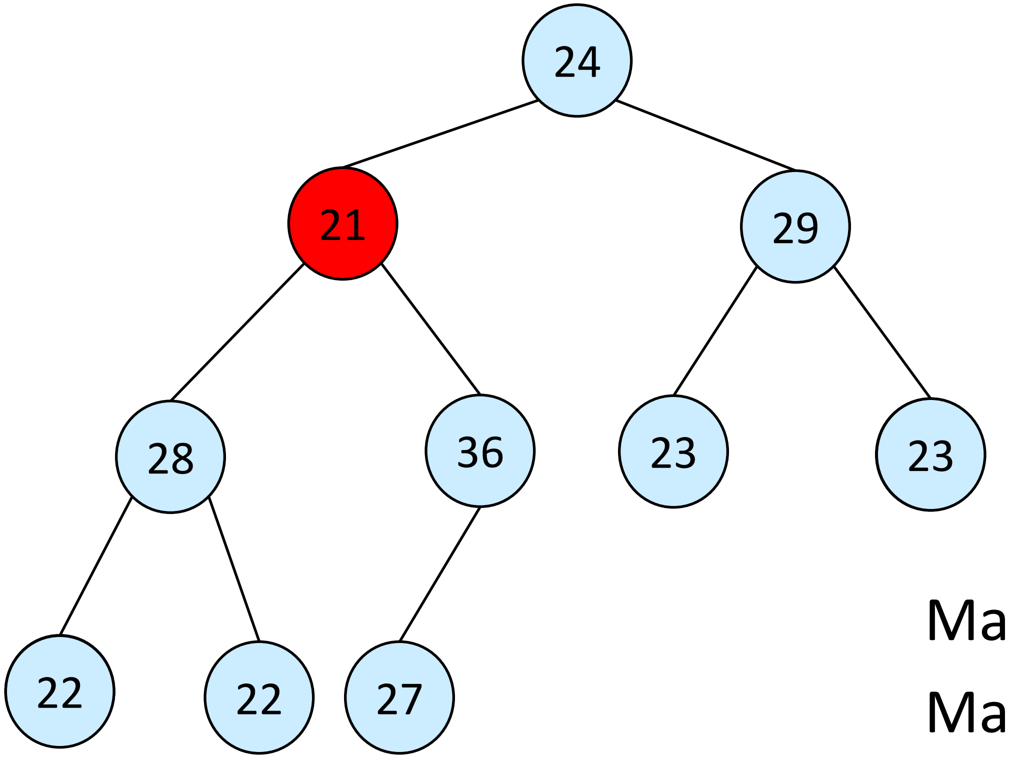


MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

BuildMaxHeap – Example



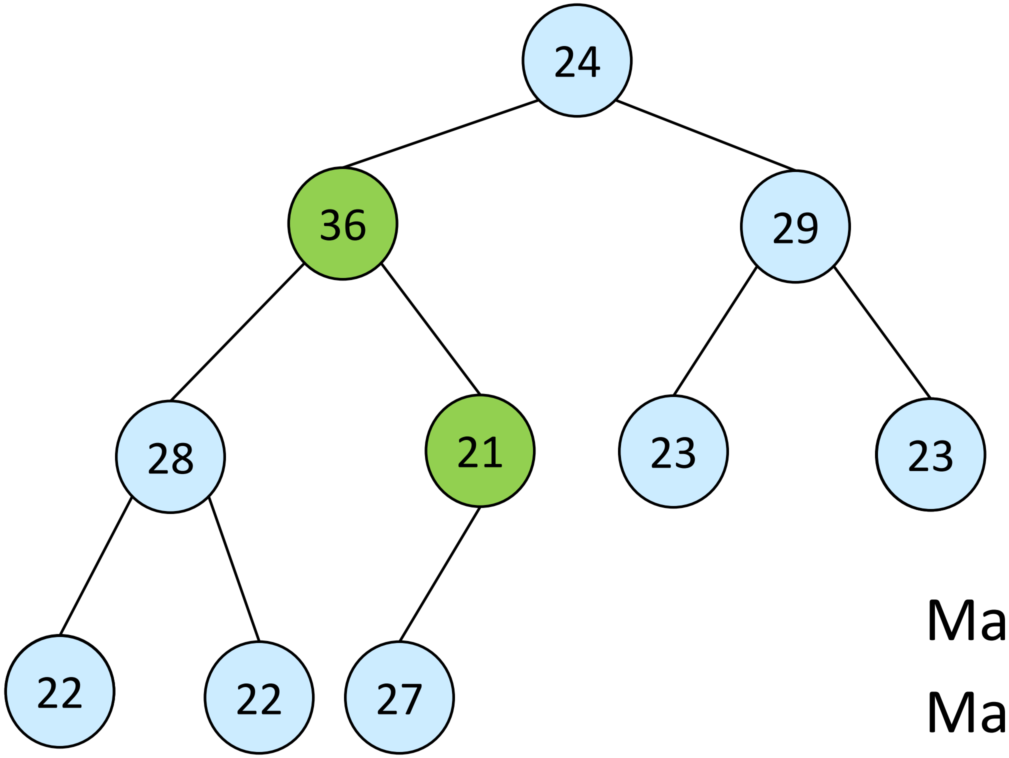
MaxHeapify(⌊10/2⌋ = 5)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

BuildMaxHeap – Example



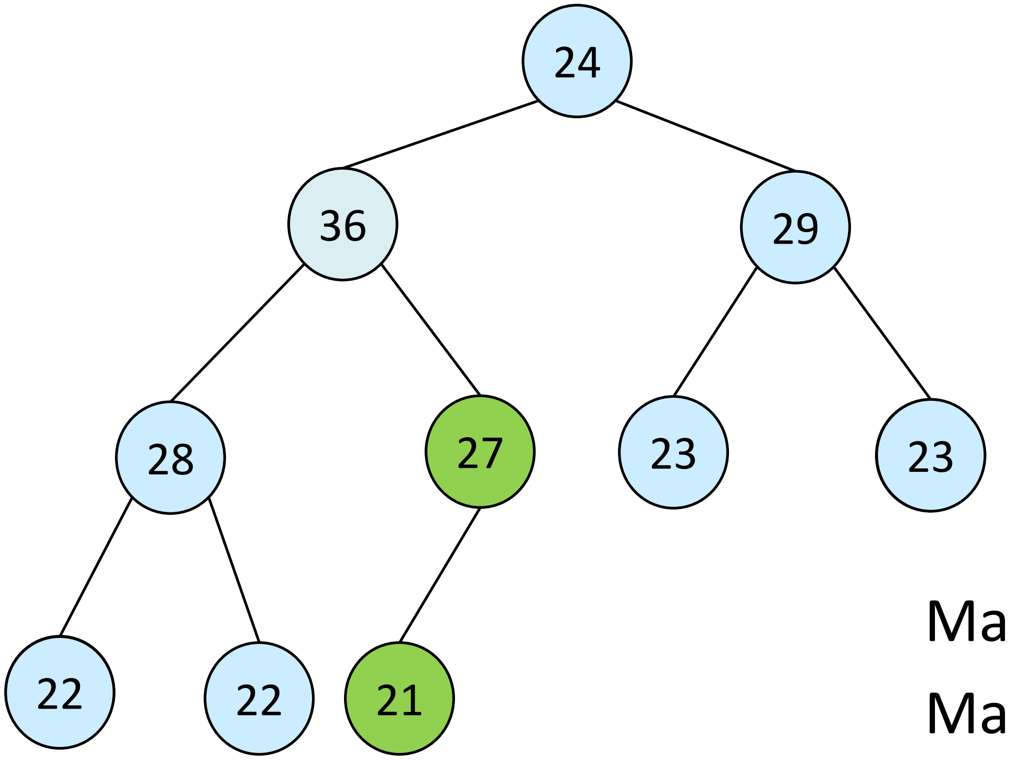
MaxHeapify(⌊10/2⌋ = 5)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

BuildMaxHeap – Example



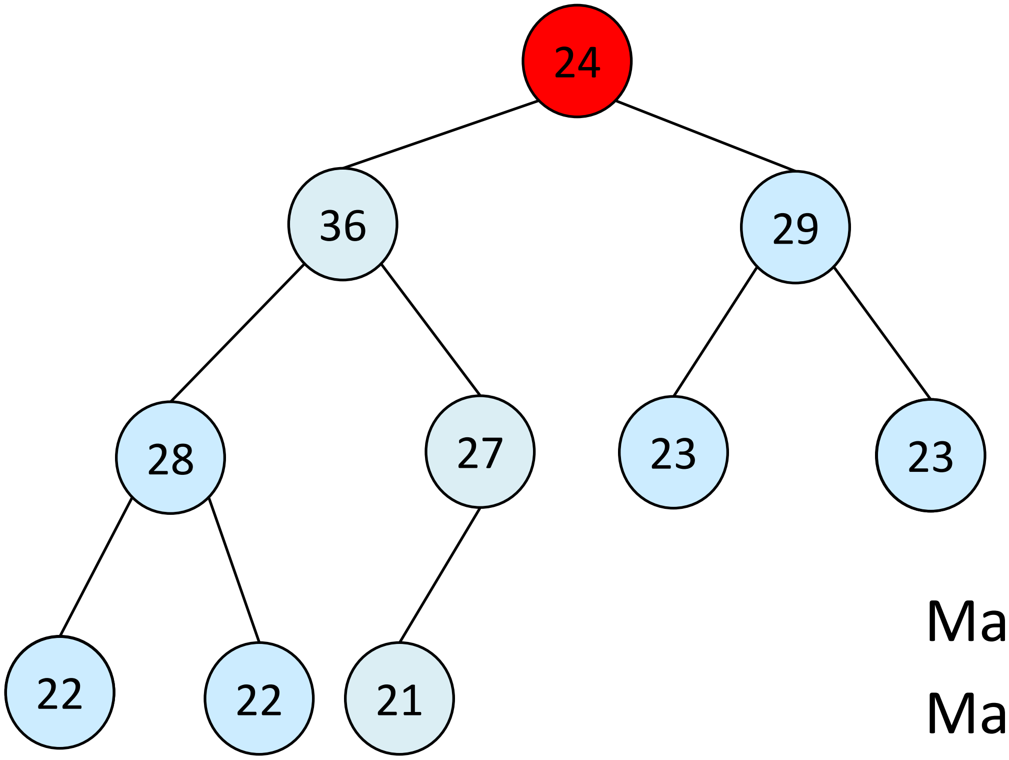
MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

BuildMaxHeap – Example



MaxHeapify($\lfloor 10/2 \rfloor$ = 5)

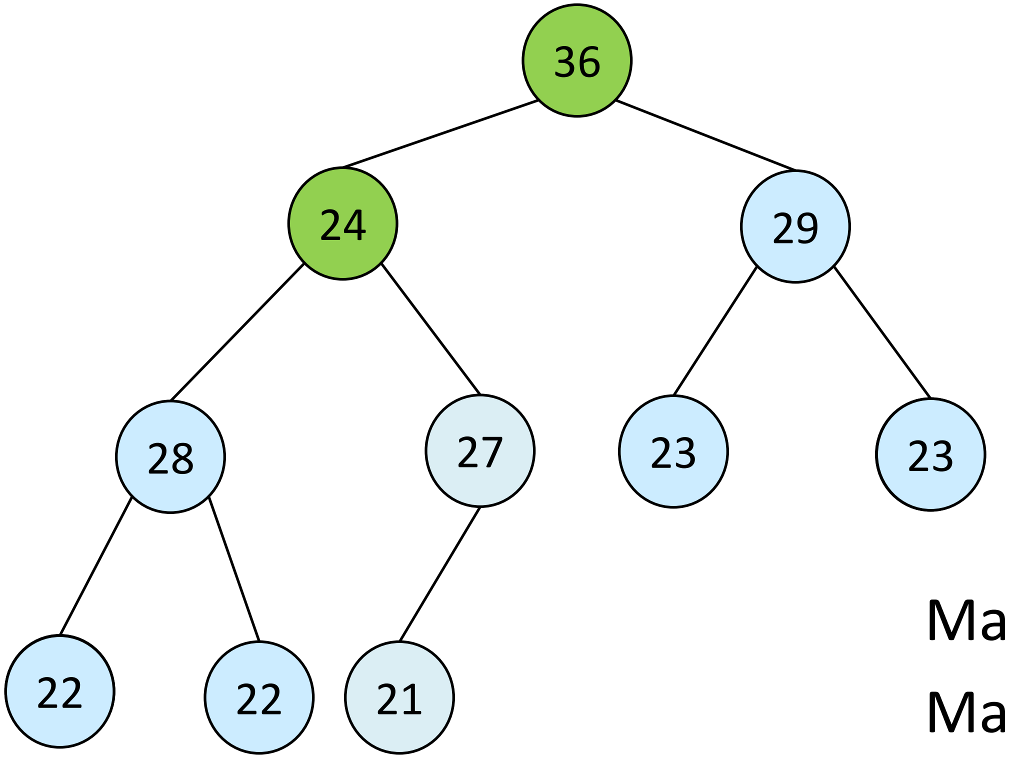
MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)

BuildMaxHeap – Example



MaxHeapify($\lfloor 10/2 \rfloor = 5$)

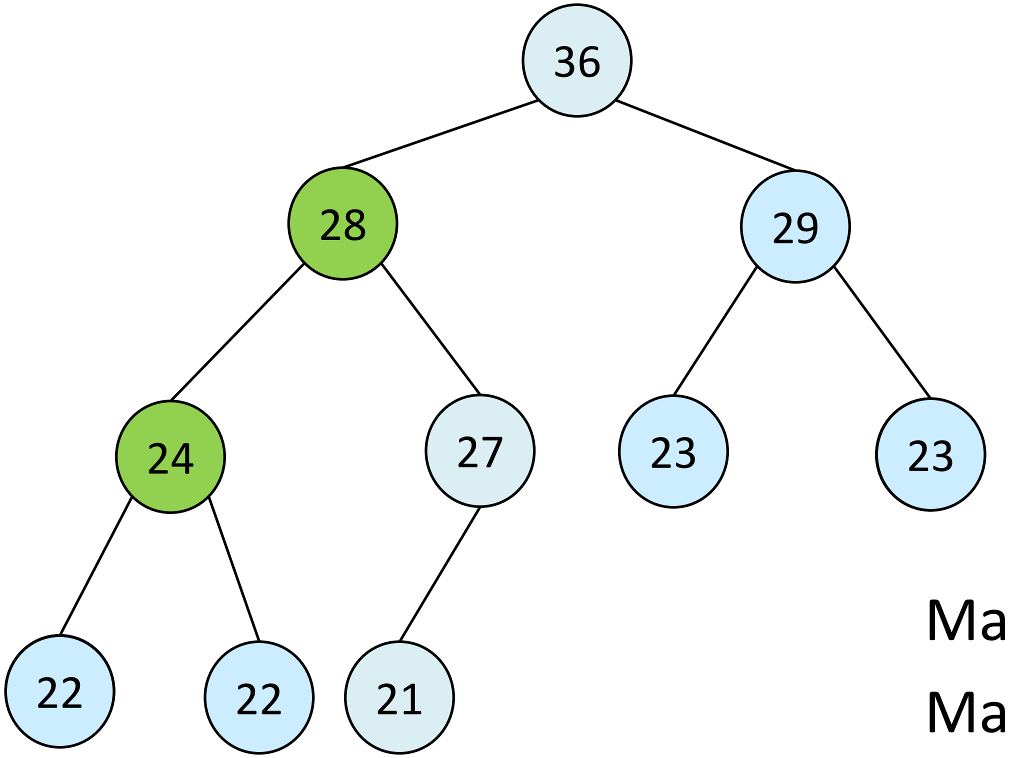
MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)

BuildMaxHeap – Example



MaxHeapify(⌊10/2⌋ = 5)

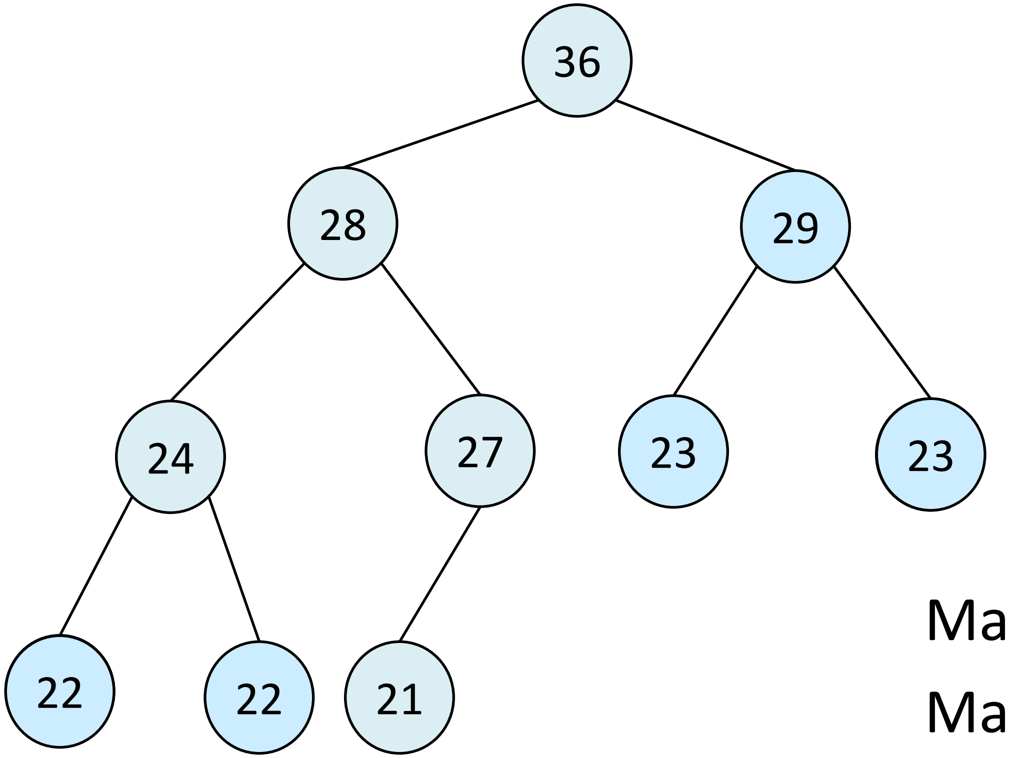
MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)

BuildMaxHeap – Example



MaxHeapify($\lfloor 10/2 \rfloor$ = 5)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)

We did a O(log n) operation O(n) times.

Correctness of *BuildMaxHeap*

- **Loop Invariant property (LI):** At the start of each iteration of the **for** loop, each node i+1, i+2, …, n is the root of a max-heap.

- **Initialization:**
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor$+1, $\lfloor n/2 \rfloor$+2, …, n are leaves, thus max-heaps.

- **Maintenance:**
  - By LI, subtrees at children of node i are max heaps.
  - Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - Decrementing i reestablishes the loop invariant for the next iteration.

- **Stop:** bounded number of calls to MaxHeapify

Running Time of *BuildMaxHeap*

- **Loose upper bound:**
  - Cost of a *MaxHeapify* call × # calls to *MaxHeapify*
  - $O(\lg n) \times O(n) = O(n \lg n)$

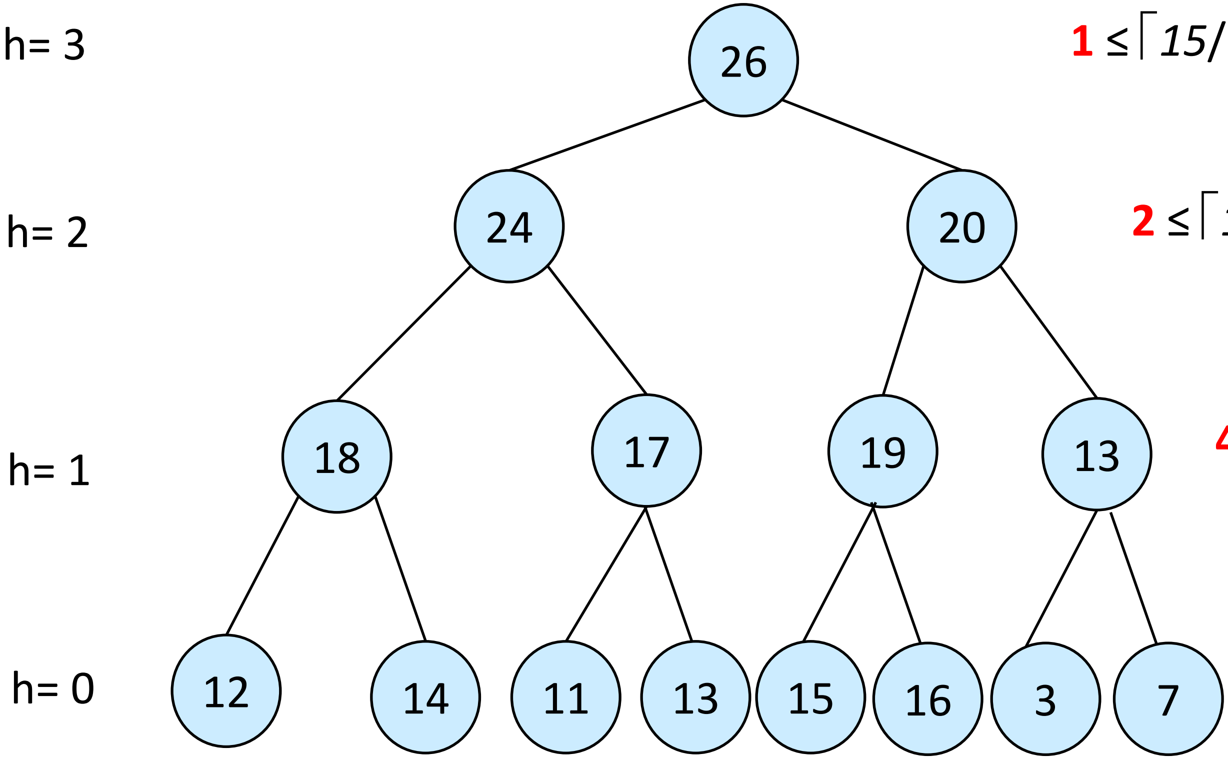
But we're not really doing O(n) work at each step since the heaps get smaller

- **Tighter bound:**
  - Cost of *MaxHeapify* is $O(h)$.
  - Height of heap is $\lfloor \lg n \rfloor$
  - $\leq \lceil n/2^{h+1} \rceil$ nodes with height h.

$\lceil n/2^{h+1} \rceil$??

n= 15

$\le \lceil n/2^{h+1} \rceil$ nodes with height *h*.



1 $\le \lceil 15/2^{3+1} \rceil = 1$ nodes w. *h=3*

2 $\le \lceil 15/2^{2+1} \rceil = 2$ nodes w. *h=2*

4 $\le \lceil 15/2^{1+1} \rceil = 4$ nodes w. *h=1*

8 $\le \lceil 15/2 \rceil = 8$ nodes w. *h=1*

h+1 is the nth row, so we know the max number of nodes per row

Running Time of *BuildMaxHeap*

- **Loose upper bound:**
  - Cost of a *MaxHeapify* call × # calls to *MaxHeapify*
  - $O(\lg n) \times O(n) = O(n \lg n)$

- **Tighter bound:**
  - Cost of *MaxHeapify* is $O(h)$.
  - Height of heap is $\lfloor \lg n \rfloor$
  - $\le \lceil n/2^{h+1} \rceil$ nodes with height h.

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O(n)$$

There are log n rows. Total work of a row is a fraction of n. When we combine all the rows, the total work done grows linearly with n

Running time of BuildMaxHeap is O(n)

Heapsort

1. Builds a max-heap from the array.
2. Put the maximum element (i.e. the root) at the correct place in the array by swapping it with the element in the last position in the array.
3. “Discard” this last node (knowing that it is in its correct place) by decreasing the heap size, and call MAX-HEAPIFY on the new root.
4. Repeat this process (goto 2) until only one node remains.

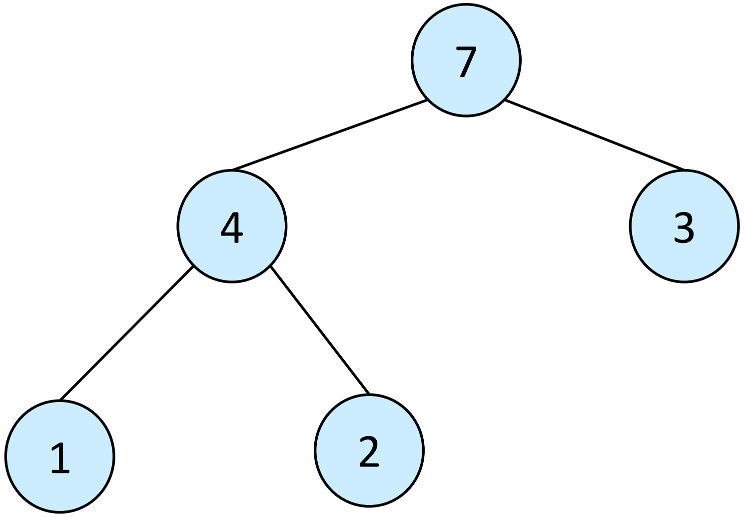
Heapsort(*A*)

```
HeapSort($A$)
    Build-Max-Heap($A$)
    for $i \leftarrow length[A]$ downto 2
        do exchange $A[1] \leftrightarrow A[i]$
           MaxHeapify(A, 1, i-1)
```

Remember insertion sort! We are progressively sorting a sub-array, this time at the end. We make a heap, take the first element (the max), swap it to the end of the list, repeat with a shorter list

Heapsort – Example

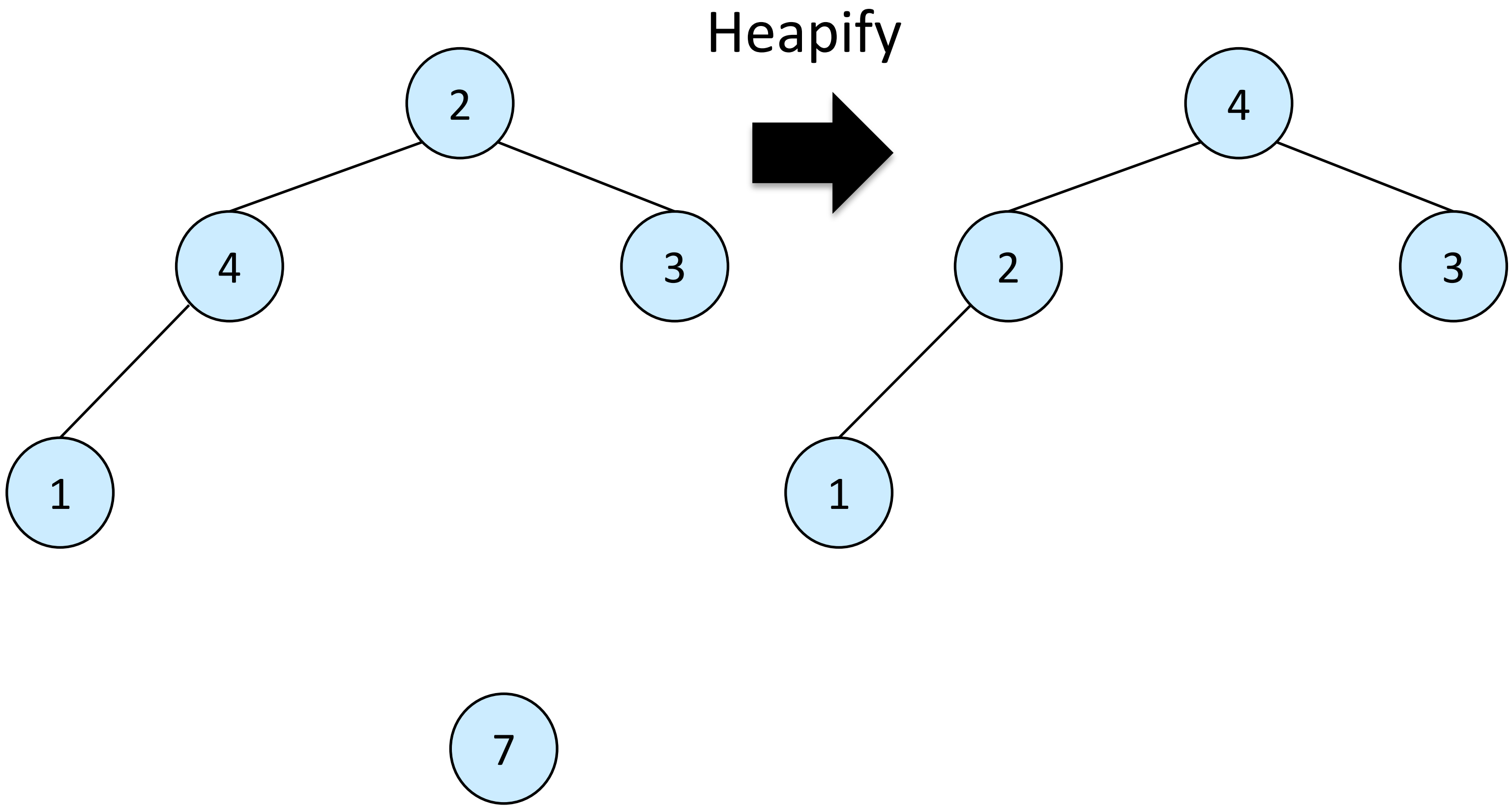
7	4	3	1	2



Heapsort – Example

2	4	3	1	7

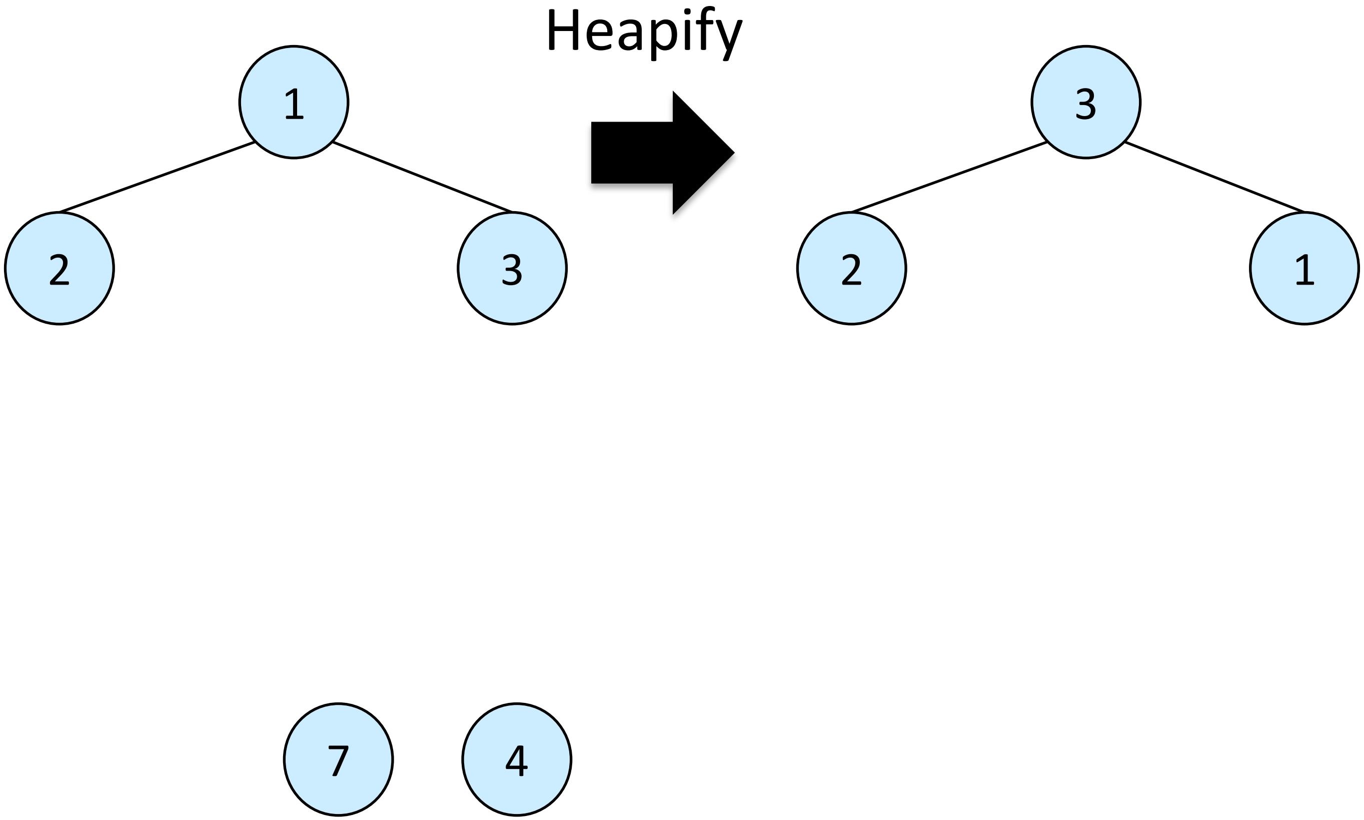
4	2	3	1	7



Heapsort – Example

1	2	3	4	7

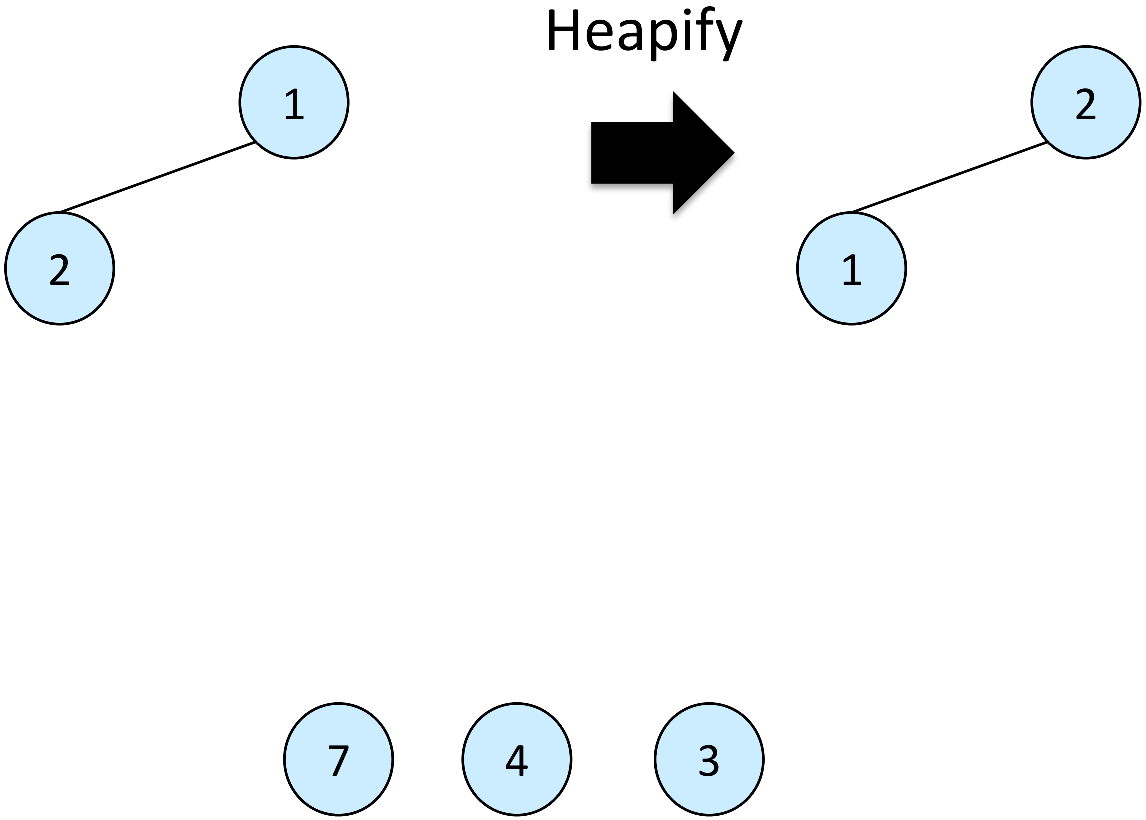
3	2	1	4	7


Heapify

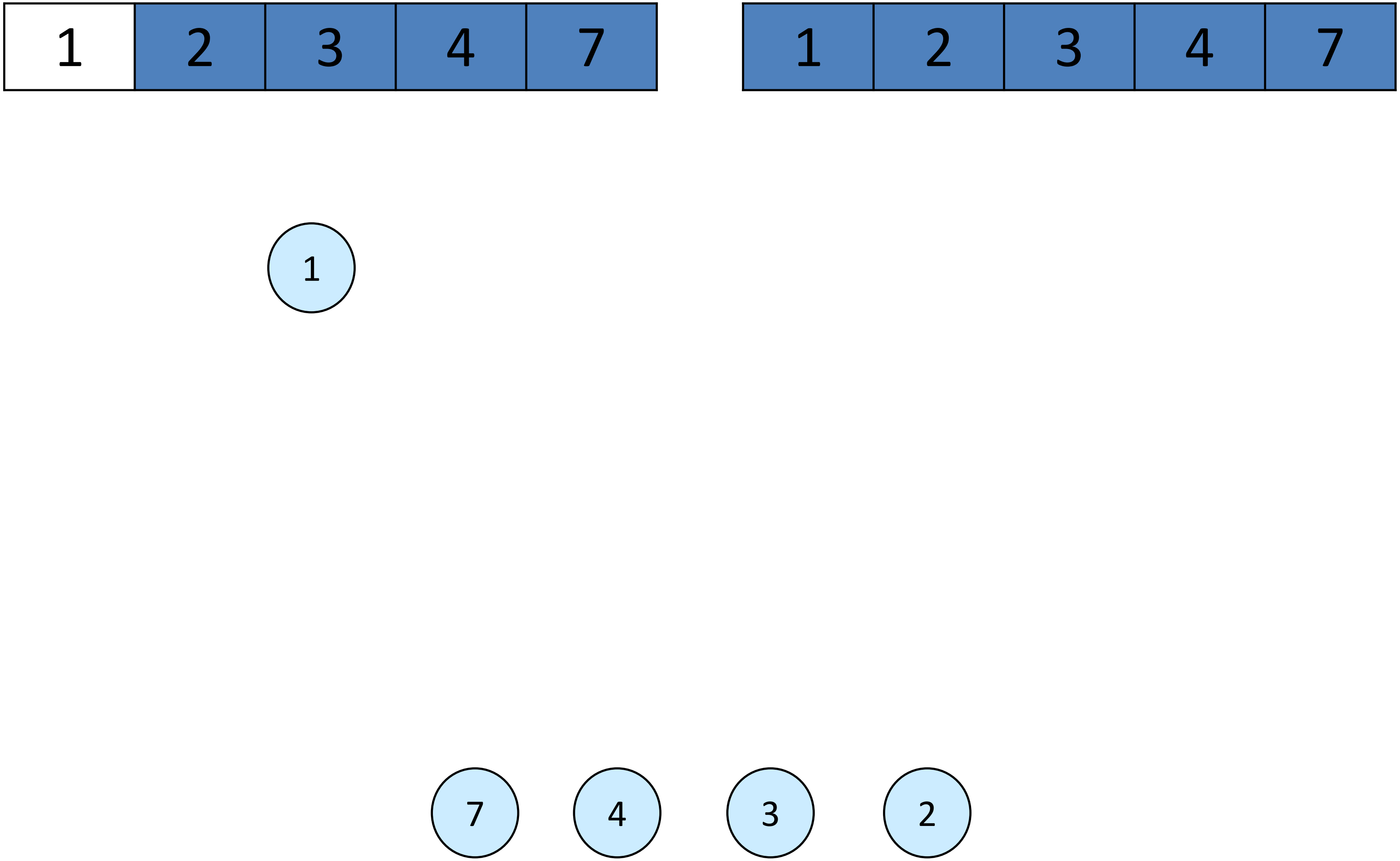
Heapsort – Example

1	2	3	4	7

2	1	3	4	7



Heapsort – Example



1	2	3	4	7

1	2	3	4	7

Heap Procedures for Sorting

- BuildMaxHeap $O(n)$
- for loop n-1 times (i.e. O(n))
  - exchange elements O(1)
  - MaxHeapify O(lg n)

Happens only once!

Because we built the max heap, at each step, the array is only one set of swaps away from being a max heap

=> HeapSort $O(n \lg n)$