COMP251: Dynamic programming (1)

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Based on (Cormen *et al.*, 2002) & (Kleinberg & Tardos, 2005)

Algorithm paradigms

- **Greedy:**
  - Build up a solution incrementally
  - Iteratively decompose and reduce the size of the problem
  - Top-down approach

- **Dynamic programming**:
  - Solve all possible sub-problems.
  - Assemble them to build up solutions to larger problems.
  - Bottom-up approach.

Although both techniques seems disconnected, we will highlight similarities.

An example?

1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = ?

20!

1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = ?

21

Principle: Use answers previously computed for a smaller instance

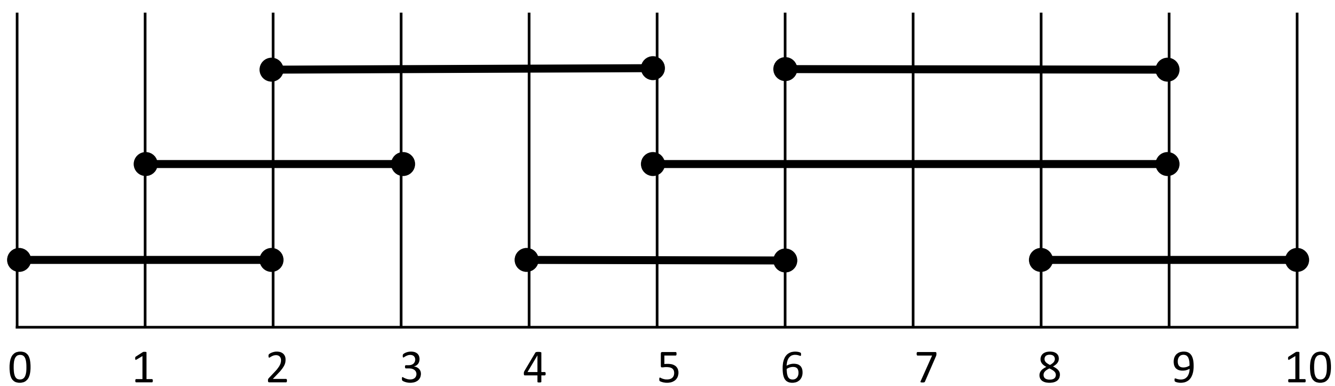
INTRODUCTION

Activity-selection Problem

- Input: Set S of n activities, $a_1, a_2, \ldots, a_n$.
  - s_i = start time of activity i.
  - f_i = finish time of activity i.
- Output: Subset A of maximum number of compatible activities.
  - 2 activities are compatible, if their intervals do not overlap.

Example:

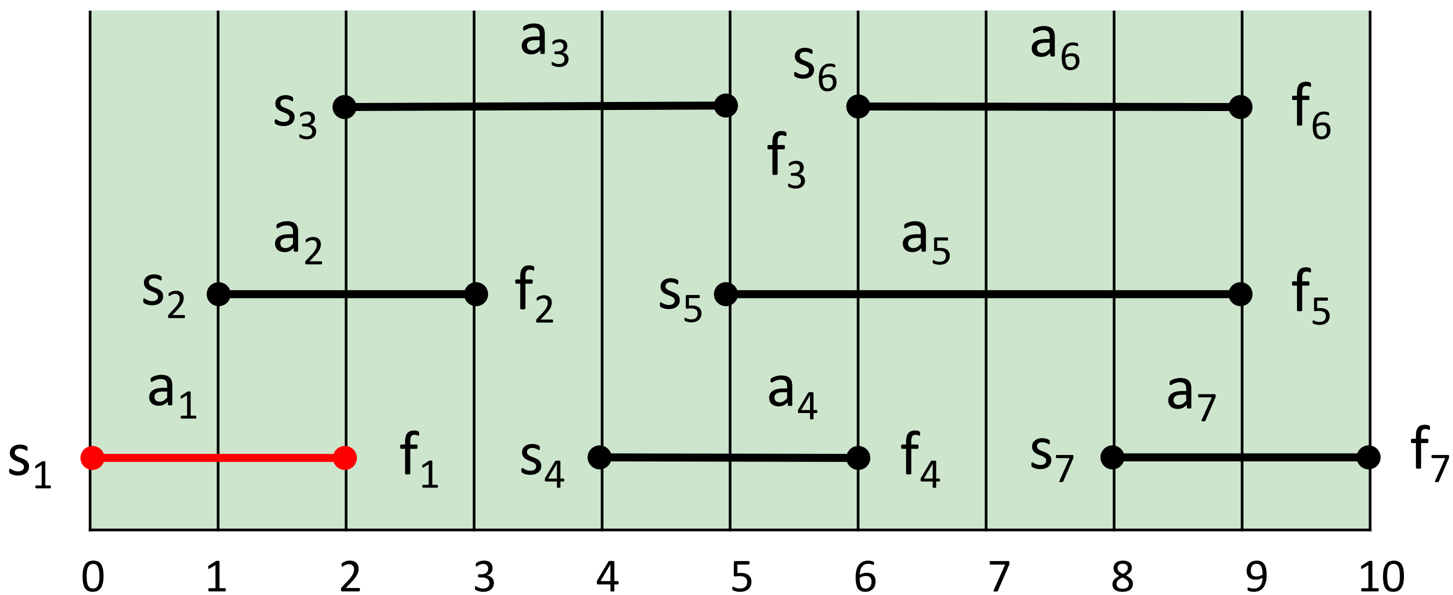
Activities in each line are compatible.



Activity-selection Problem

i	1	2	3	4	5	6	7
s_i	0	1	2	4	5	6	8
f_i	2	3	5	6	9	9	10

Activities sorted by finishing time.



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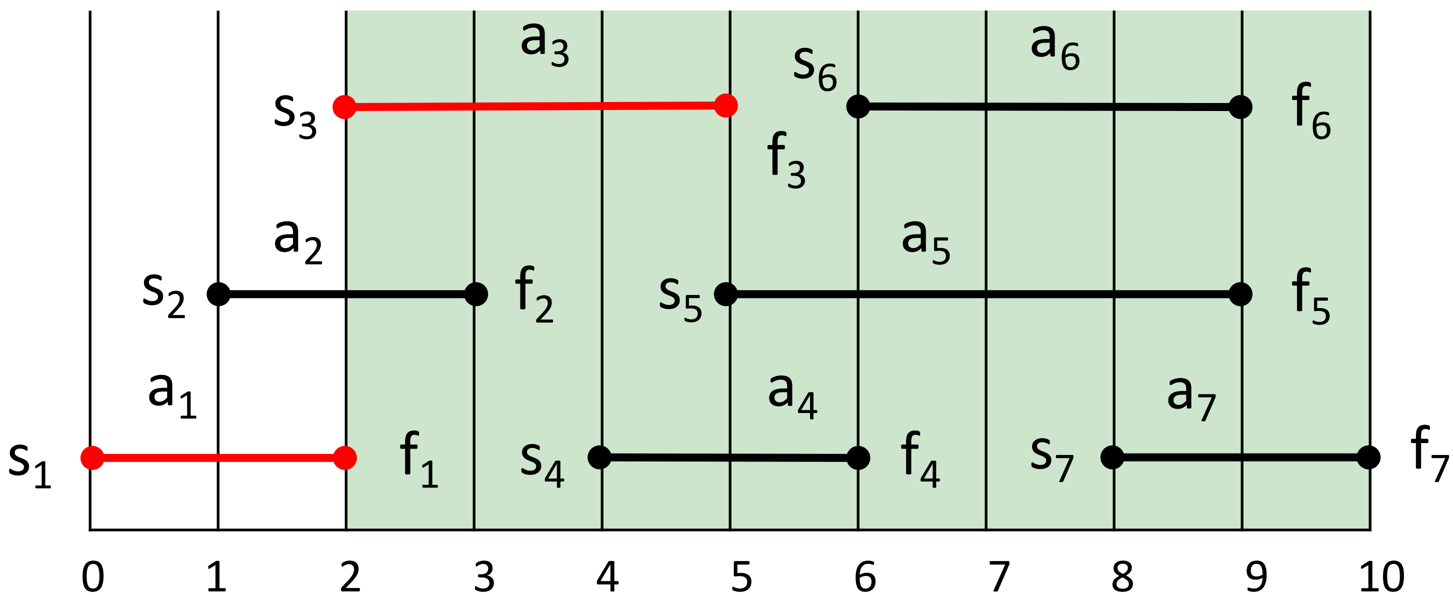

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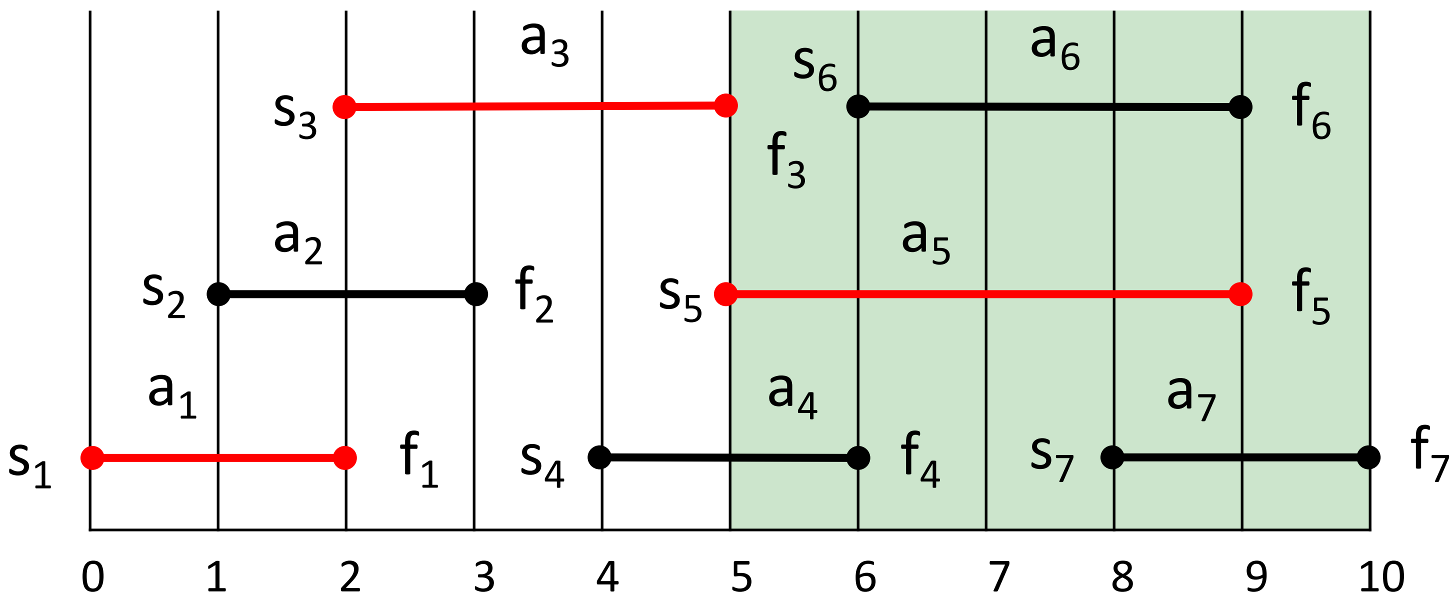

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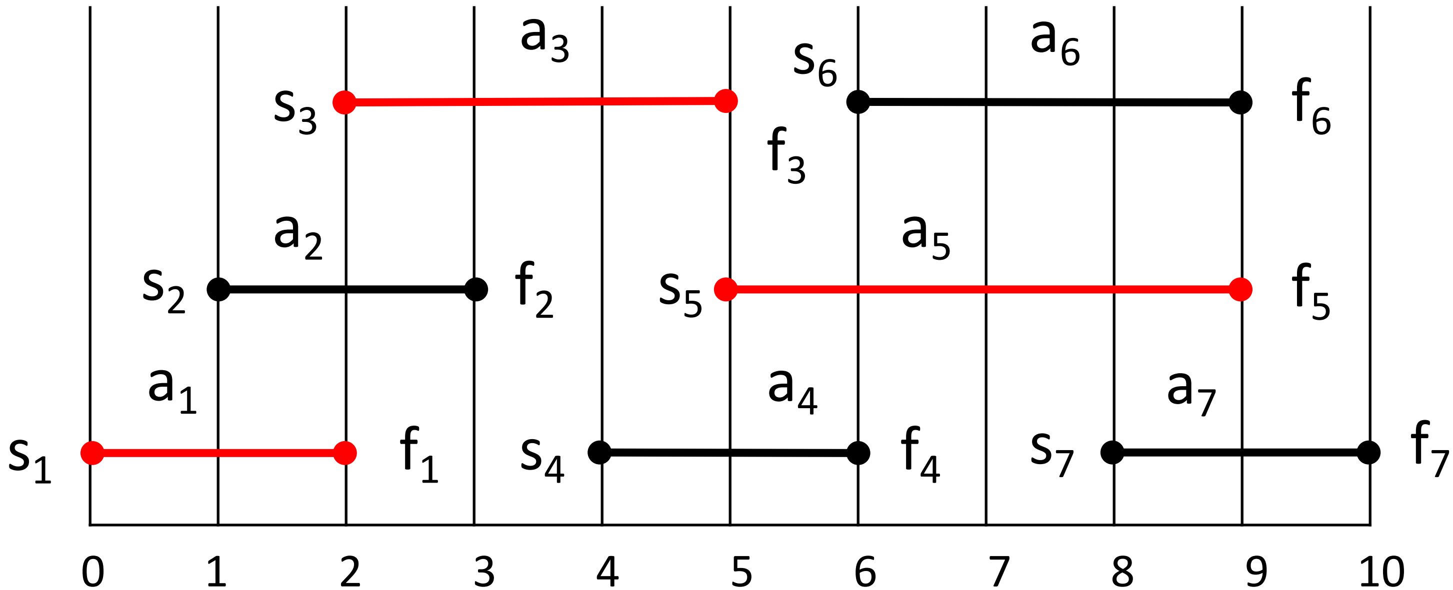


Optimal sub-structure

- Let S_{ij} = subset of activities in S that start after a_i finishes and finish before a_j starts.

$$S_{ij} = \left\{ a_k \in S : \forall i, j \quad f_i \leq s_k < f_k \leq s_j \right\}$$

- A_{ij} = optimal solution to S_{ij}
- **$A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$**

Greedy choice

	Before theorem
# subproblems in optimal solution	2
# choices to consider	j-i-1

$A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$

We can solve the problem S_{ij} top-down:

- Consider all $a_k \in S_{ij}$
- Solve S_{ik} and S_{kj}
- Pick the best m such that $A_{ij} = A_{im} \cup \{ a_m \} \cup A_{im}$

Greedy choice

Theorem:

Let $S_{ij} \neq \emptyset$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\{ f_k : a_k \in S_{ij}\}$. Then:

1. a_m is used in some maximum-size subset of mutually compatible activities of S_{ij}.

2. $S_{im} = \emptyset$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem.

Greedy choice

	Before theorem	After theorem
# subproblems in optimal solution	2	1
# choices to consider	j-i-1	1
	$A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$	$A_{ij} = \{ a_m \} \cup A_{mj}$

We can now solve the problem S_{ij} top-down:

- Choose $a_m \in S_{ij}$ with the earliest finish time (greedy choice).
- Solve S_{mj}.

Objective

- A greedy algorithm can compute an optimal solution if we identify:
  - a greedy choice
  - an optimal substructure property
- A greedy choice is not always available.
- What can we do if we have an optimal substructures property but not a greedy choice?
- How can we use the optimal substructures property to design an efficient algorithm?

We will illustrate this approach on a variant of the interval scheduling problem. Next week, we will review more examples.

WEIGHTED INTERVAL SCHEDULING

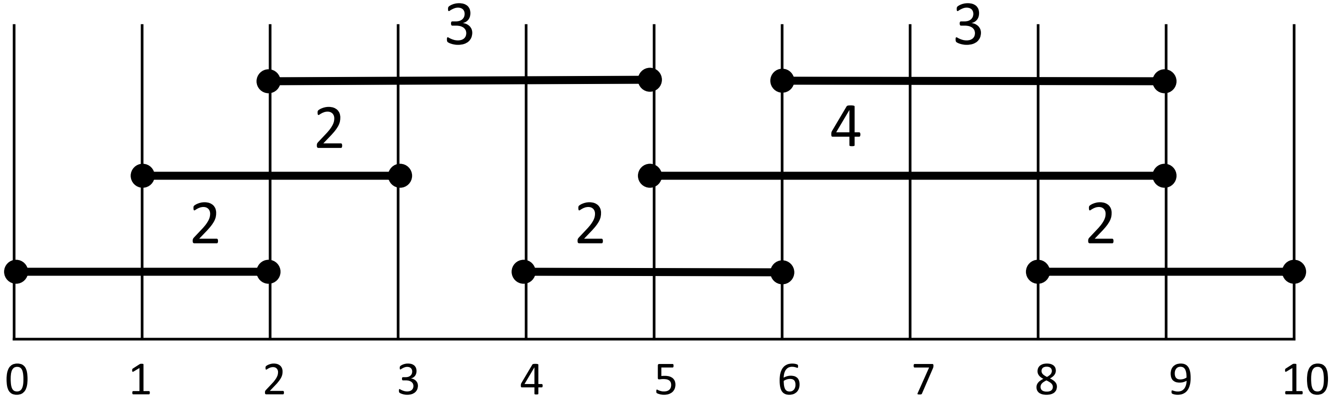
Weighted interval scheduling

- **Input:** Set S of n activities, $a_1, a_2, \ldots, a_n$.
  - s_i = start time of activity i.
  - f_i = finish time of activity i.
  - w_i= weight of activity i

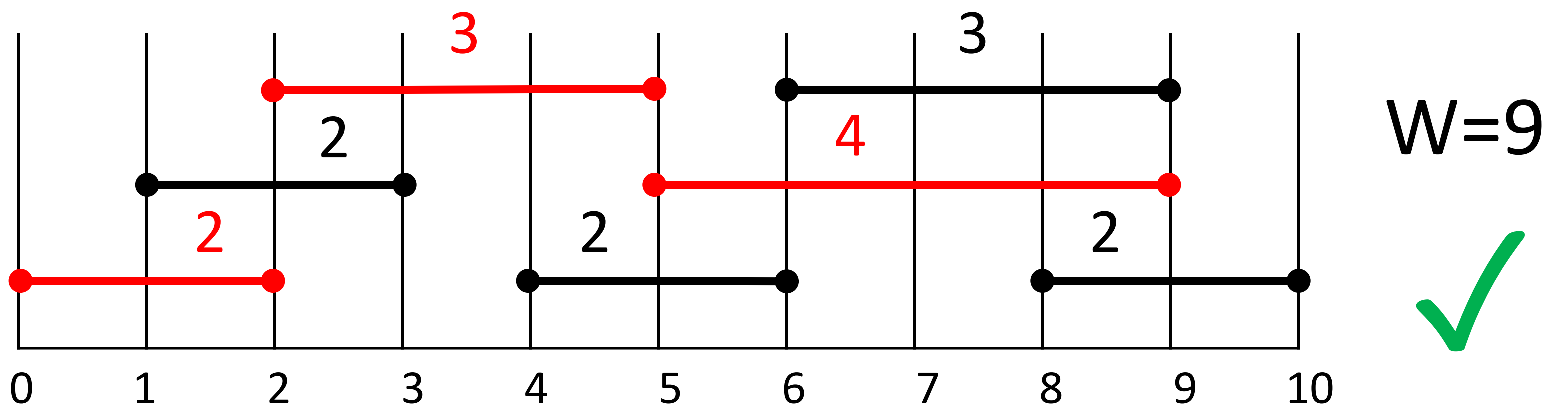
The weight can be anything

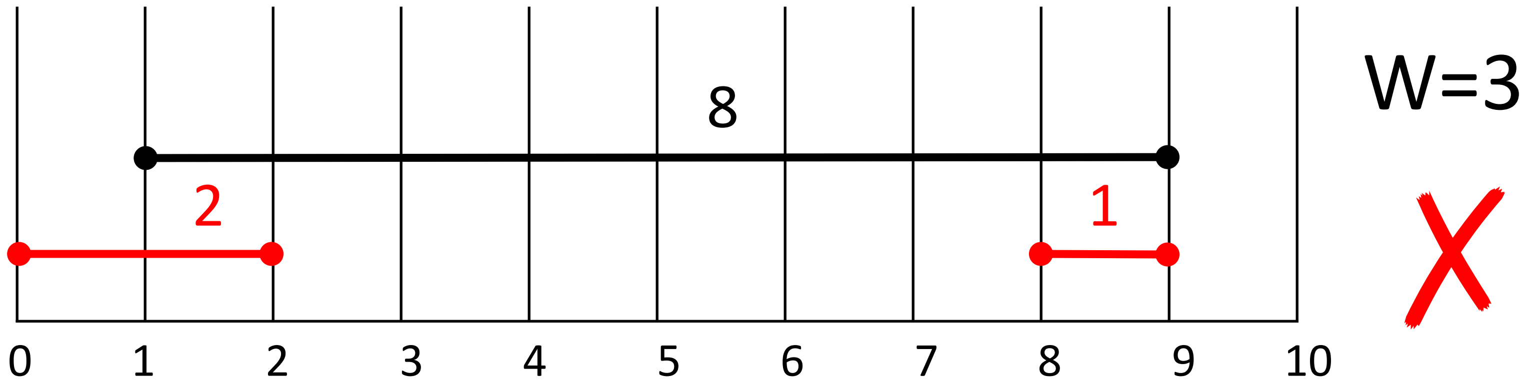
- **Output:** find maximum weight subset of mutually compatible activities.
  - 2 activities are compatible, if their intervals do not overlap.

Example:



Application of the greedy algorithm





Discussion

- **Optimal substructure:** ✓
  - A_{ij} = optimal solution to S_{ij}
  - $A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$

- **Greedy Choice:** ✗
  - Select the activity with earliest finishing time.

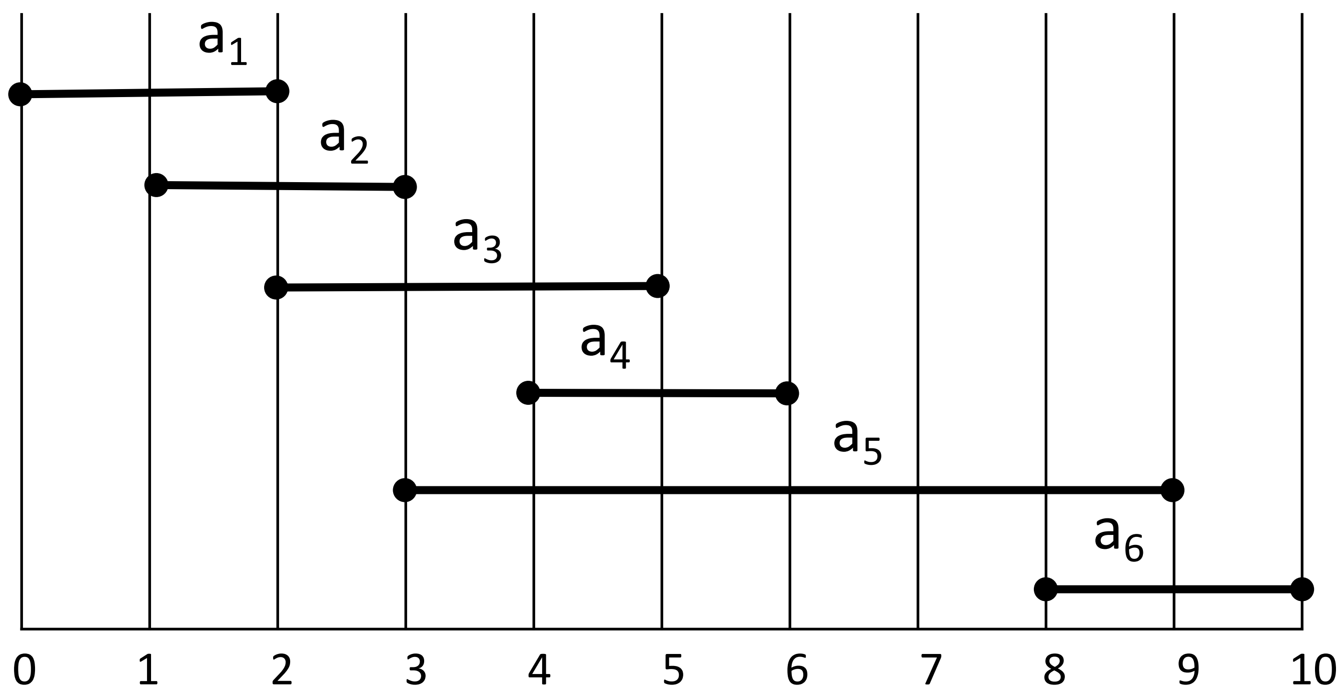
Without the greedy choice property, we need to consider all possible decompositions of A_{ij} to find the optimal one.

Data structure (1)

Notation: All activities are sorted by finishing time $f_1 \leq f_2 \leq \ldots \leq f_n$

Definition: $p(j)$ = largest index $i < j$ such that activity/job i is compatible with activity/job j.

Examples: $p(6)=4$, $p(5)=2$, $p(4)=2$, $p(2)=0$.



OPT() stores the value we want to optimize

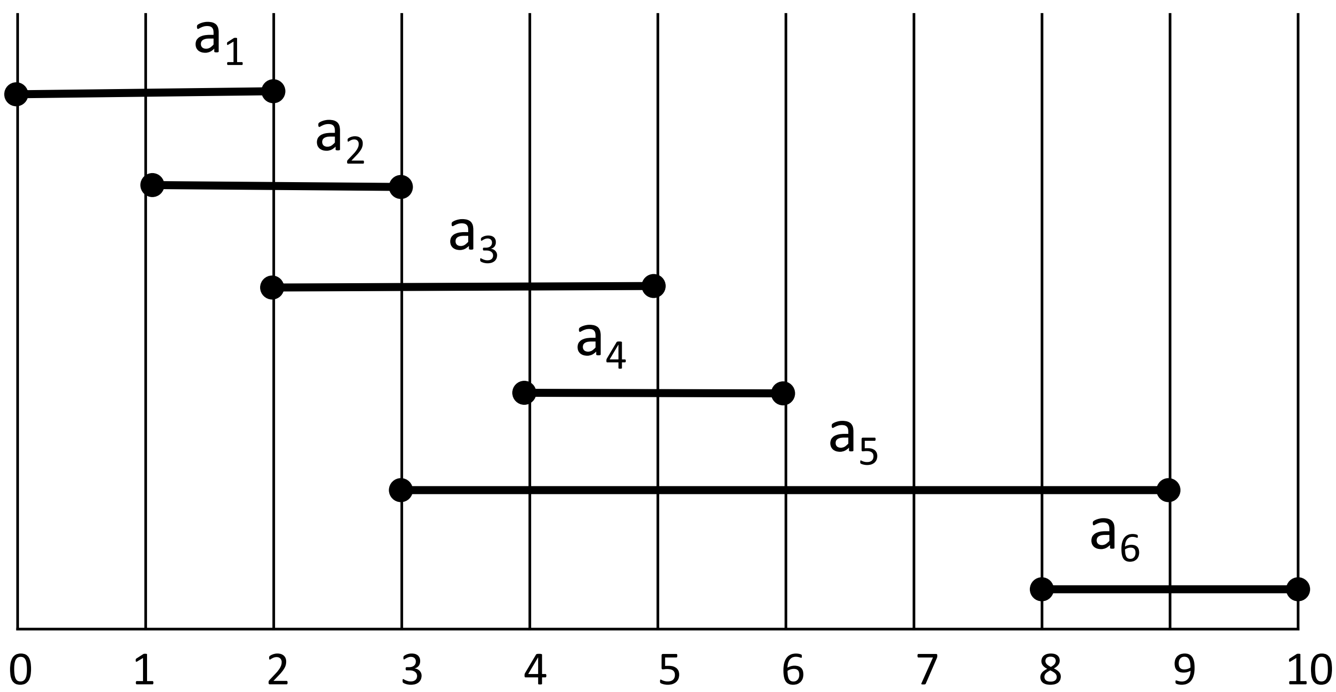
Data Structure (2)

Note: OPT(n) is the solution to our problem, where n is the number of activities.

OPT(j) = value of the optimal solution to the problem including activities 1 to j

= max total weight of compatible activities 1 to j

Examples: OPT(6) = 8, OPT(3)=5, OPT(1)=2



Binary Choice

Objective: We want to recursively compute OPT.

Question: Is activity j used in the optimal solution OPT(j)?

Case 1: *OPT uses activity j*

- The weight w_j is used to compute OPT(j)
- We cannot use activities NOT compatible with j
- We build an optimal solution with activities { 1, 2, … , p(j) }
- The weight of the optimal solution is w_j + OPT(p(j))

Case 2: *OPT does not use activity j*

- We build an optimal solution with other activities {1, … , j-1}
- The weight of the optimal solution is OPT(j-1)

A recursive solution

Base case: If there is no activity to select from, the weight is null.

$$OPT(j) = \begin{cases} 0 & if\ j = 0 \\ max\{w_j + OPT(p(j)), OPT(j-1)\} & Otherwise \end{cases}$$

Recursive case: We determine if it is best to use or not activity j.

Recursive Algorithm

Input: `n, s[1..n], f[1..n], w[1..n]`

Number of activities, starting and finishing times, weights

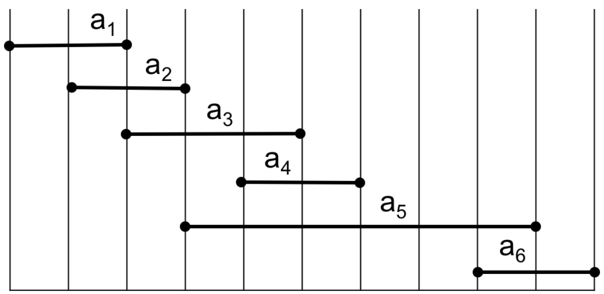
Preprocessing:

- `Sort activities by finishing time f[1] ≤ ... ≤ f[n]`
- `Compute p[1], p[2], ..., p[n]`

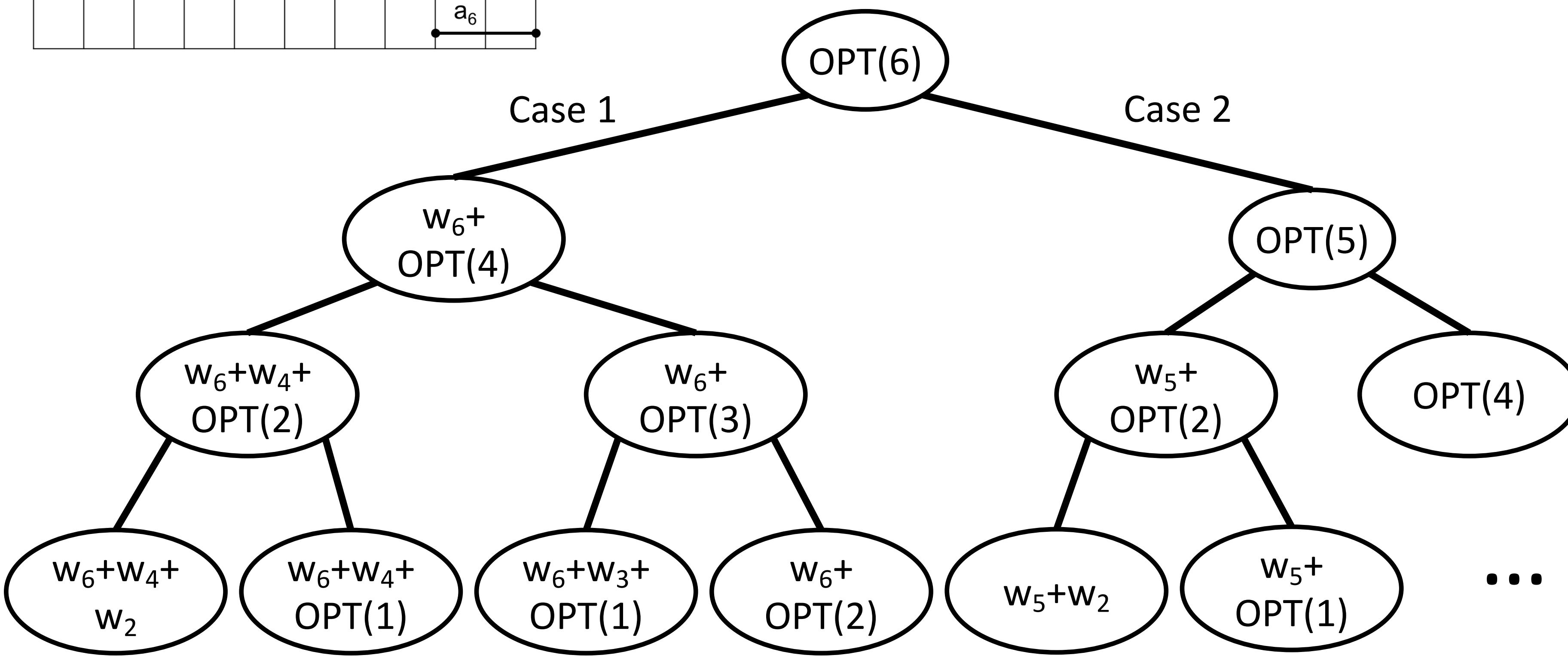
Main:

```
Compute-Opt(j)
if j = 0
   return 0
else
   return max(w[j] + Compute-Opt(p[j]), Compute-Opt(j–1))
```

Brute Force Approach



Observation: OPT(j) is calculated multiple times…



Memoization

Memoization: Cache results of each subproblem; lookup as needed.

```
Input: n, s[1..n], f[1..n], w[1..n]
Sort jobs by finish time so that f[1]≤f[2]≤ ... ≤f[n].
Compute p[1], p[2], ..., p[n].

for j = 1 to n
   OPT[j] ← empty.
OPT[0] ← 0.
```

Initialization of OPT.

```
Compute-Opt(j)
if OPT[j] is empty
   OPT[j]←max(w[j]+Compute-Opt(p[j]),Compute-Opt(j–1))
return OPT[j].
```

We store the values of OPT(j) in a table, so that we can re-use them instead of computing them again.

Running time

Claim: Memoized version of the algorithm takes *O(n log n)* time

- Sort by finishing time: *O(n log n)*
- *Computing p(): O(n log n)* via sorting by starting time

- Compute-Opt(j): each invocation takes *O(1)* time, and either:
  1. Returns an existing value OPT(j)
  2. Fills in one new entry OPT(j) and makes two recursive calls

- Progress measure ϕ = # non-empty entries of OPT
  1. Initially ϕ=0, throughout $\phi \leq$ n
  2. Increases ϕ by 1 $\Rightarrow$ 2 recursive calls
  3. At most 2n recursive calls

- Overall running time of Compute-Opt(n) is *O(n)*

Note: O(n) if the activities are presorted

DYNAMIC PROGRAMMING

Bottom-up

Observation: When we compute OPT[j], we only need values OPT[k] for k<j.

```
BOTTOM-UP (n;s_1,...,s_n;f_1,...,f_n;w_1,...,w_n):

Sort jobs by finish time so that f_1≤f_2≤...≤f_n
Compute p(1), p(2), ..., p(n).

OPT[0]←0
for j = 1 to n
   OPT[j] ← max { w_j + OPT[p(j)], OPT[j–1] }
```

Main Idea of Dynamic Programming: Solve the sub-problems in an order that makes sure when you need an answer, it's already been computed.

For now, you can see it as a variant of the memoization algorithm that incrementally compute the OPT(k)

Finding a solution

Dyn. Prog. algorithm computes the optimal value.

Q: How to find a solution that reaches this optimal value?

A: Bactracking!

Knowing the optimal solution, we determine if activity j has been used (or not) to obtain it

```
Find-Solution(j)
if j = 0
   return ∅
else if (v[j] + M[p[j]] > M[j–1])
   return { j } ∪ Find-Solution(p[j])
else
   return Find-Solution(j–1)
```

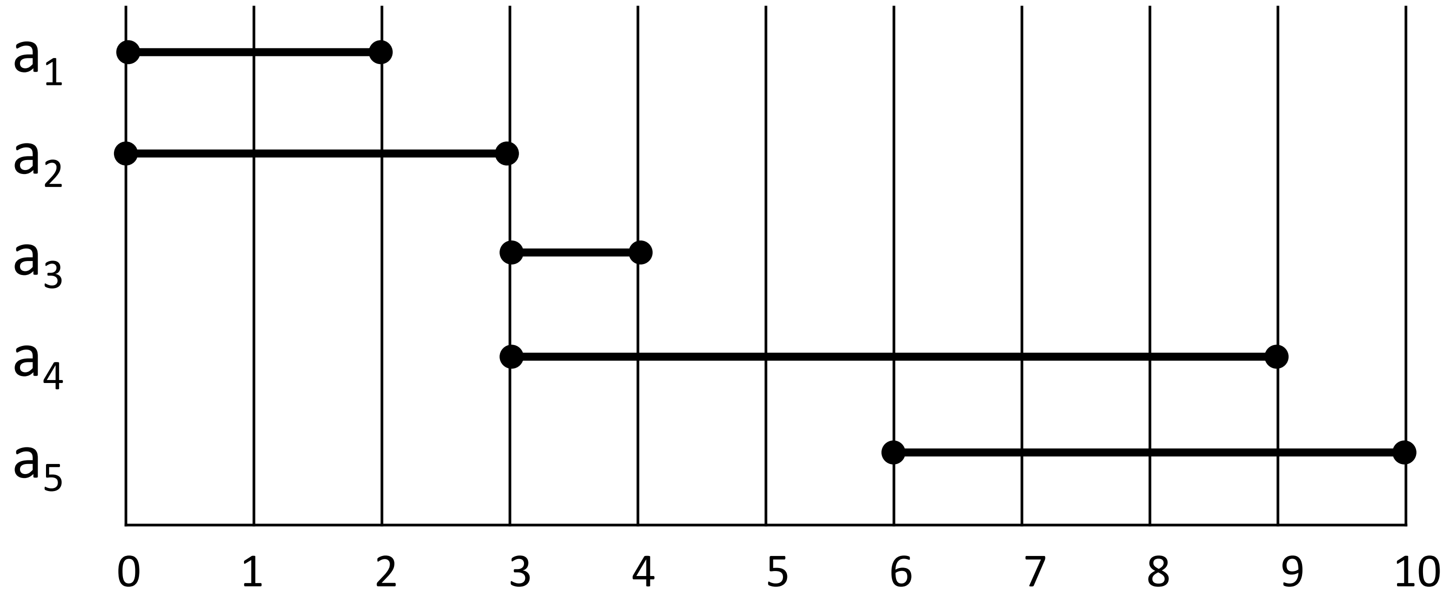
Analysis. # of recursive calls ≤ $n \Rightarrow O(n)$.

Example: Computing solution

This is p()

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	-	-	-	-	-
w_j+OPT[p(j)]	-	-	-	-	-
OPT[j-1]	-	-	-	-	-

(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.

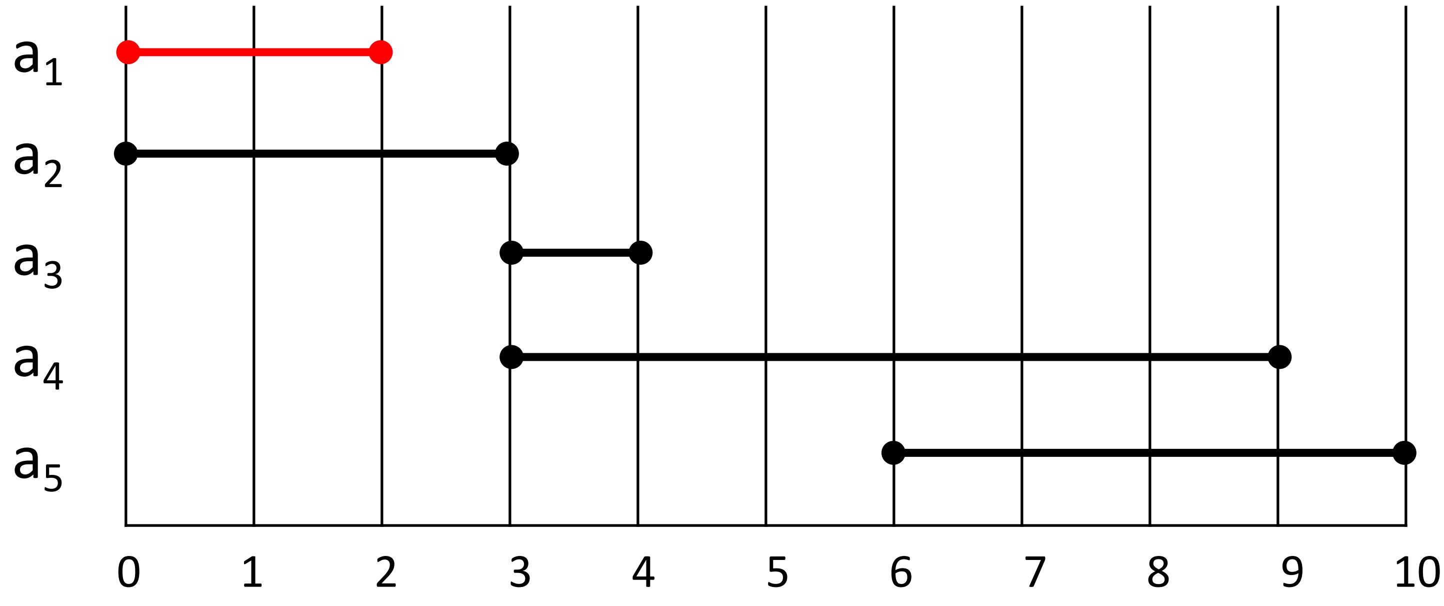


Example: Computing solution

M[0]=0

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	-	-	-	-
w_j+OPT[p(j)]	**2**	-	-	-	-
OPT[j-1]	0	-	-	-	-

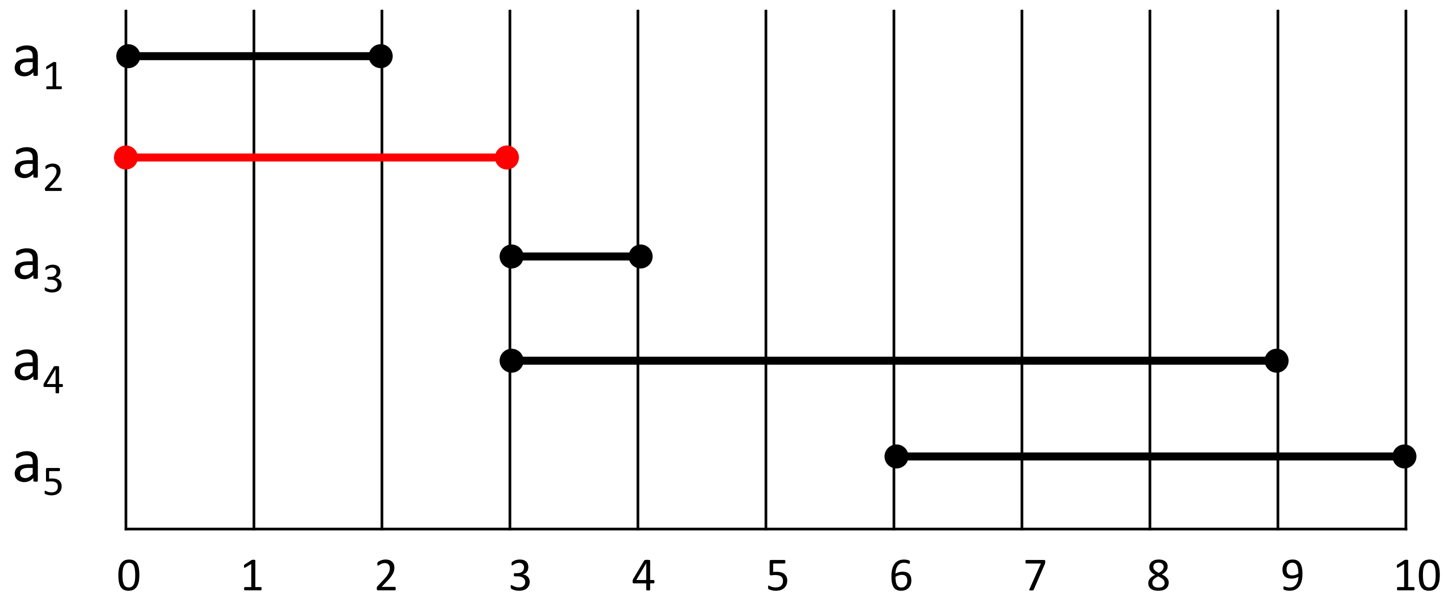
(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.



Example: Computing solution

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	-	-	-
w_j+OPT[p(j)]	2	**3**	-	-	-
OPT[j-1]	0	2	-	-	-

(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.



Example: Computing solution

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	4	-	-
w_j+OPT[p(j)]	2	3	**4**	-	-
OPT[j-1]	0	2	3	-	-

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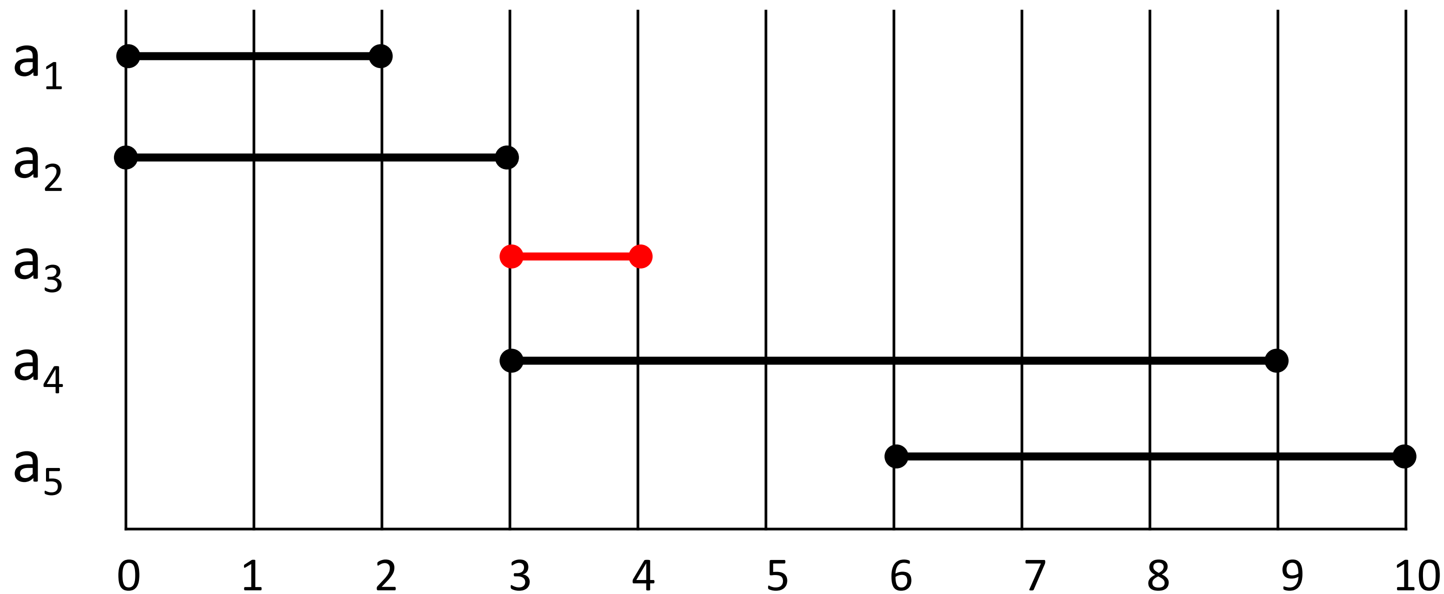
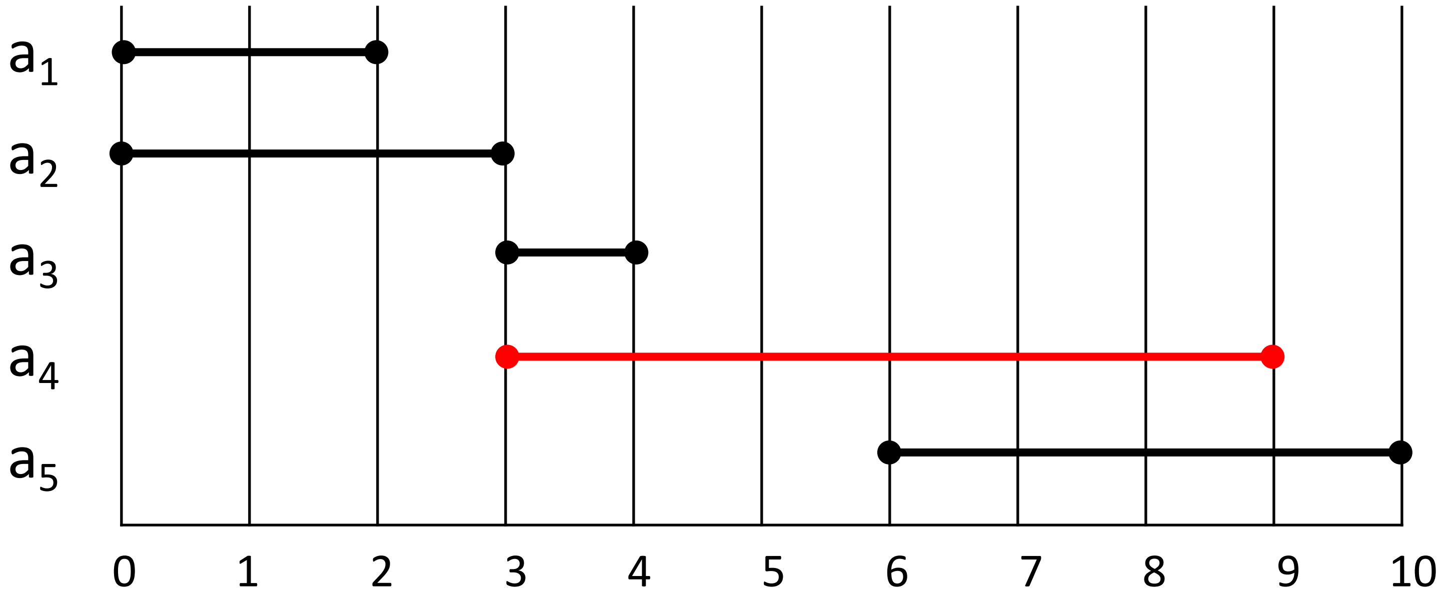


Example: Computing solution

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	4	9	-
w_j+OPT[p(j)]	2	3	4	**9**	-
OPT[j-1]	0	2	3	4	-

(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.

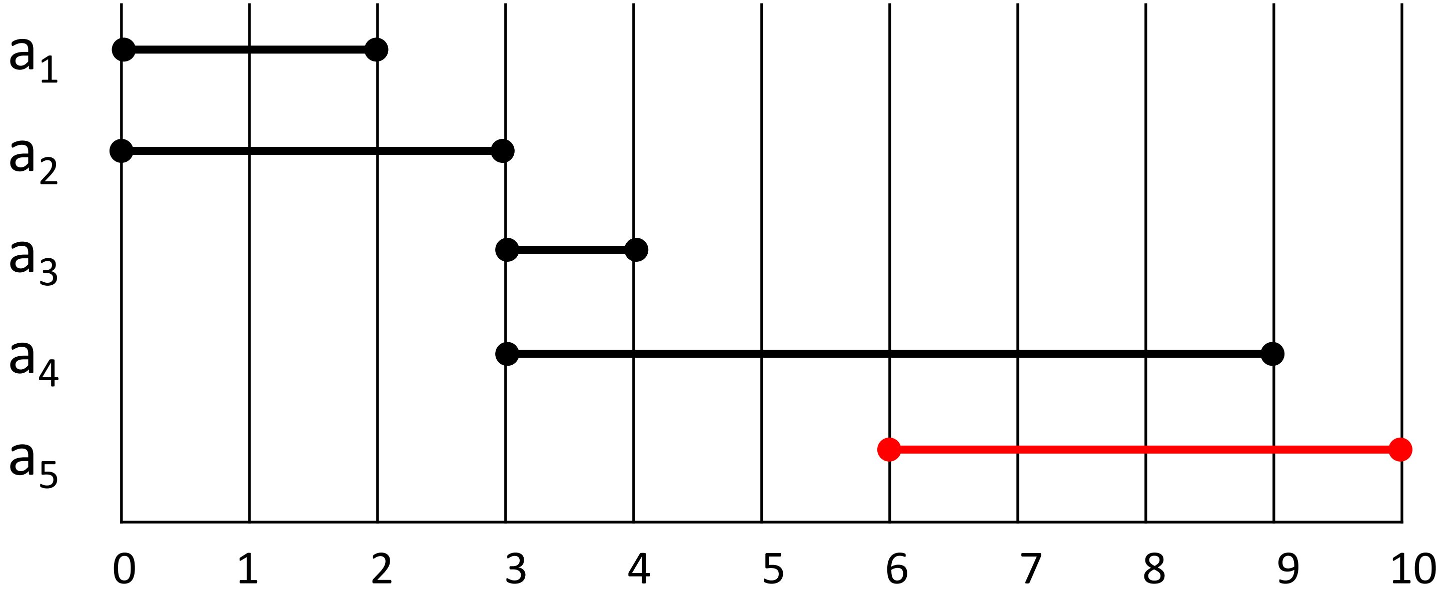


Example: Computing solution

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	4	9	9
w_j+OPT[p(j)]	2	3	4	9	8
OPT[j-1]	0	2	3	4	**9**

Optimal solution

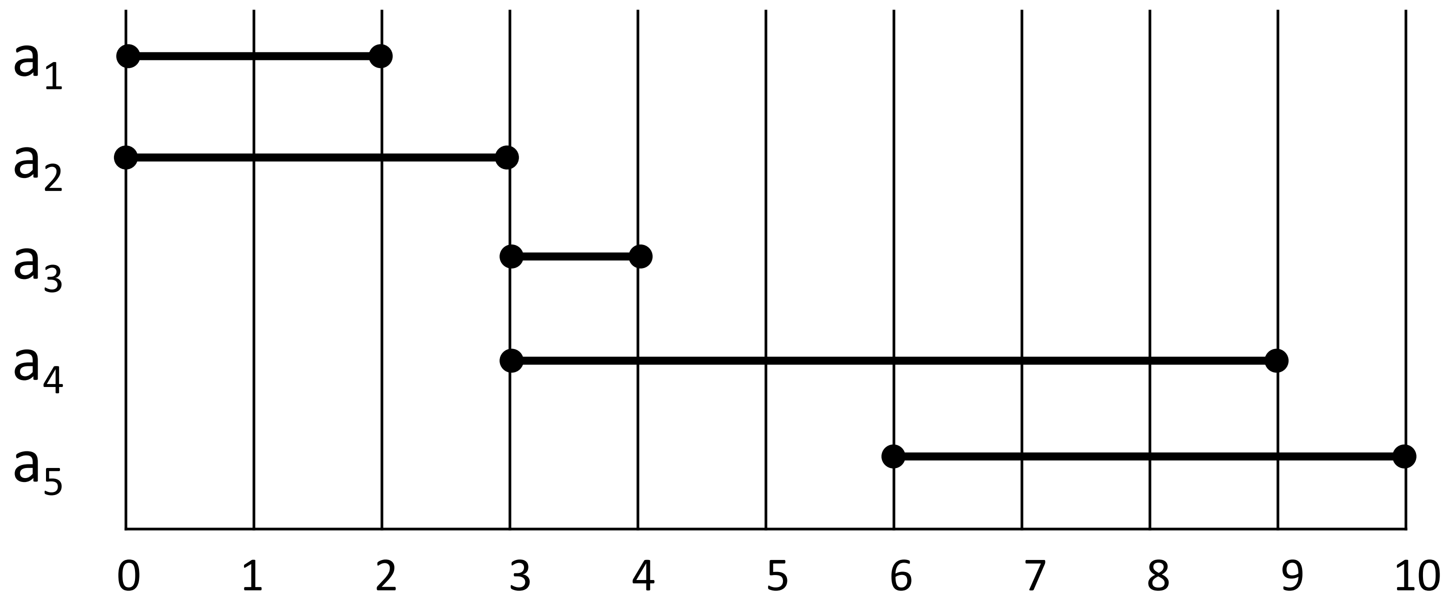
(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.



Example: Reconstruction

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	4	9	9
w_j+OPT[p(j)]	2	3	4	9	8
OPT[j-1]	0	2	3	4	9

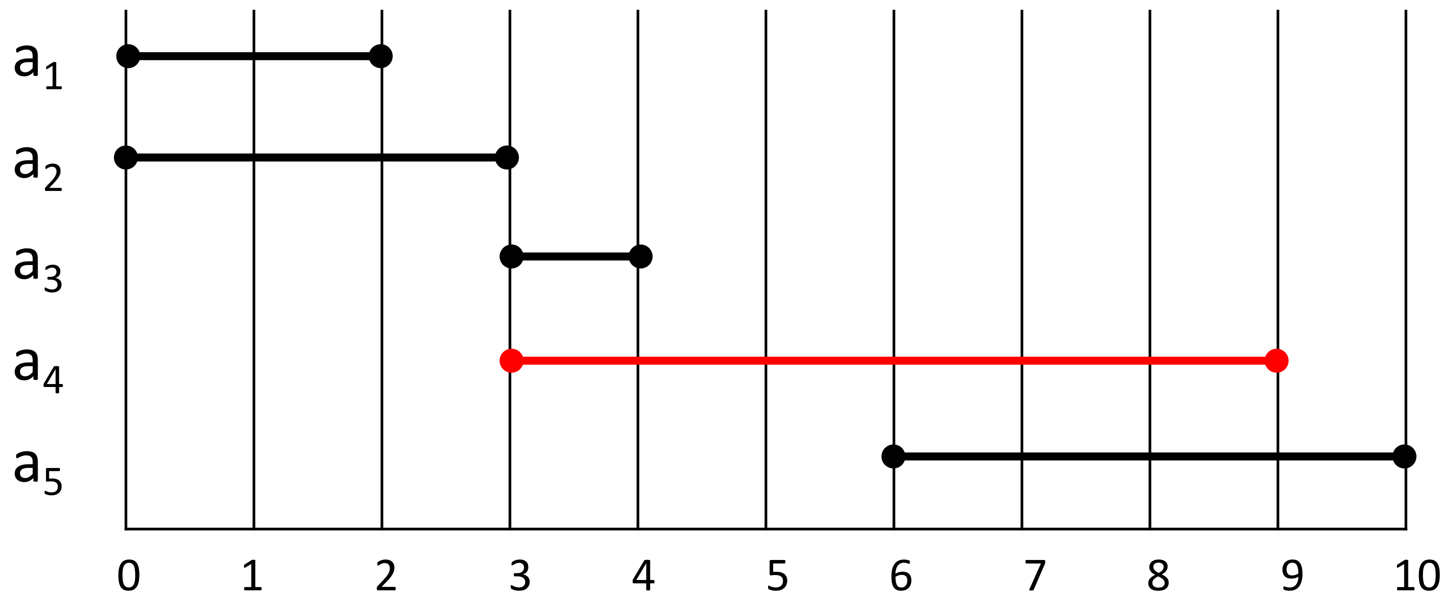
(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.



Example: Reconstruction

activity	1	2	3	4	5
predecessor	0	0	2	2	3
OPT[j]	2	3	4	9	9
w_j+OPT[p(j)]	2	3	4	9	8
OPT[j-1]	0	2	3	4	9

(1) Activities sorted by finishing time. (2) Weight equal to the length of activity.



Example: Reconstruction

activity	1	2	3	4	5
predecessor	0	0	2	2	3
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OPT[j-1]	0	2	3	4	9

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