COMP251: Network flows (2)

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Based on slides from M. Langer (McGill)

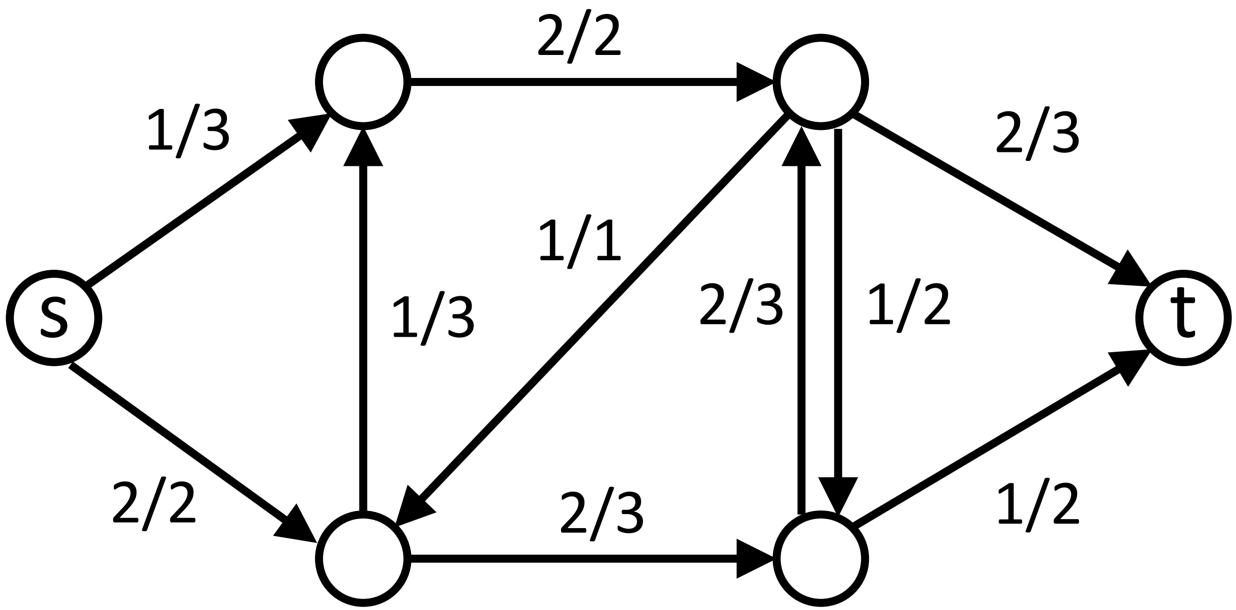
Recap Network Flows

$G = (V, E)$ directed.

Each edge (u, v) has a ***capacity*** $c(u, \mathrm{v}) \geq 0$.

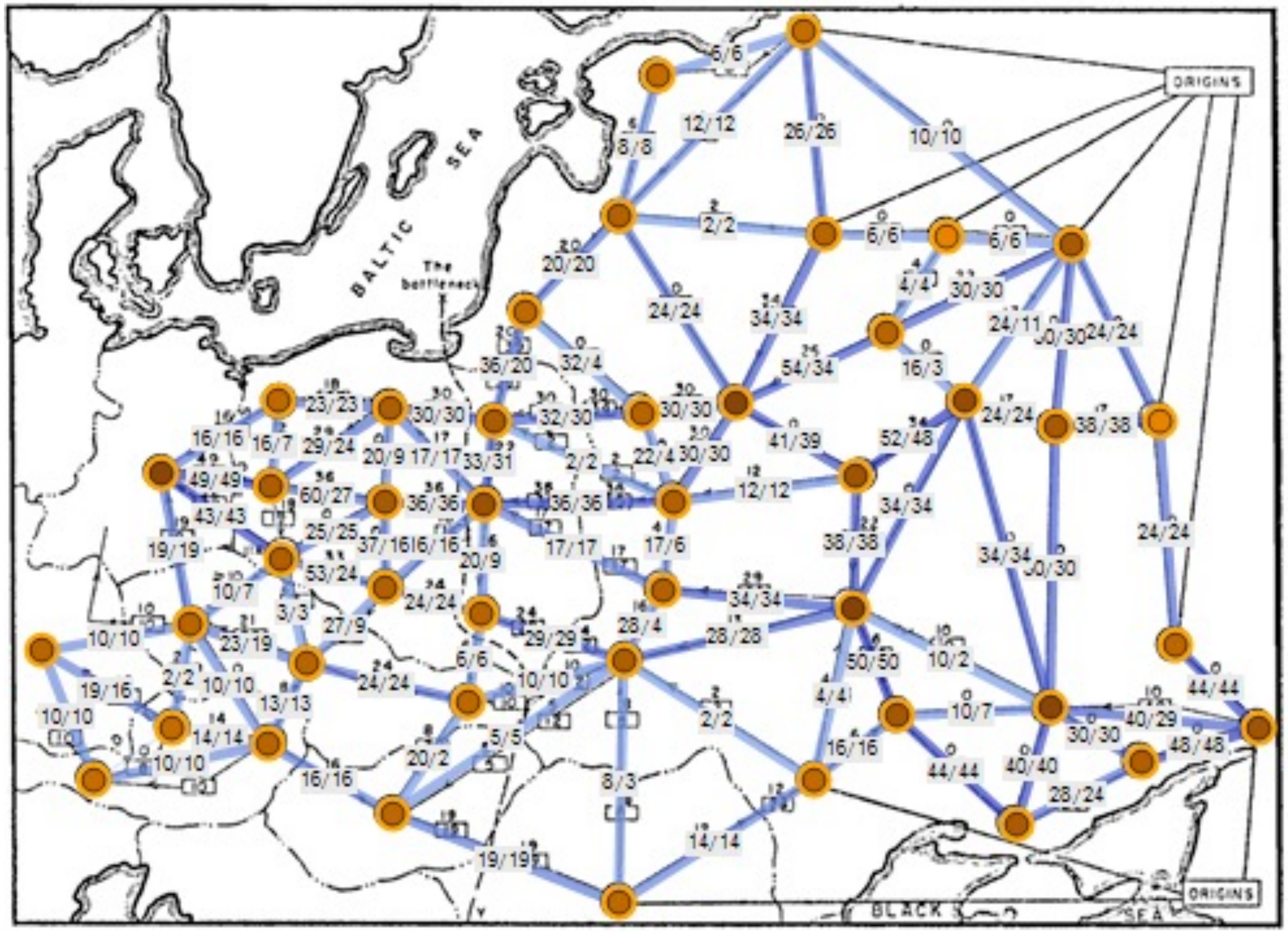
If $(u,\mathrm{v}) \notin E$, then $c(u,\mathrm{v}) = 0$.

Source vertex s, ***sink*** vertex t, assume $s \rightsquigarrow \mathrm{v} \rightsquigarrow t$ for all $\mathrm{v} \in V$.



Problem: Given G, s, t, and c, find a flow whose value is maximum.

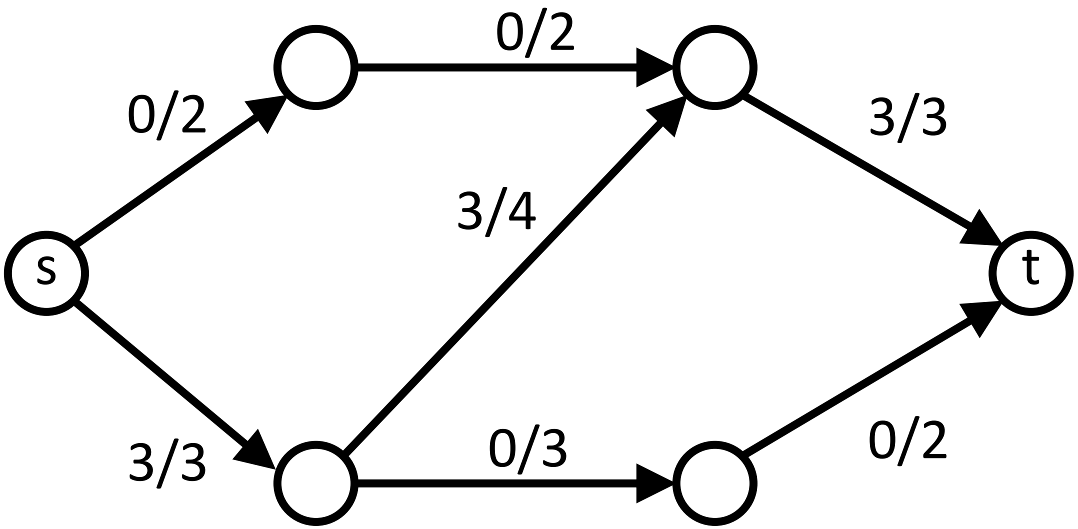
Application: a cold war example



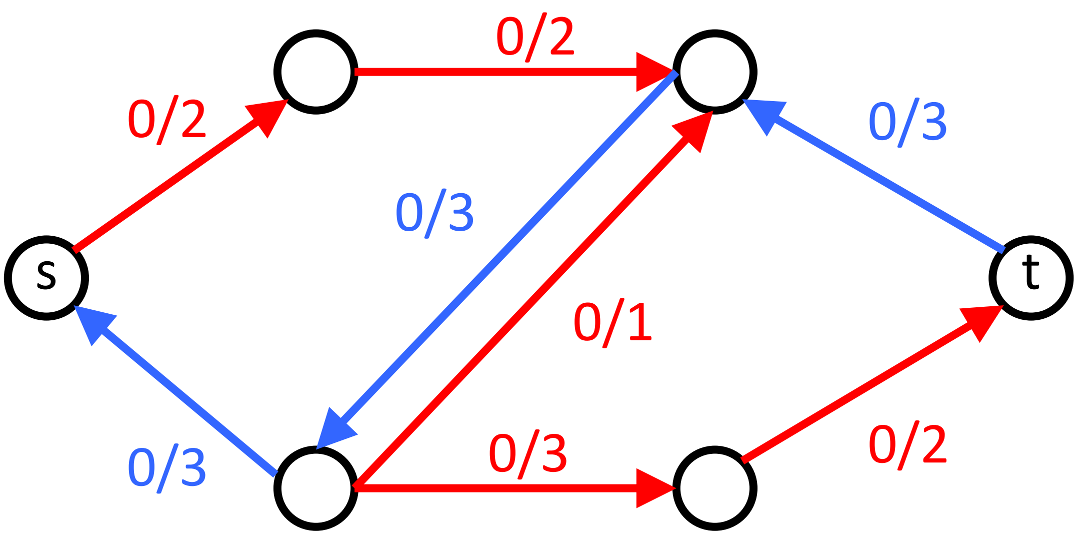
Maximizing flow of supplies in eastern europe

Recap (residual graphs)

Flow



Residual graph



Recap (Ford-Fulkerson algorithm)

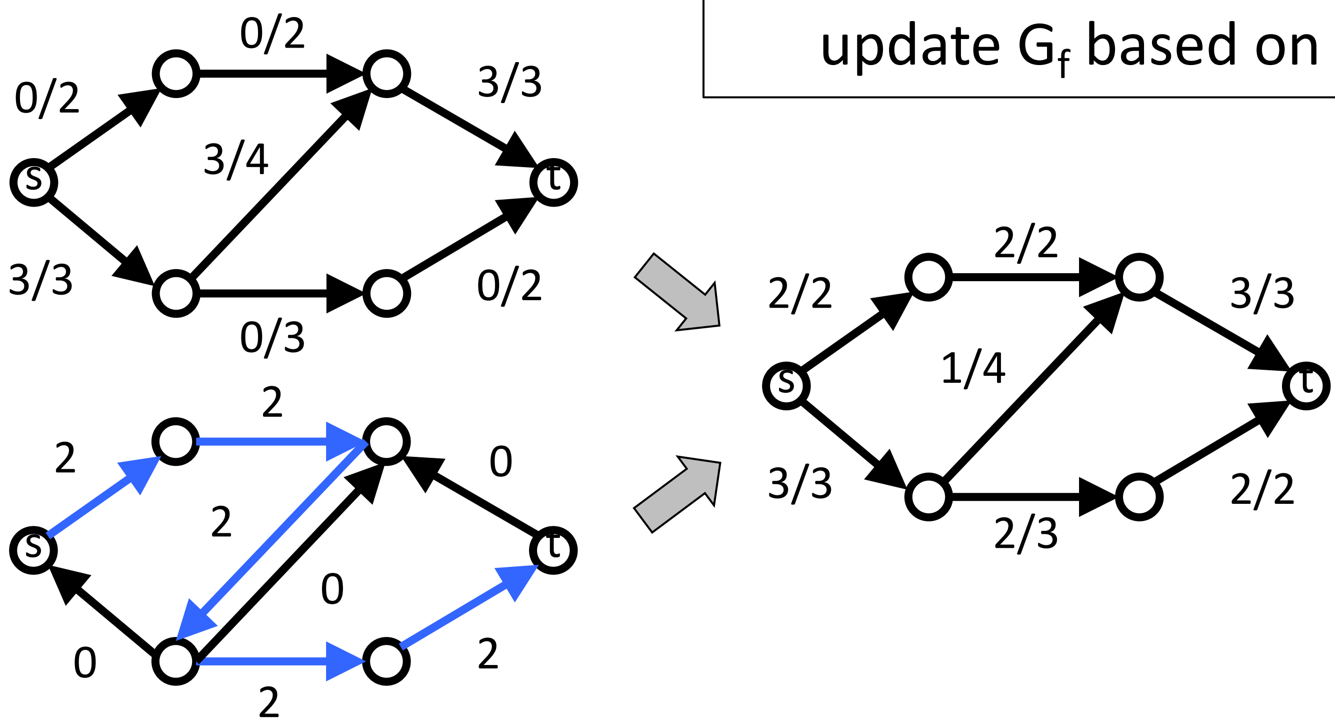
$f \leftarrow 0$

$G_f \leftarrow G$

`while` (there is a s-t path in G_f) `do`

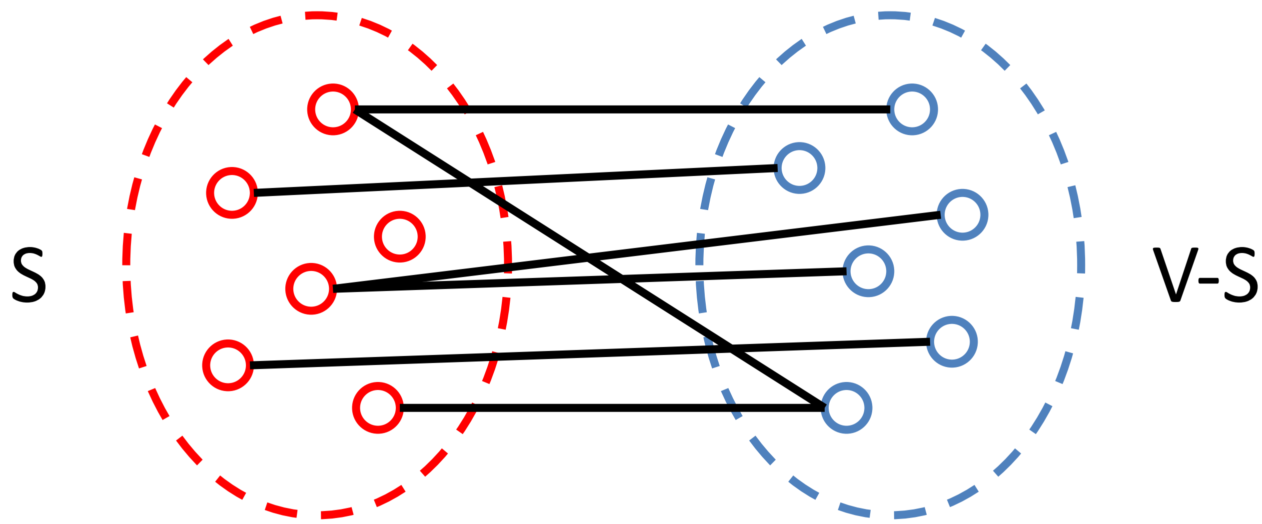
    `f.augment(P)`

    update G_f based on new f



Recap graph cuts

A graph cut is a partition of the graph vertices into two sets.



The crossing edges from S to V-S are { (u,v) | u ∈ S, v ∈ V-S }, also called the cut set.

The lowest weight edge from the cut set is the light edge.

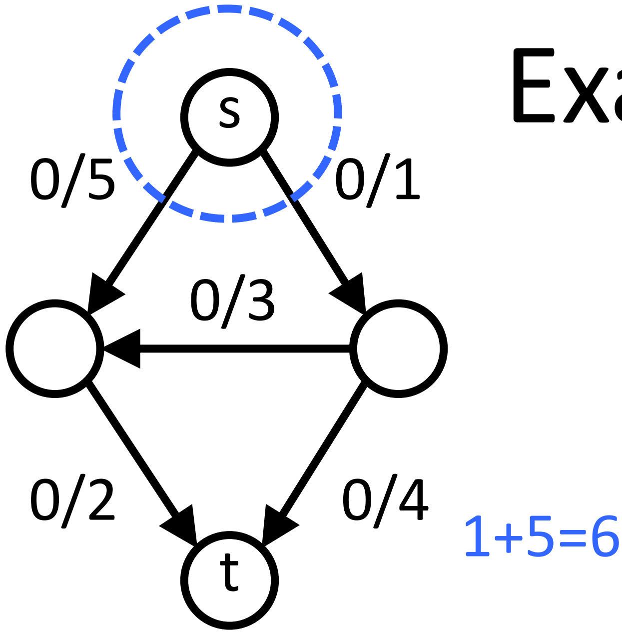
Cuts in flow networks

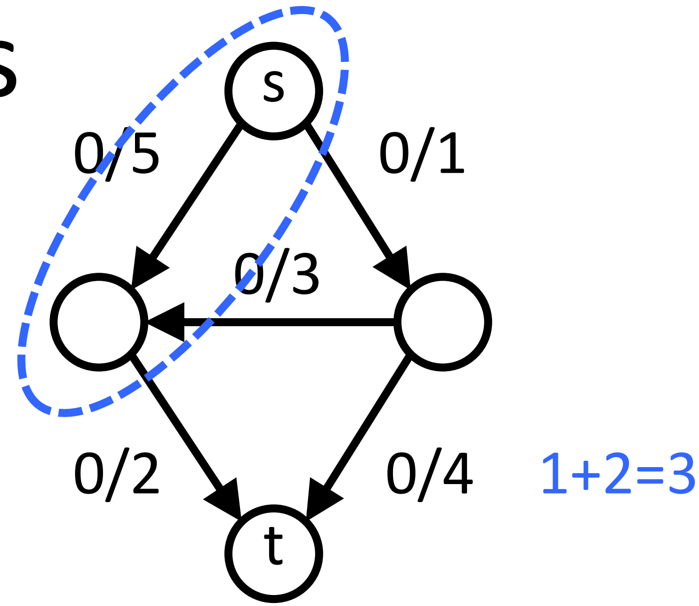
Definition: An s-t cut of a flow network is a cut A, B such that $s \in A$ and $t \in B$.

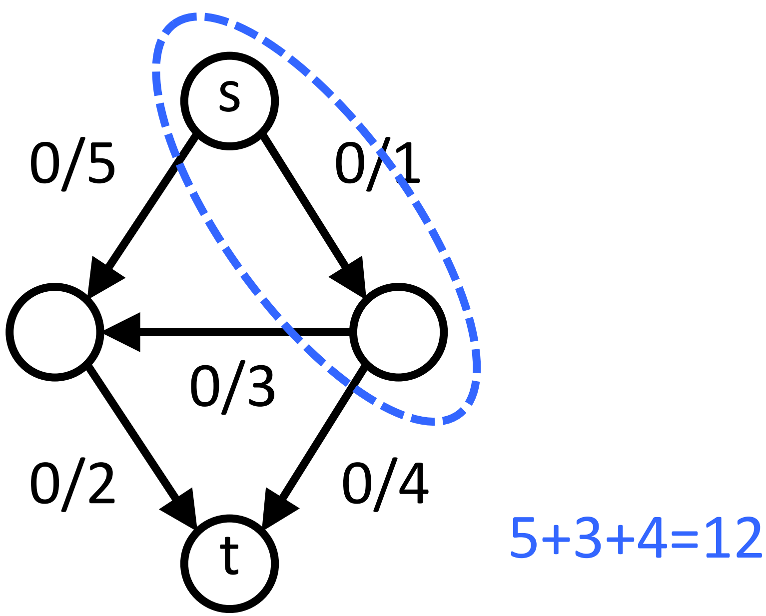
Notation: We write **cut(A,B)** the set of edges from A to B.

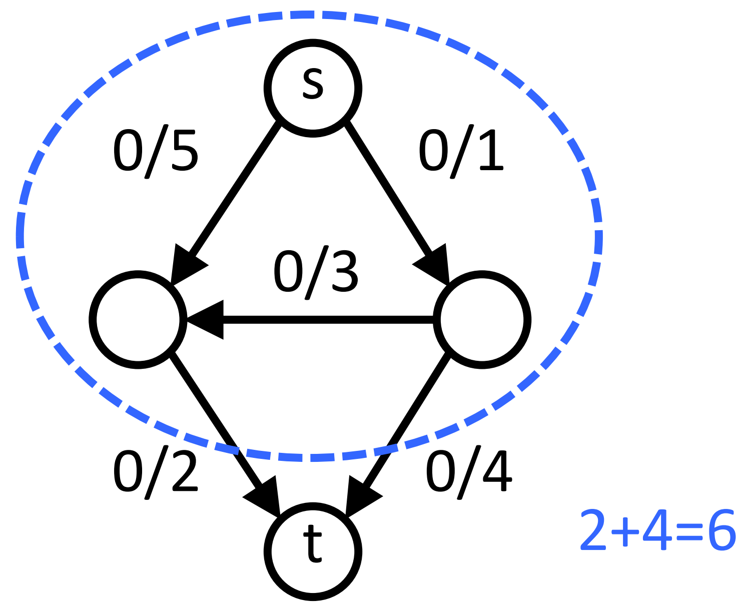
Definition: The capacity of an s-t cut is $\sum_{e \in cut(A,B)} c(e)$

Examples


s
0/5
0/1
0/3
0/2
0/4
t
1+5=6


s
0/5
0/1
0/3
0/2
0/4
t
1+2=3


s
0/5
0/1
0/3
0/2
0/4
t
5+3+4=12


s
0/5
0/1
0/3
0/2
0/4
t
2+4=6

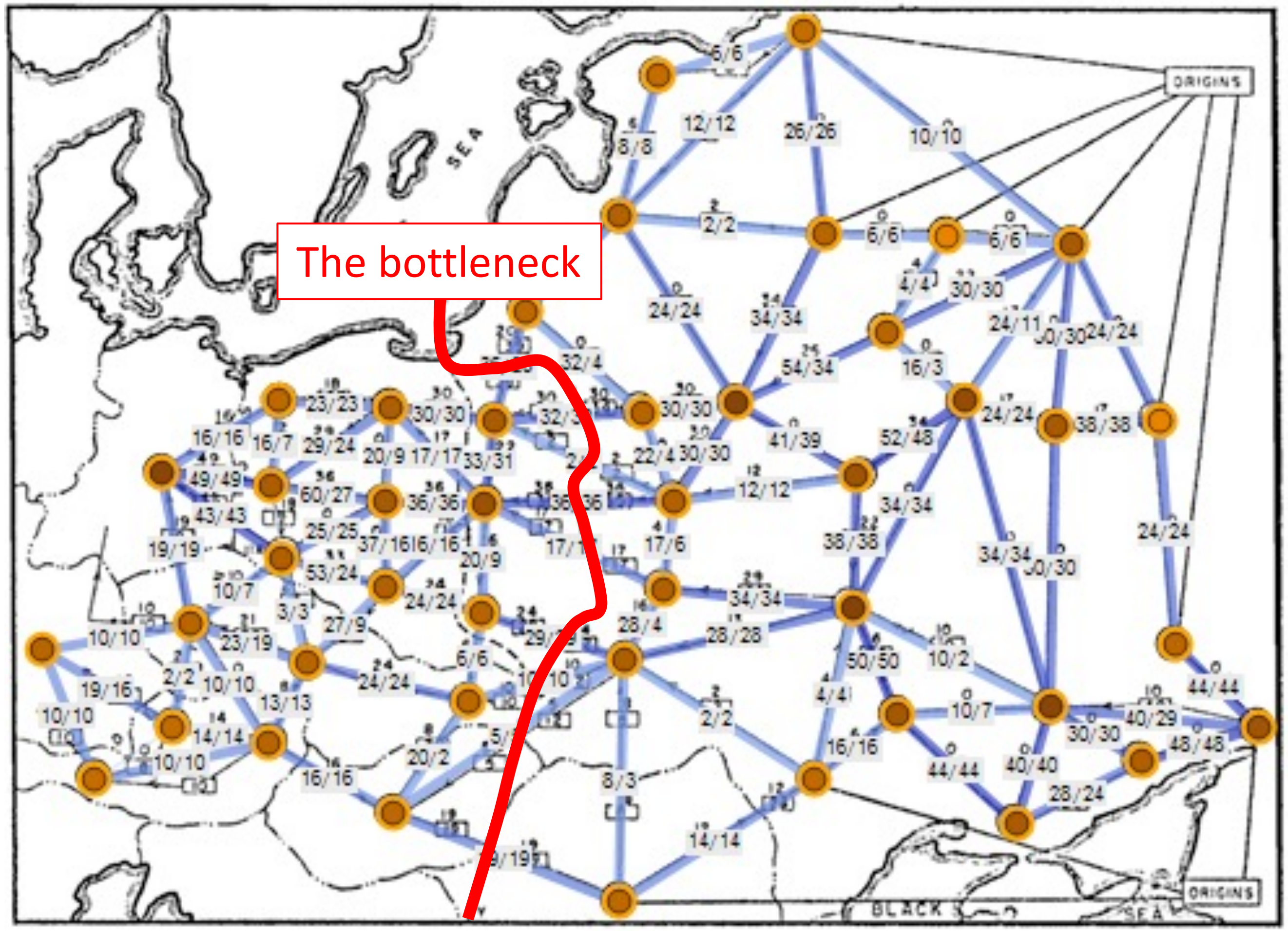
Objectives

For any flow network:

- Maximum value of a flow = the minimum capacity of any cut.
- Ford-Fulkerson gives the “max flow” and the “min cut”.

If a cut has lower capacity than the max flow, then how did the max flow go from one side of that cut to the other? This is what we call the bottleneck effect!

Application: a cold war example



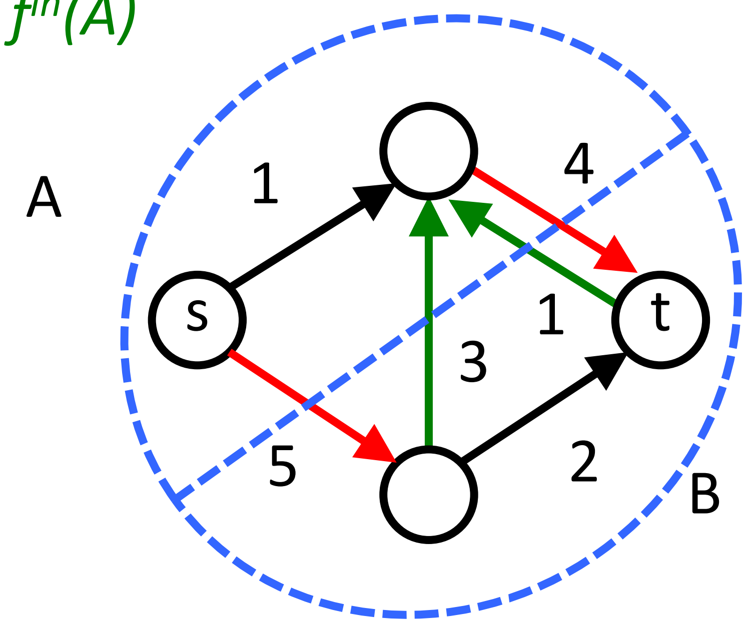
How to cut supplies if cold war turns into real war!

Flow through a cut

Claim: Given a flow network. Let f be a flow and A, B be a s-t cut. Then,

$$|f| = \sum_{e \in cut(A,B)} f(e) - \sum_{e \in cut(B,A)} f(e)$$

Notation: $|f| = f^{out}(A) - f^{in}(A)$



Intuition: remember the net flow through an edge, which is the difference between the two positive flows

$|f| = 9 - 4 = 5$

Flow through a cut

Proof:

- for any $u \in V-\{s,t\}$, we have $f^{out}(u)=f^{in}(u)$.

Flow into a vertex is equal to flow out of a vertex

- Summing over $u \in A\text{-}\{s\}$: $$\sum_{u\in A-\{s\}} f^{out}(u) = \sum_{u\in A-\{s\}} f^{in}(u)$$

Remember A is the left side of a cut

- $$|f| = f^{out}(s) = \sum_{u\in A} f^{out}(u) - \sum_{u\in A} f^{in}(u)$$

The excess is the flow that comes out of s!

s is the source, which has only flow out

- Each edge $e=(u,v)$ with $u,v \in A$ contributes to both sums, and can be removed (Note: f^{in}(s) = 0).

$$|f| = \sum_{e\in cut(A,B)} f(e) - \sum_{e\in cut(B,A)} f(e)$$

$$\equiv f^{out}(A) - f^{in}(A)$$

Upper bound on flow through cuts

Claim: For any network flow *f*, and any s-t cut(A,B)

$$|f| \le \sum_{e \in cut(A,B)} c(e)$$

Proof:

$$\begin{aligned} |f| &= f^{out}(A) - f^{in}(A) \\ &\le f^{out}(A) \\ &\le \sum_{e \in cut(A,B)} c(e) \end{aligned}$$

If a cut has lower capacity than the max flow, then how did the max flow go from one side of that cut to the other? This is what we call the bottleneck effect!

Observations

- Some cuts have greater capacities than others.
- Some flows are greater than others.
- **But every flow must be ≤ capacity of every s-t cut.**
- Thus, the value of the maximum flow is less than capacity of the minimum cut.

If a cut has lower capacity than the max flow, then how did the max flow go from one side of that cut to the other? This is what we call the bottleneck effect!

Value of flow in Ford-Fulkerson

- Ford-Fulkerson terminates when there is no augmenting path in the residual graph G_f.
- Let A be the set of vertices reachable from s in G_f, and B=V-A.
- A,B is a s-t cut in G_f.
- A,B is an s-t cut in G (G and G_f have the same vertices).
- $|f|=f^{out}(A)-f^{in}(A)$
- We want to show: $$\left|f\right| = \sum_{e \in cut(A,B)} c(e)$$
- And in particular:

1 $$f^{out}(A) = \sum_{e \in cut(A,B)} c(e)$$ 2 $$f^{in}(A) = 0$$

If the flow uses 100% of the capacity of the minimum cut, no higher flow is possible

Value of flow in Ford-Fulkerson

$$f^{out}(A) = \sum_{e \in cut(A,B)} c(e)$$

(1) For any e=(u,v) ∈ cut(A,B), f(e)=c(e).

Recall the definition of A!

- $f(e)<c(e) \Rightarrow e=(u,v)$ would be a forward edge in the residual graph G_f with capacity $c_f(e)=c(e)-f(e)>0$.
- v reachable from s in G_f ⇒ contradiction. ■

$$f^{in}(A) = 0$$

(2) $f^{in}(A)=0$: ∀ e=(v,u)∈E such that v∈B, u∈A, we have f(e)=0.

- $f(e)>0 \Rightarrow \exists$ backward edge (u,v) in G_f such that $c_f(e)=f(e)$
- v is reachable from s in G_f ⇒ contradiction. ■

Max flow = Min cut

- Ford-Fulkerson terminates when there is no path s-t in the residual graph G_f
- This defines a cut in A,B in G (A = nodes reachable from s)
- $|f| = f^{out}(A) - f^{in}(A)$ From our first claim.

$$= \sum_{e \in cut(A,B)} f(e) - \sum_{e \in cut(B,A)} f(e)$$

- Ford-Fulkerson flow = $\sum_{e \in cut(A,B)} c(e) - 0$ Use the theorem we just proved.

= capacity of cut(A,B) The full capacity of the cut is met. We cannot have a lower cut otherwise the flow should be lower.

Note: We did not prove uniqueness.

Computing the min cut

Q: Given a flow network, how can we compute a minimum cut?

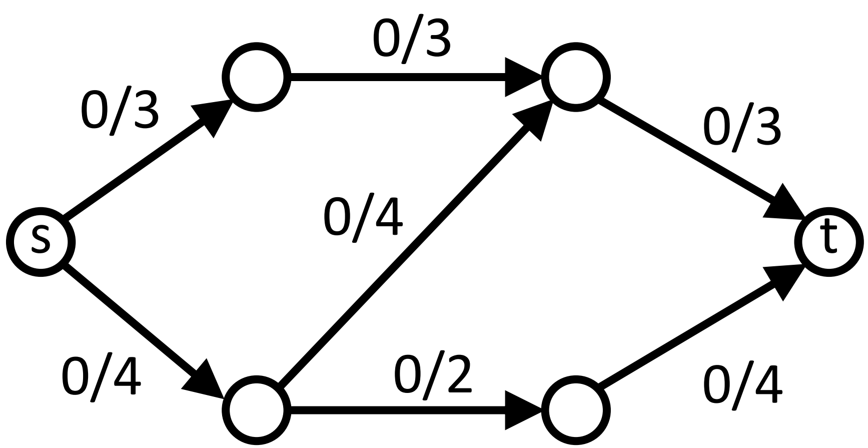
Answer:

- Run Ford-Fulkerson to compute a maximum flow (it gives us G_f)
- Run BFS or DFS starting at s.
- The reachable vertices define the set A for the cut

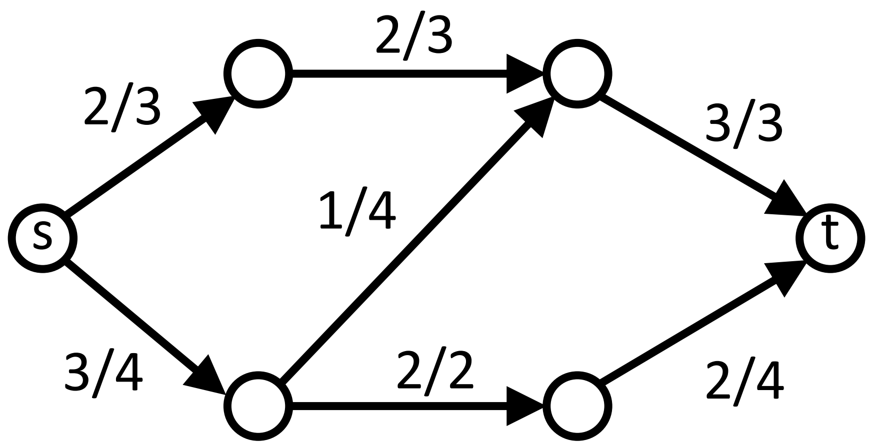
When ford-fulkerson terminates, it means there is no path between s and t on the residual graph. The “bottleneck” that stops a path from being found is the min cut!

Example (min cut with Ford-Fulkerson)

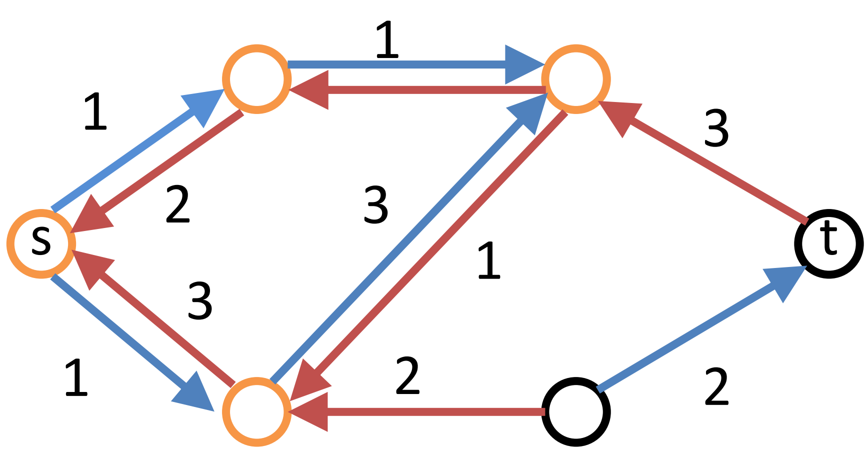
(1) Initial flow net G



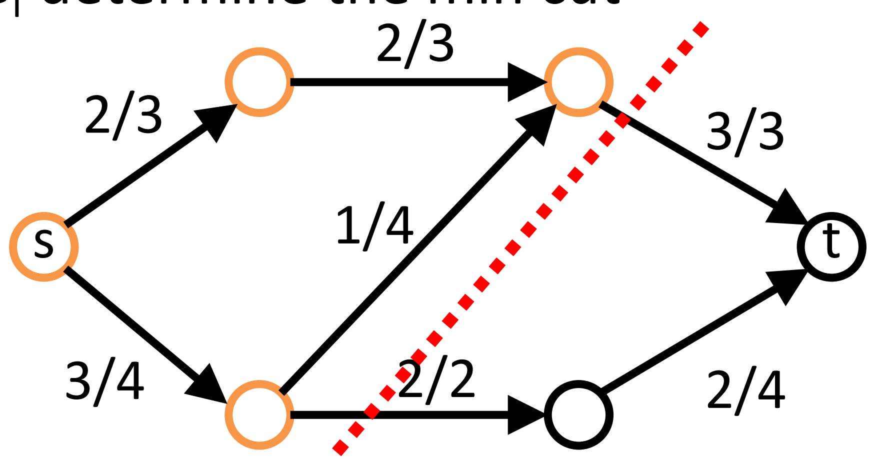
(2) Compute max flow (FF)



(3) Compute G_f and vertices accessible from s

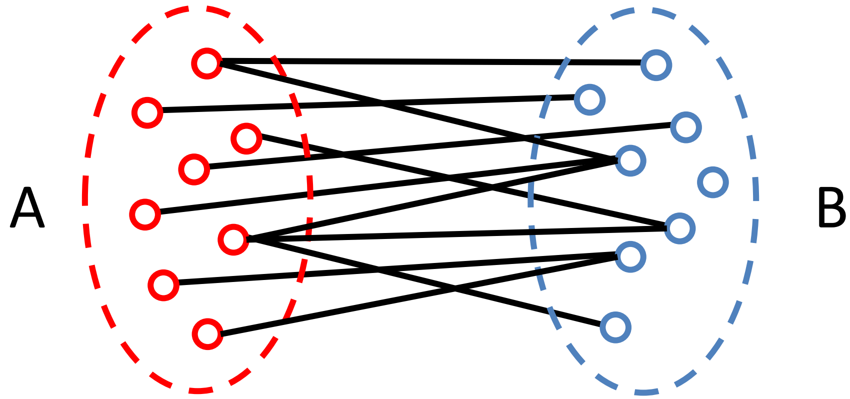


(4) Vertices accessible from s in G_f determine the min cut

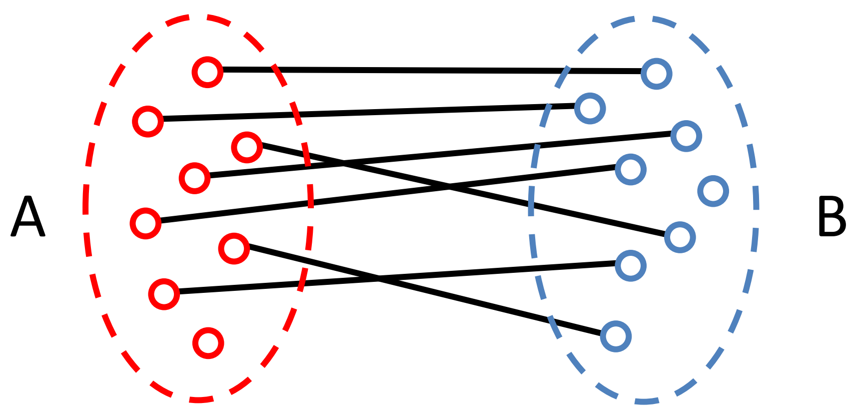


Bipartite matching

Suppose we have an undirected graph bipartite graph G=(V,E).



Q: How can we find the maximal matching?

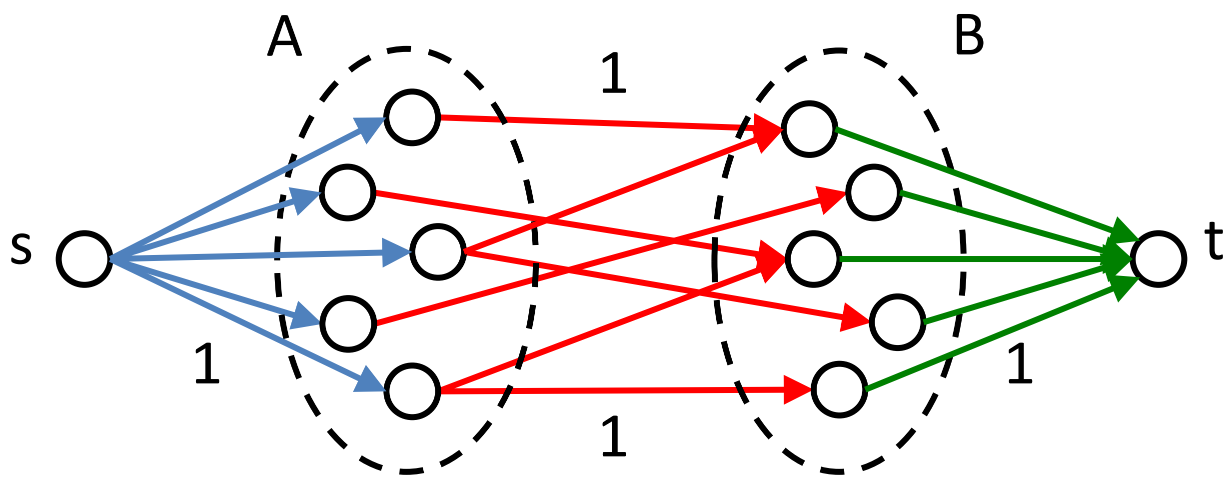


A maximal matching is a matching M of a graph G that is not a subset of any other matching

Bipartite matching with network flows

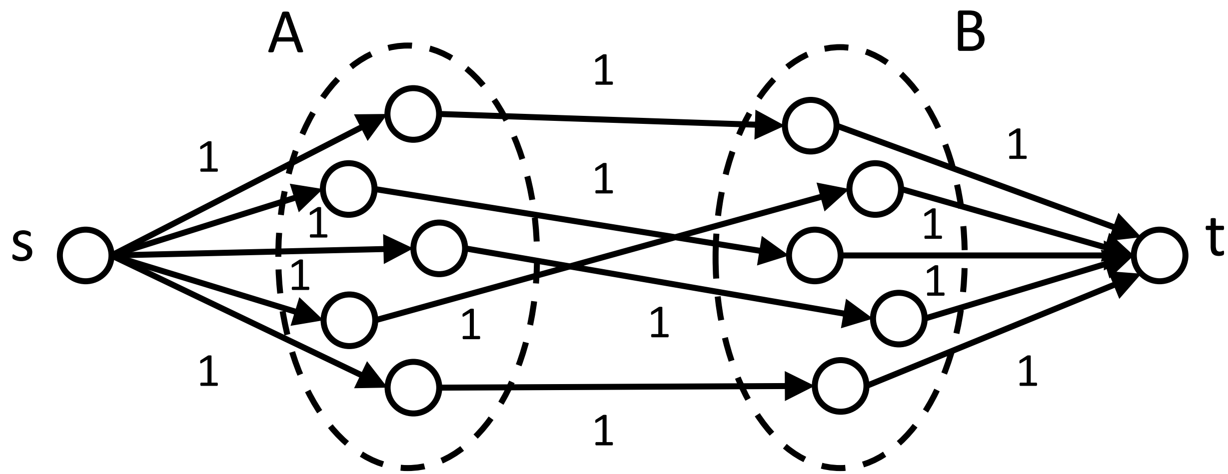
Define a flow network $G'=(V',E')$ such that:

- $V' = V \cup \{s,t\}$
- $E' = \{ (u,v) \mid u \in A, v \in B, (u,v) \in E\} \cup \{ (s,u) \mid u \in A\} \cup \{ (v,t) \mid v \in B\}$
- Capacities of every edge = 1.



Motivation: Max flow ⇒ max matching.

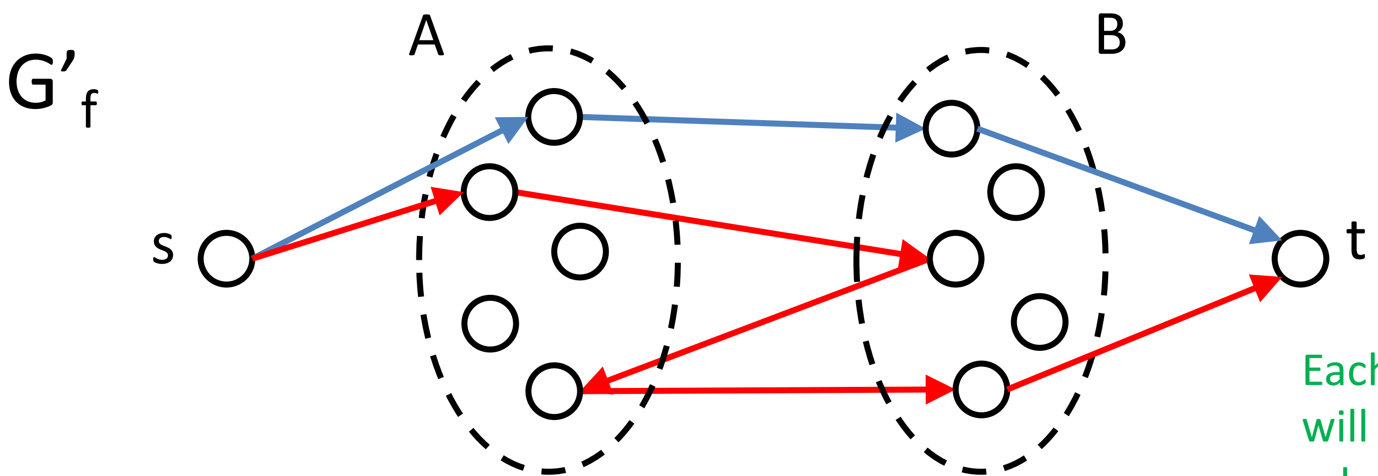
Max flow in bipartite graphs



Exercise: The maximal flow found by Ford-Fulkerson defines a maximal matching in the original graph G (the maximal set of edges (u,v) u∈A & v∈B such that f(u,v)=1).

Max matching with Ford-Fulkerson

Ford-Fulkerson will find an augmenting path with β=1 at each iteration. They are of the form:



Each augmenting path will add one more edge to the match.

Or have more than one zig-zag in G_f

Notes:

- No edge from B to A in E'. The back edges are in the residual graph.
- Edges e such that c(e)=0 are not shown.
- By definition of DFS, an augmenting path cannot visit s twice.

Running time

Q: How long will it take to find a maximal matching with Ford-Fulkerson?

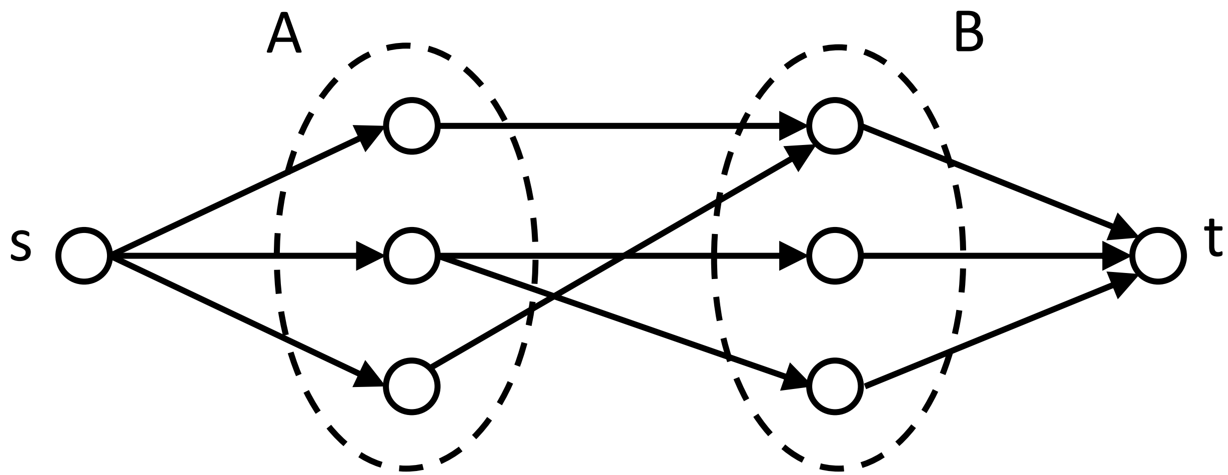
- The general complexity of Ford-Fulkerson is O(C ・|E|), where $$C = \sum_{u} c(s,u)$$
- Suppose |A|=|B|=n
- Then, C=|A|=n and |E'|= |E|+2n = m+2n (Assume m>n)
- Thus, C|E'| = n (m + 2n)
- Running time is O(nm)

m is the number of edges in the initial graph

This is O(nm + 2n^2), so you could also argue is O(n^2), but if you are asked in an exam, you must justify your answer for the one you pick

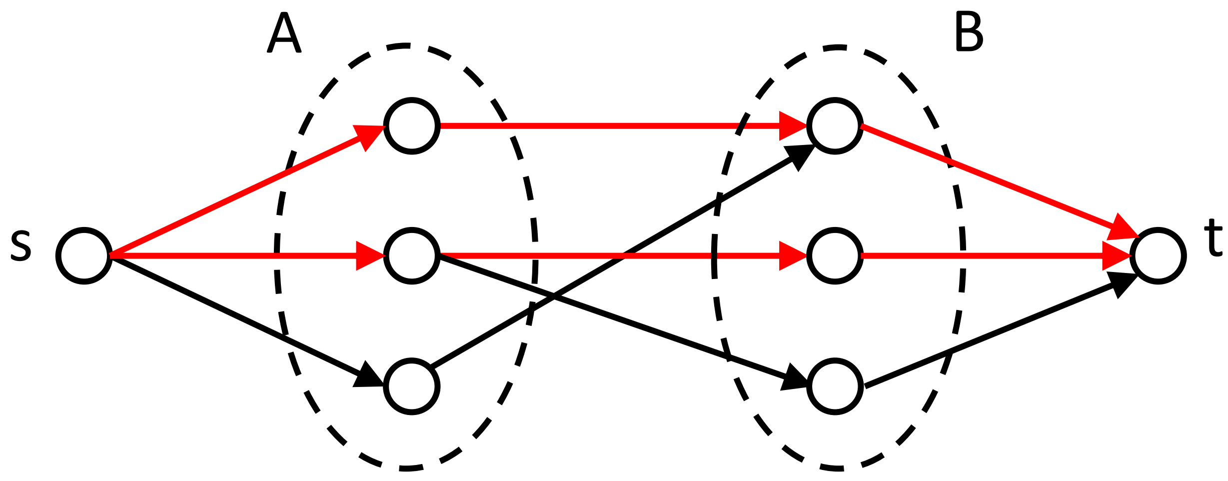
Example

What is max flow? What is min cut?



Example

What is the max flow? What is the min cut?



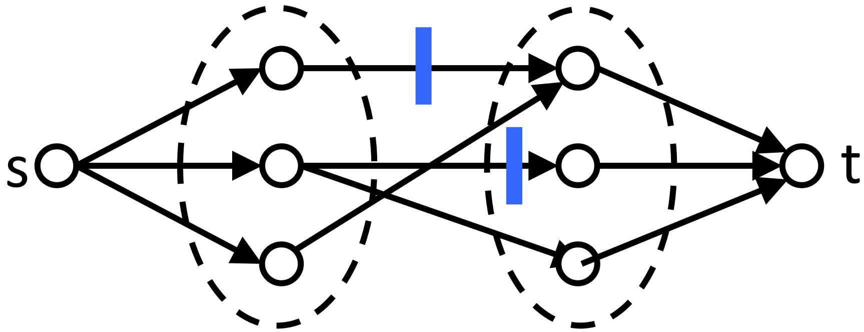
Max flow |f|=2.

Note: there are other flows with |f|=2.

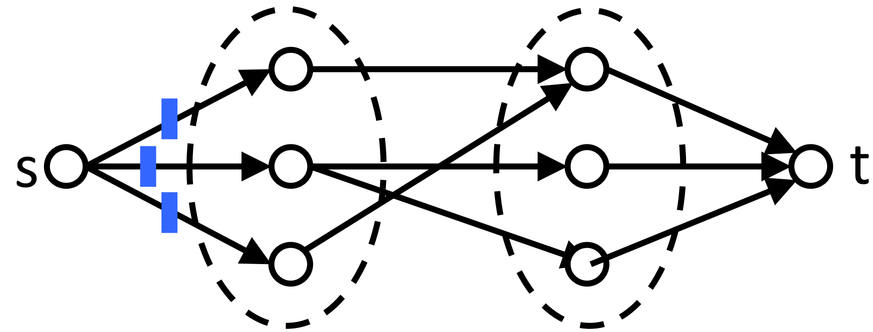
What is the minimum cut?

Example

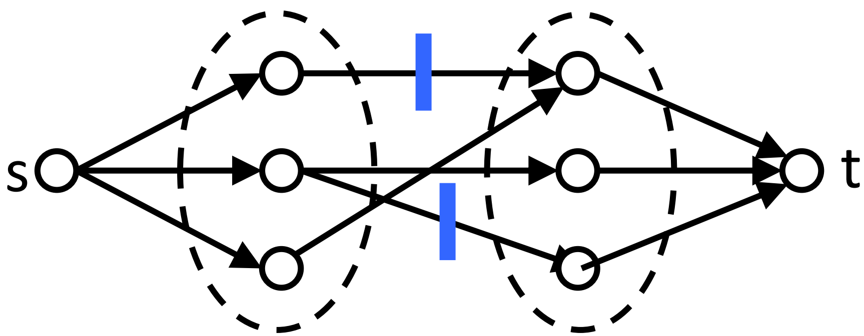
Find any min cut with capacity 2.



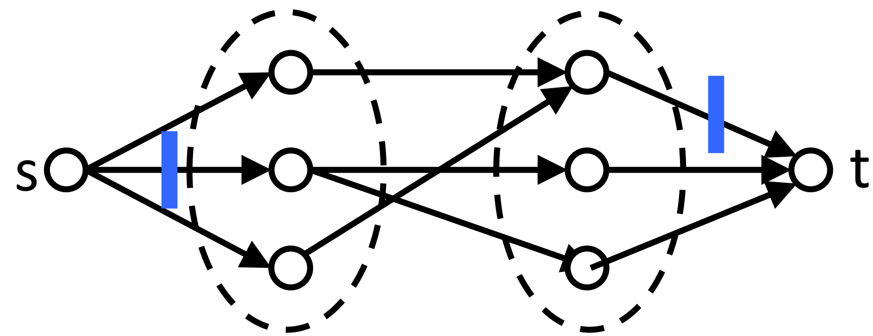
Not a cut!



Not a min cut!



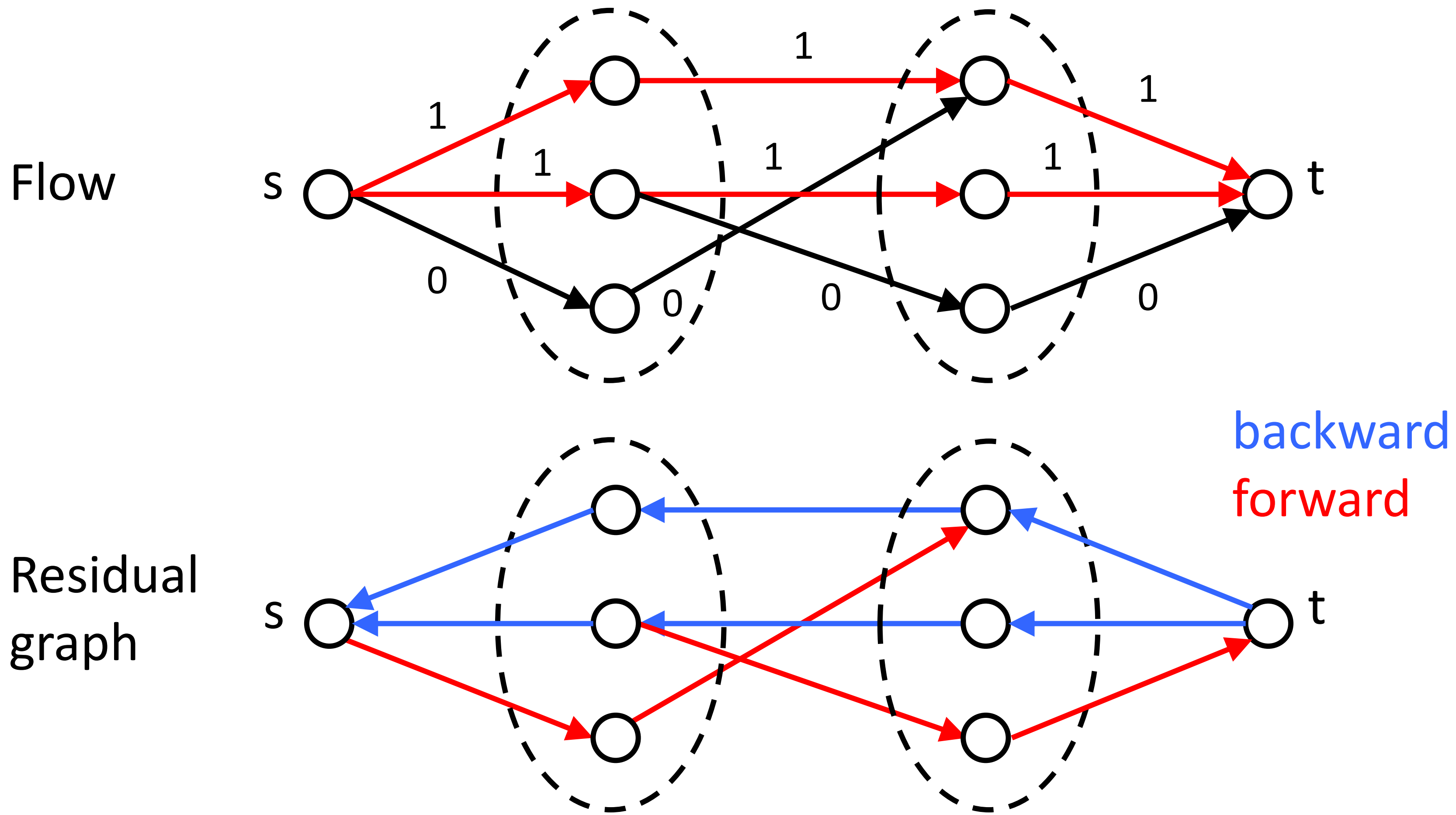
Not a cut!



min cut!

Example

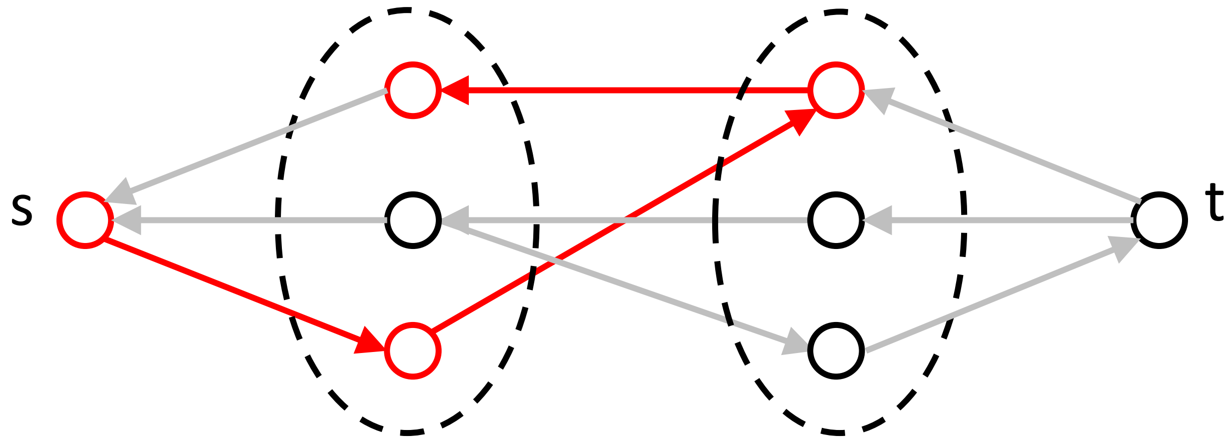
To find a min cut compute a max flow.



Example

To find the cut run BFS (or DFS) from s on the residual graph.
The reachable vertices define the (min) cut.

Residual graph with DFS



Min cut (in G!)