COMP251: Single source shortest paths

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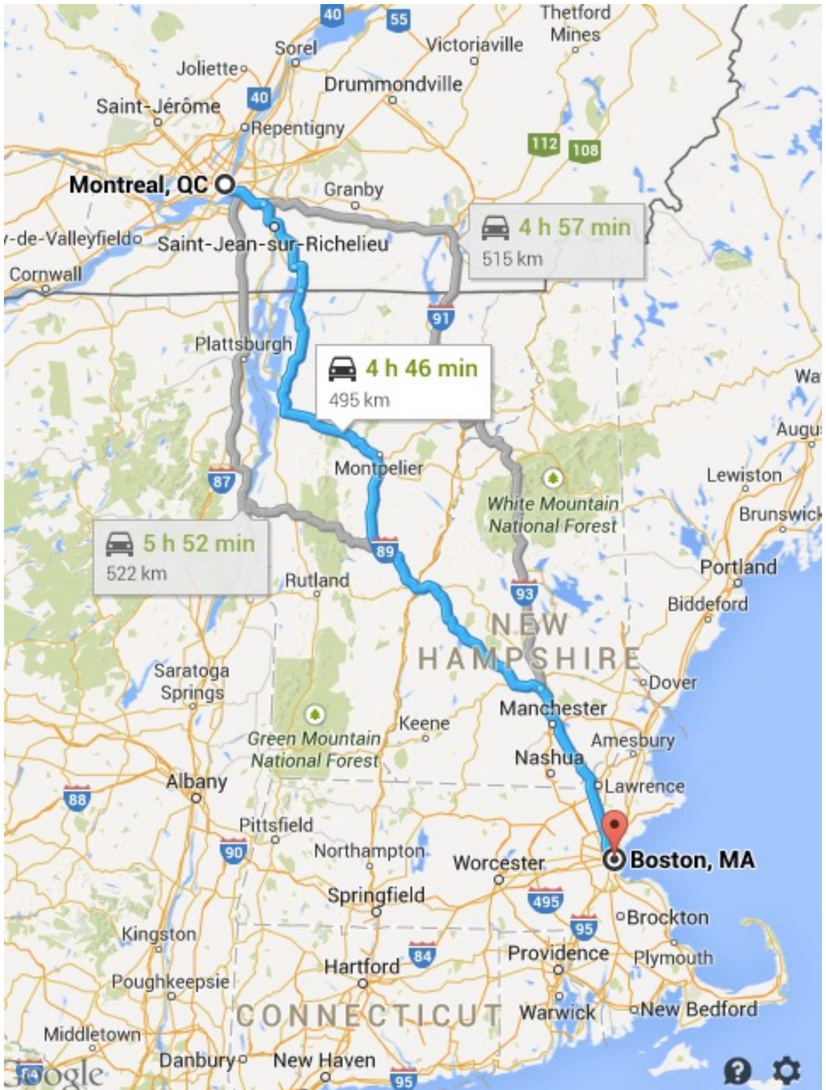
Based on (Cormen *et al.*, 2002)

Outline

- Introduction
- Optimal Substructure
- Definitions & properties
  - Edge relaxation
  - Triangle inequality
  - Upper bound property
  - No-path property
  - Convergence property
  - Path relaxation property
- Single source shortest path in DAGs
- Dijkstra's Algorithm

Problem



What is the shortest road to go from one city to another?

Example: Which road should I take to go from Montréal to Boston (MA)?

Variants:

- What is the fastest road?
- What is the cheapest road?

Modeling as graphs

Input:

- Directed graph $G = (V, E)$
- Weight function w: $E \rightarrow \mathbb{R}$

Weight of path $p = \langle v_0, v_1, \ldots, v_n \rangle$

$$w(p) = \sum_{k=1}^{n} w(v_{k-1}, v_k)$$

= sum of edges weights on path *p*

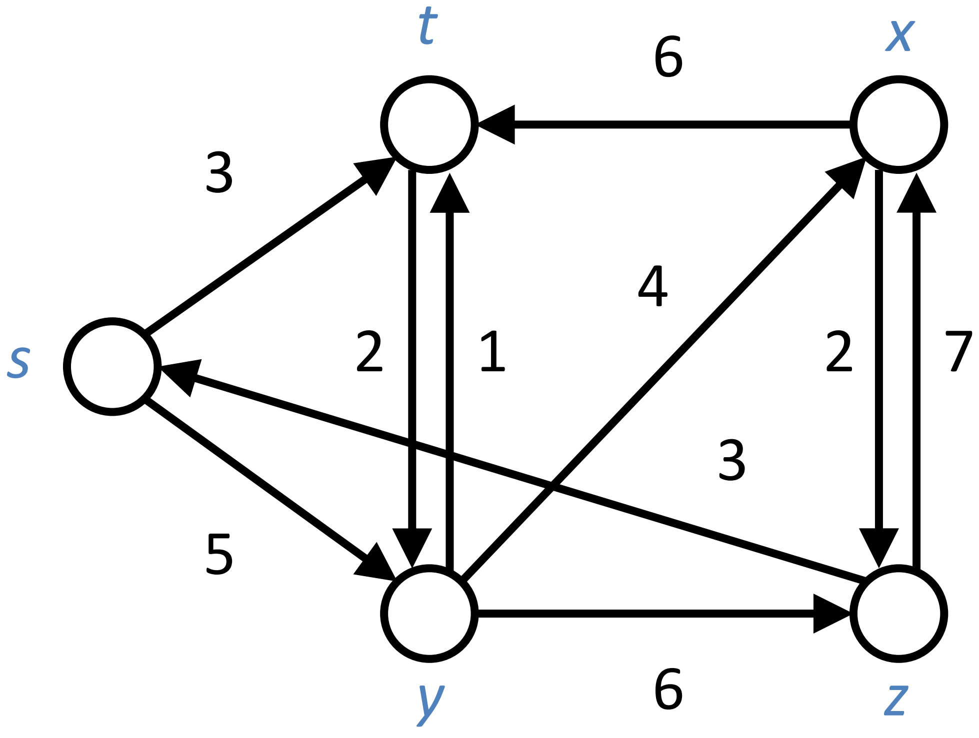
Shortest-path weight u to v:

$$\delta(u,v) = \begin{cases} \min\left\{ w(p) : u \overset{p}{\mapsto} v \right\} & \text{If there exists a path } u \leadsto v. \\ \infty & \text{Otherwise.} \end{cases}$$

Shortest path u to v is any path p such that $w(p) = \delta(u, v)$

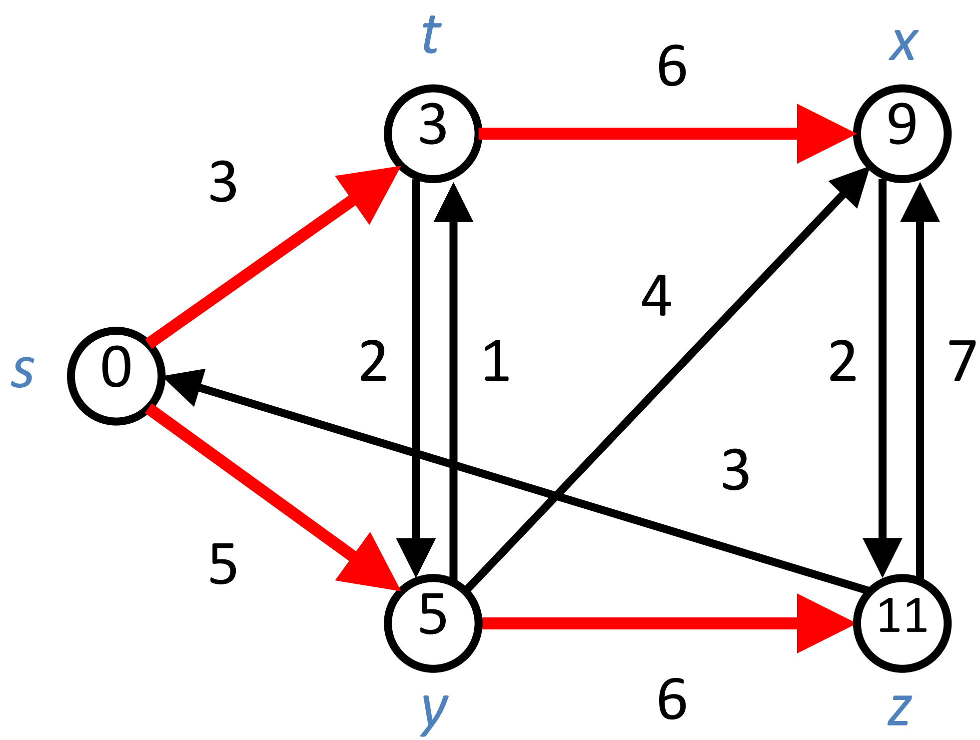
Generalization of breadth-first search to weighted graphs

Example



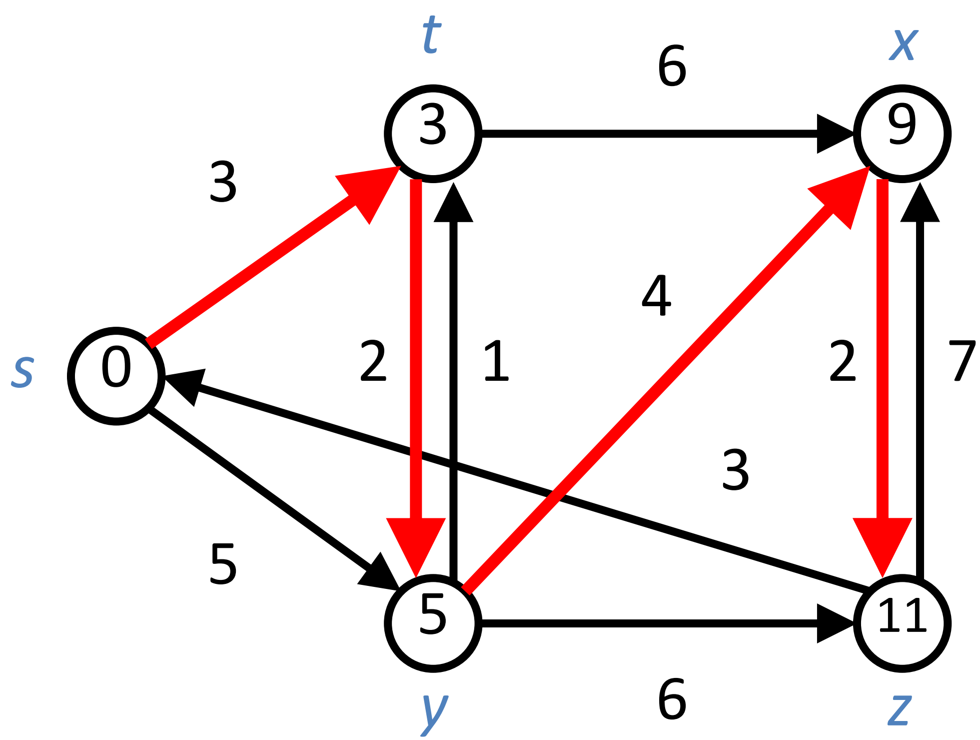
Shortest path from s?

Example



Shortest paths are organized as a tree.
Vertices store the length of the shortest path from s.

Example



Shortest paths are not necessarily unique!

Variants

- **Single-source:** Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.
- **Single-destination:** Find shortest paths to a given destination vertex.
- **Single-pair:** Find shortest path from u to v.

  Note: No way to known that is better in worst case than solving the single-source problem!
- **All-pairs:** Find shortest path from u to v for all $u, v \in V$.

Negative weight edges

Negative weight edges can create issues.

Why? If we have a negative-weight cycle, we can just keep going around it, and get $w(s, v) = -\infty$ for all v on the cycle.

When? If they are reachable from the source (Corollary: It is OK to have a negative-weight cycles if it is not reachable from the source).

What? Some algorithms work only if there are no negative-weight edges in the graph. We must specify when they are allowed and not.

Cycles

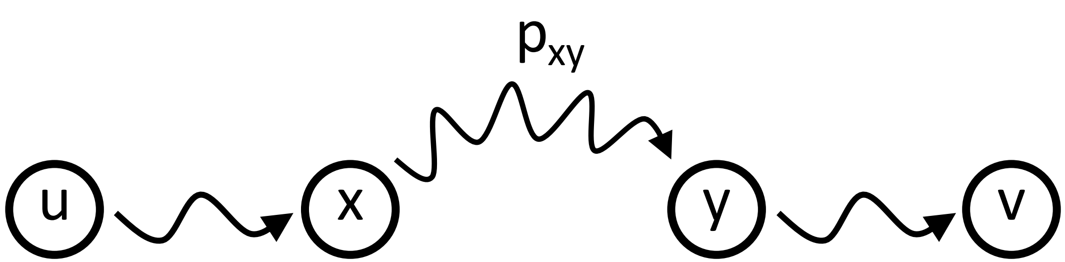
Shortest paths cannot contain cycles:

- Negative-weight: Already ruled out.
- Positive-weight: we can get a shorter path by omitting the cycle.
- Zero-weight: no reason to use them $\Rightarrow$ assume that our solutions will not use them.

Optimal substructure

> ***Lemma***
> Any subpath of a shortest path is a shortest path.

Proof (contradiction using cut and paste approach):



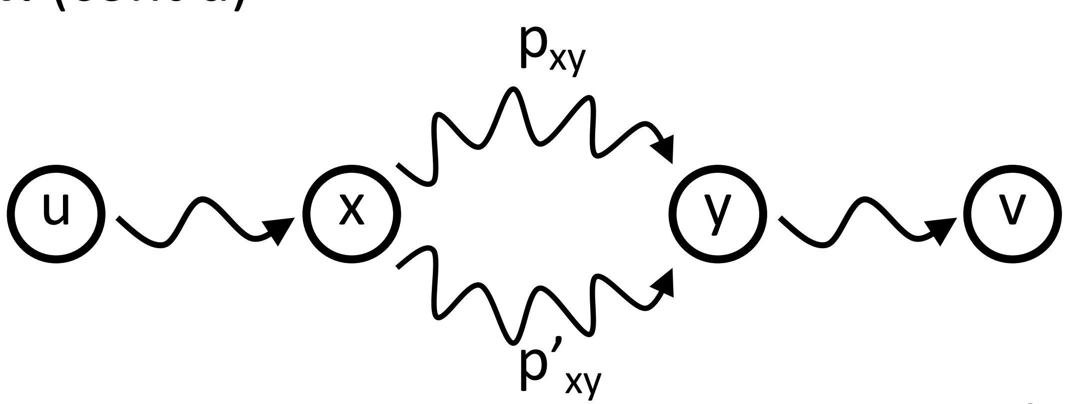
Suppose this path *p* is a shortest path from *u* to *v*.

Then $\delta(u, v) = w(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv})$.

Optimal substructure

> ***Lemma***
> Any subpath of a shortest path is a shortest path.

Proof: (cont’d)



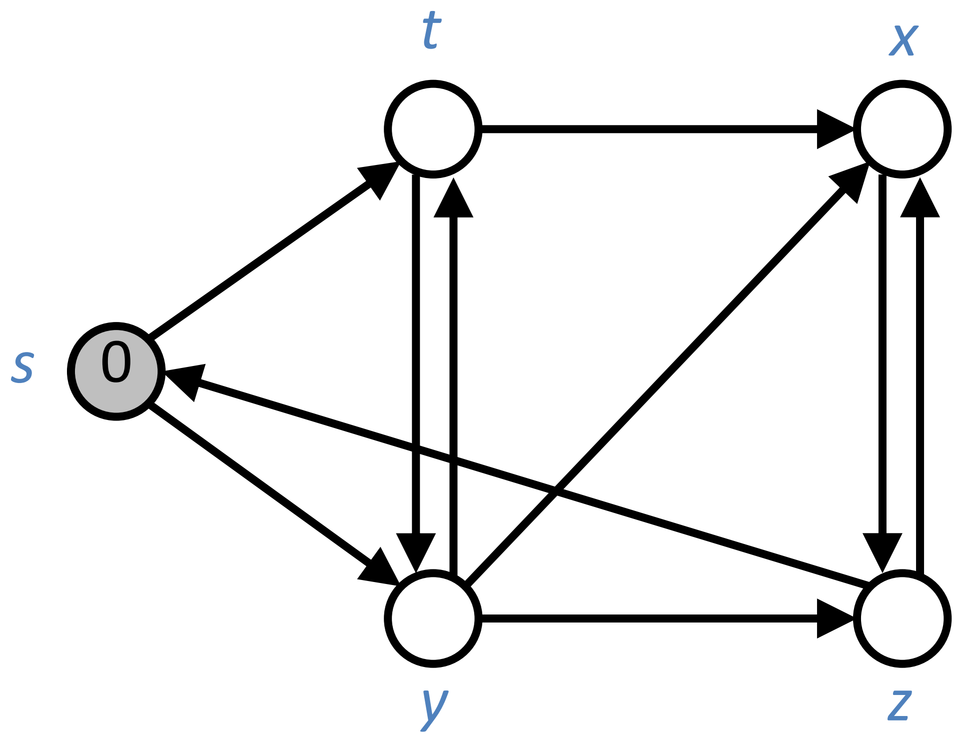
Now suppose there exists a shorter path $x \overset{p'_{xy}}{\rightsquigarrow} y : w(p'_{xy}) < w(p_{xy})$.

$$w(p') \;=\; w(p_{ux}) \;+\; w(p'_{xy}) \;+\; w(p_{yv})$$

$$\quad\quad < \; w(p_{ux}) \;+\; w(p_{xy}) \;+\; w(p_{yv}) \;=\; w(p)$$

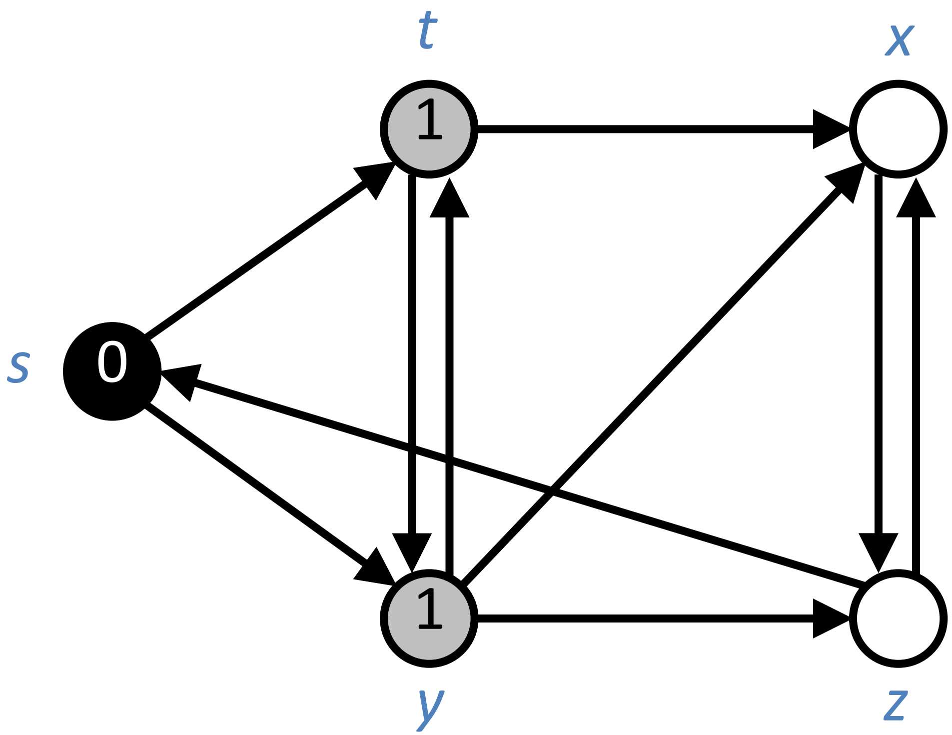
Contradiction of the hypothesis that p is the shortest path!

Customized breadth-first search

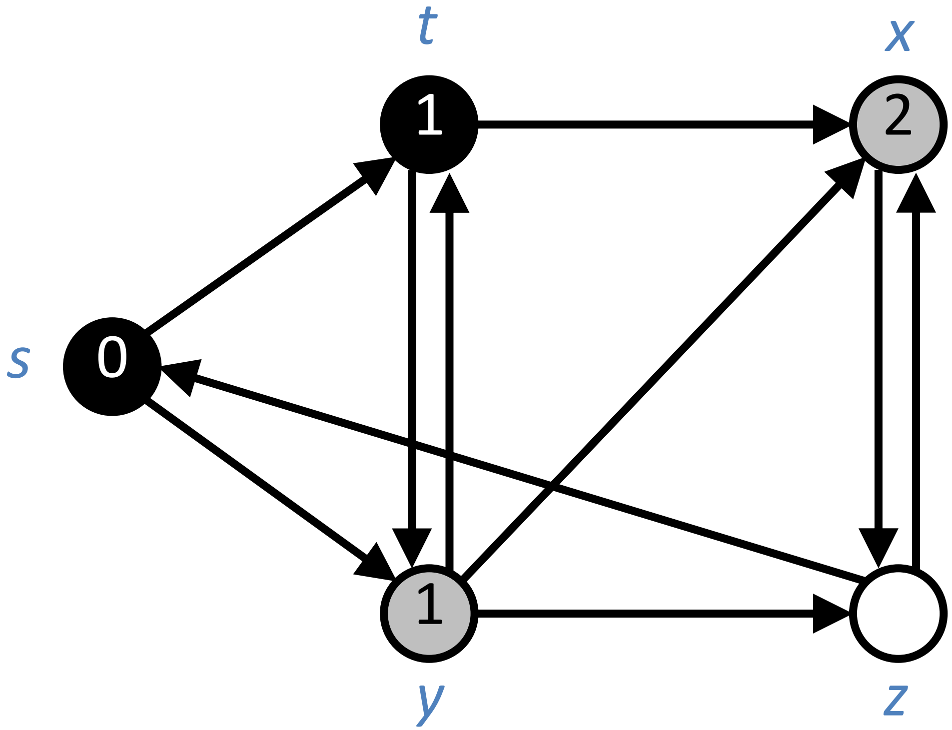


Vertices count the number of edges used to reach them.

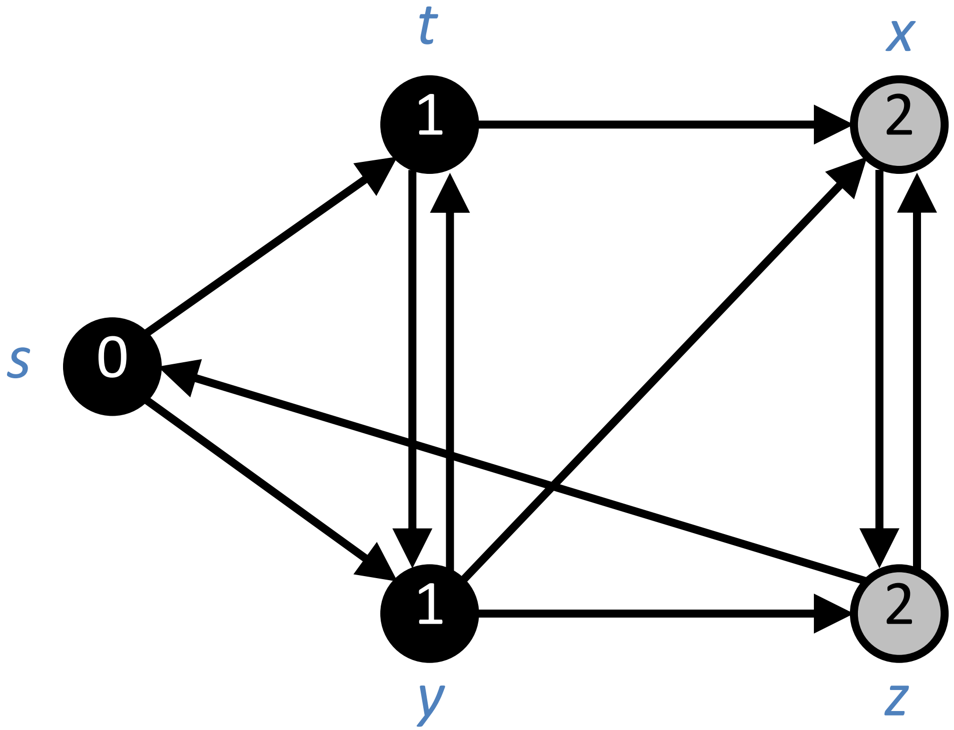
Customized breadth-first search



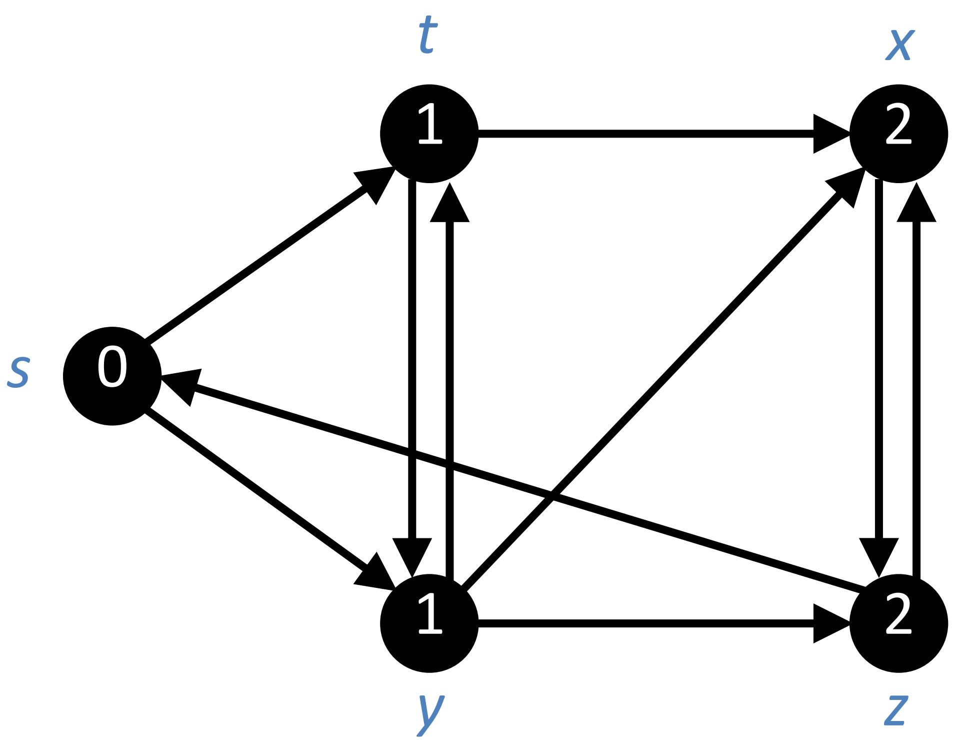
Customized breadth-first search



Customized breadth-first search



Customized breadth-first search



Can we generalize BFS to use edge weights?

Principle of a single-source shortest-path algorithm

For each vertex $v \in V$:

- $d[v] = \delta(s, v)$.
  - Initially, $d[v] = \infty$.
  - Reduces as algorithms progress, but always maintain $d[v] \geq \delta(s, v)$.
  - Call $d[v]$ a **shortest-path estimate**.
- $\pi[v]$ = predecessor of v on a shortest path from s.
  - If no predecessor, $\pi[v] = NIL$.
  - π induces a tree - **shortest-path tree** (see proof in textbook).

$\delta(s, v)$ is the absolute shortest path
$d[v]$ is our current estimate of the shortest path

Generic algorithm structure

1. Initialization
2. Scan vertices and relax edges

The algorithms differ in the order and how many times they relax each edge.

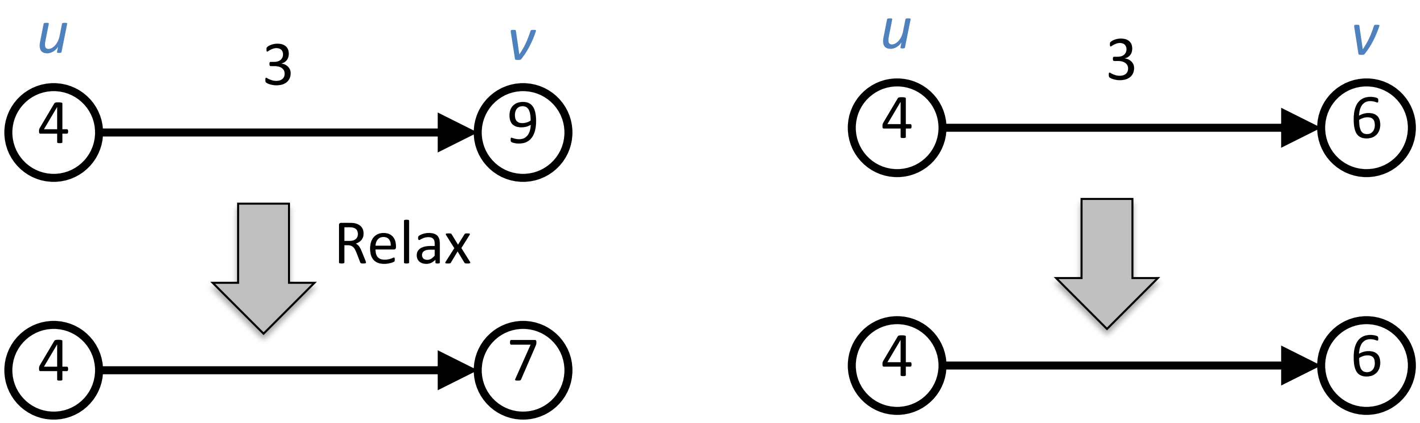
Initialization

```
INIT-SINGLE-SOURCE(V,s)
  for each v ∈ V do
     d[v]← ∞
     π[v]← NIL
  d[s]←0
```

Relaxing an edge

This is used to reduce $d[v]$ during the execution of the algorithm.

```
RELAX(u,v,w)
  if d[v]>d[u]+w(u,v) then
    d[v] ← d[u]+w(u,v)
    π[v] ← u
```



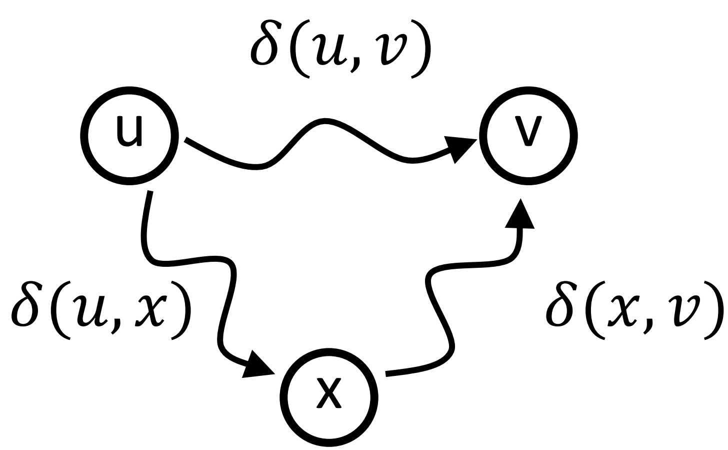
Triangle inequality

For all $(u, v) \in E$, we have $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$.

Proof:

Weight of shortest path $u \leadsto v$ is ≤ weight of any path $u \leadsto v$.

Path $u \leadsto x \leadsto v$ is a path $u \leadsto v$, and if we use a shortest path $u \leadsto x$ *and* $x \leadsto v$, its weight is $\delta(u, x) + \delta(x, v)$.



Upper bound property

Always have $\delta(s, v) \le d[v]$ for all v.
Once $d[v] = \delta(s, v)$, it never changes.

Proof:

WLOG = Without Loss Of Generality

- Initially true.
- Then, assume it exists a vertex v such that $d[v] < \delta(s, v)$.
  WLOG, v is the first vertex for which this happens.
  Let u be the vertex that causes $d[v]$ to change.
  Then, after relaxation $d[v] = d[u] + \delta(u, v)$. But we also have:

$$d[v] < \underbrace{\delta(s, v) \le \delta(s, u) + \delta(u, v)}_{\text{(triangle inequality)}} \le d[u] + \delta(u, v)$$

(v is first violation, thus inequality is valid for u)

$\Rightarrow d[v] < d[u] + \delta(u, v)$.

Note the strict inequality!

Contradicts $d[v] = d[u] + \delta(u, v)$.

No-path property

$\boxed{\text{If } \delta(s,v) = \infty\text{, then } d[v] = \infty \text{ always.}}$

Proof: $d[v] \geq \delta(s,v) = \infty \Rightarrow d[v] = \infty$.

Convergence property

We need all 3 conditions!

In 1: (u, v) is the last edge on the shortest path

If:

1. $s \rightsquigarrow u \rightarrow$ v is a shortest path,
2. $d[u] = \delta(s, u)$,
3. we call $RELAX(u, v, w)$,

then $d[v] = \delta(s, v)$ afterward.

In other words, after RELAX $d[v]$ is guaranteed to be the shortest path value.

Proof:

After relaxation:

$d[v] \le d[u] + w(u, v)$ (RELAX code)

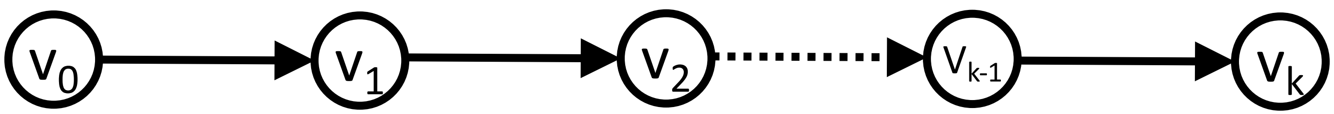
$= \delta(s, u) + w(u, v)$ ($d[u] = \delta(s, u)$ by 2.)

$= \delta(s, v)$ ($s \rightsquigarrow u \rightarrow$ v is a shortest path & lemma sub-optimal structure)

Since $d[v] \ge \delta(s, v)$, must have $d[v] = \delta(s, v)$.

Path-relaxation property

Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from $s = v_0$ to v_k. If we relax, *in order*, (v_0, v_1), (v_1, v_2),..., (v_{k-1}, v_k), even intermixed with other relaxations, then $d[v_k] = \delta(s, v_k)$.



Proof:

Induction to show that $d[v_i] = \delta(s, v_i)$ after (v_{i-1}, v_i) is relaxed.

Base Case: $i = 0$. Initially, $d[v_0] = 0 = \delta(s, v_0) = \delta(s, s)$.

Inductive step: Assume $d[v_{i-1}] = \delta(s, v_{i-1})$. Relax (v_{i-1}, vi). By *convergence property*, $d[v_i] = \delta(s, v_i)$ afterward and $d[v_i]$ never changes.

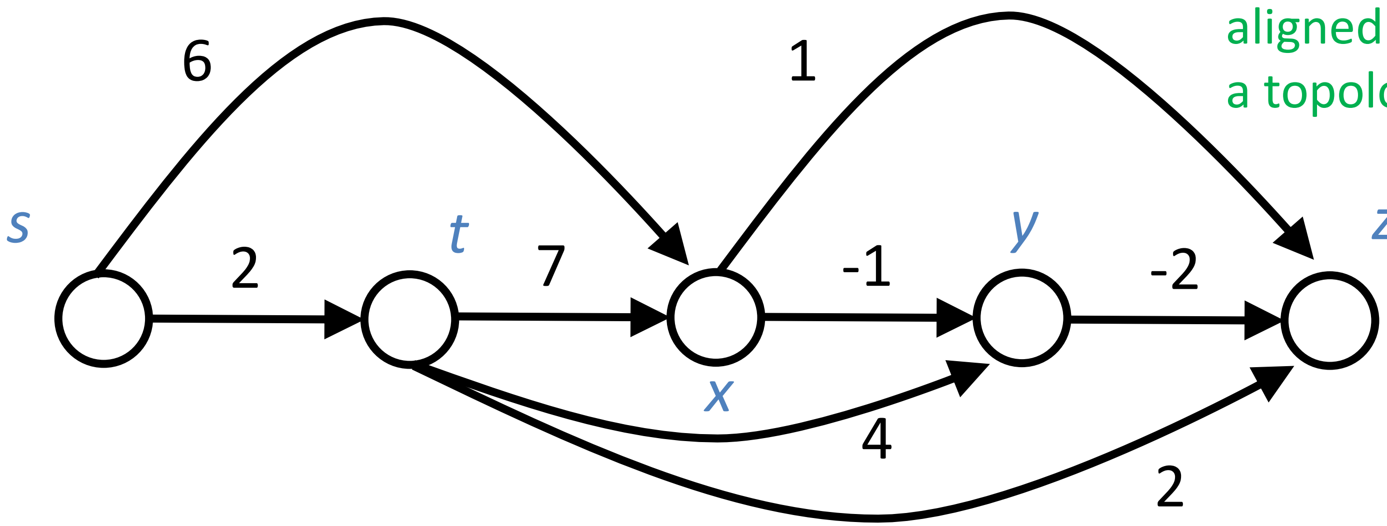
Single-source shortest paths in a DAG

DAG ⇒ no negative-weight cycles.

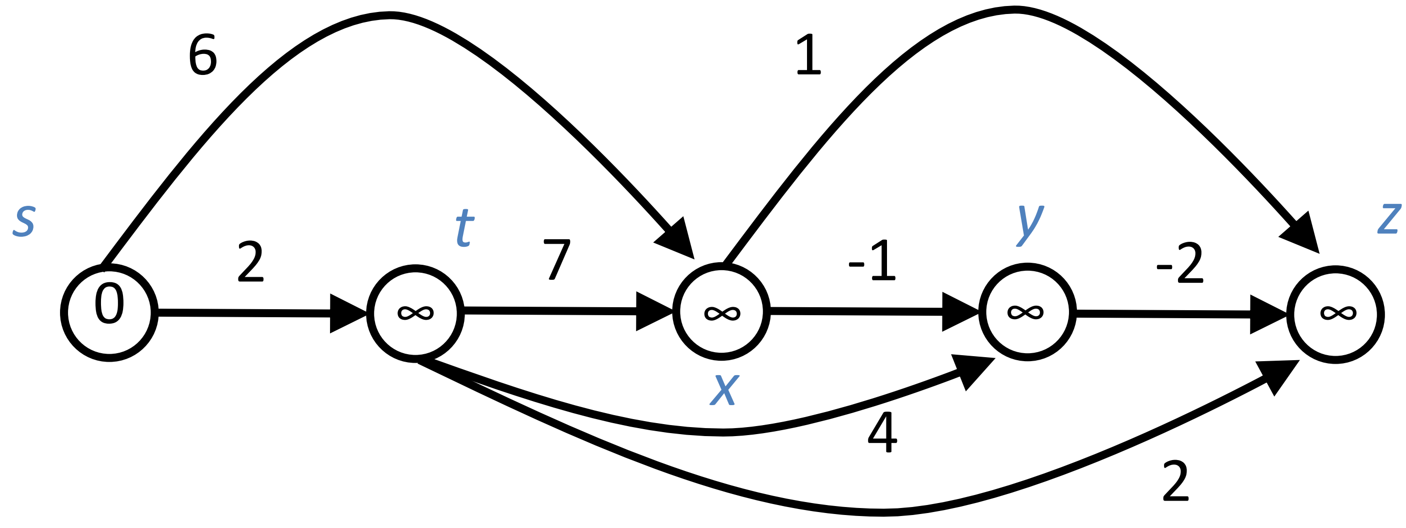
Note: the edges can have any weight you want.

```
DAG-SHORTEST-PATHS(V,E,w,s)
topologically sort the vertices
INIT-SINGLE-SOURCE(V,s)
for each vertex u in topological order do
   for each vertex v ∈ Adj[u] do
      RELAX(u,v,w)
```

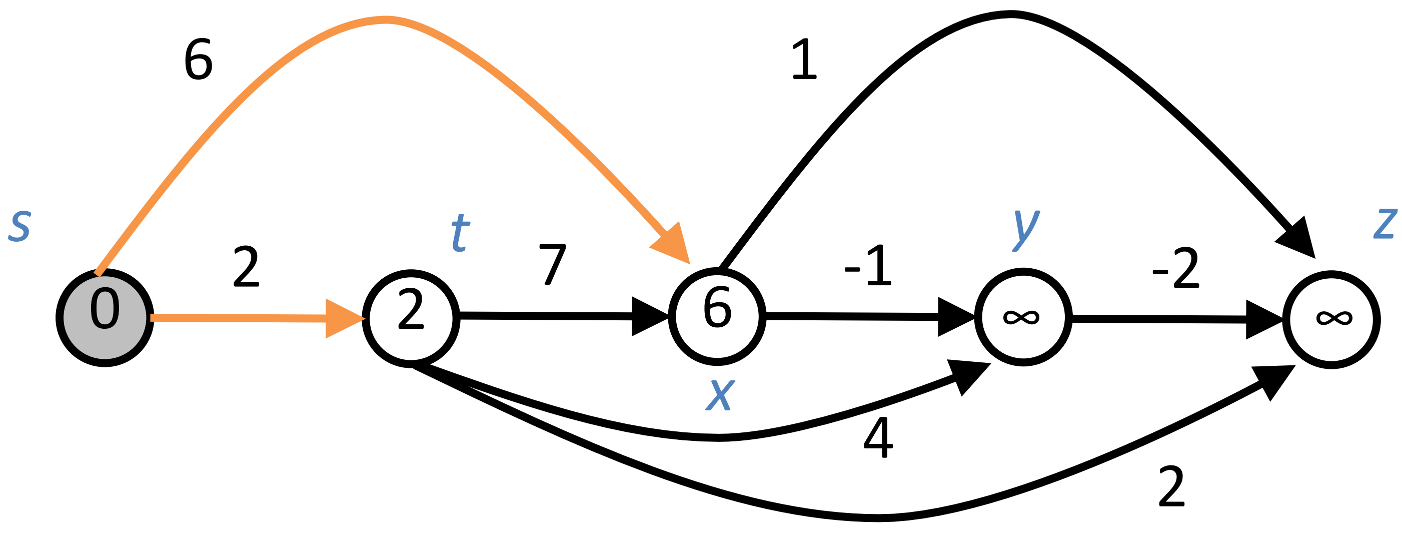
The vertices are aligned according to a topological order.



Example



Example



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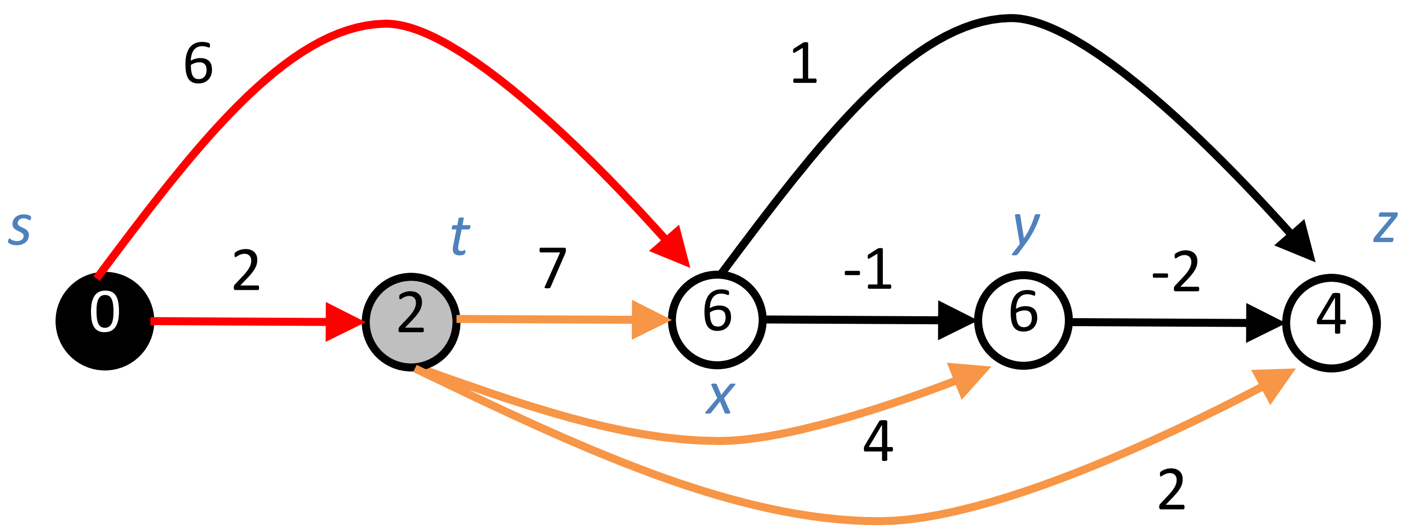
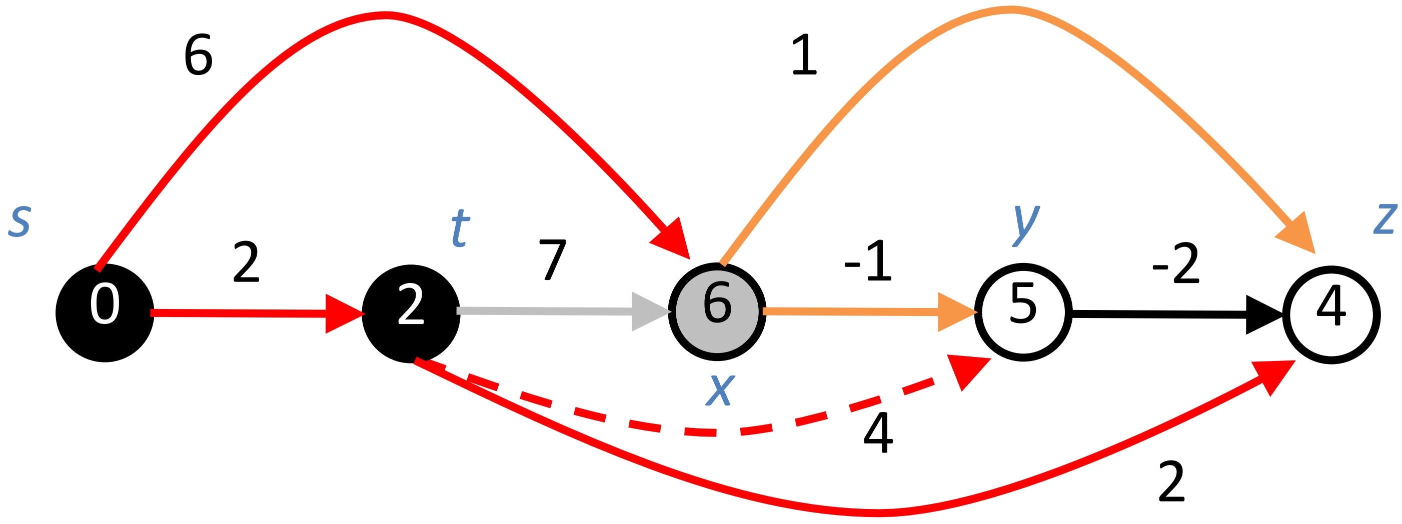


Example



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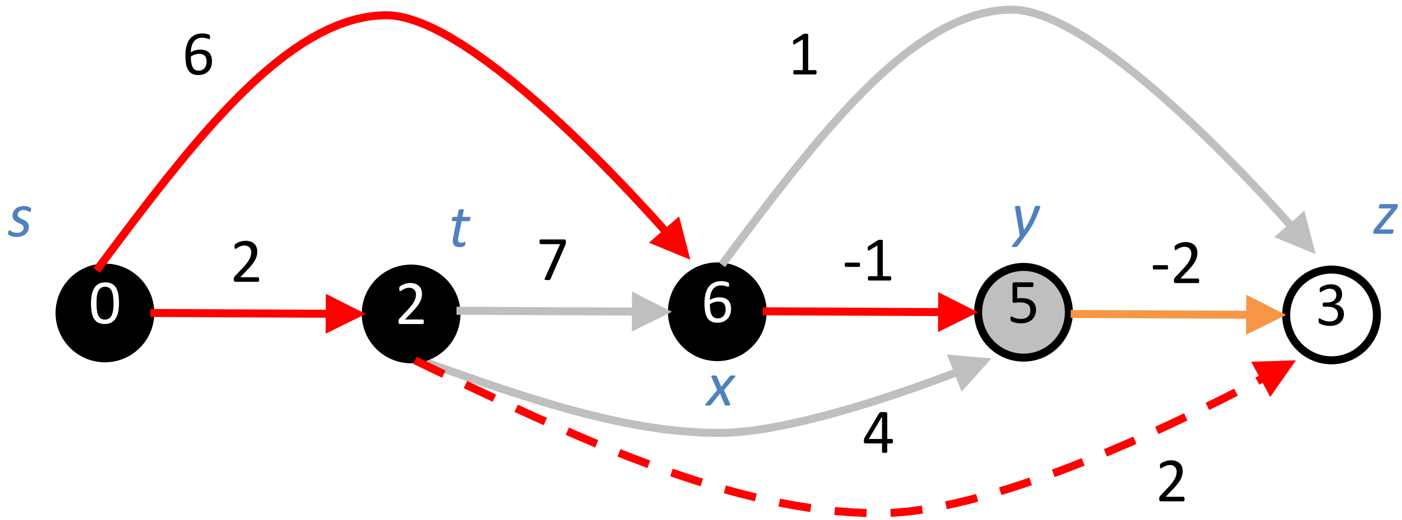
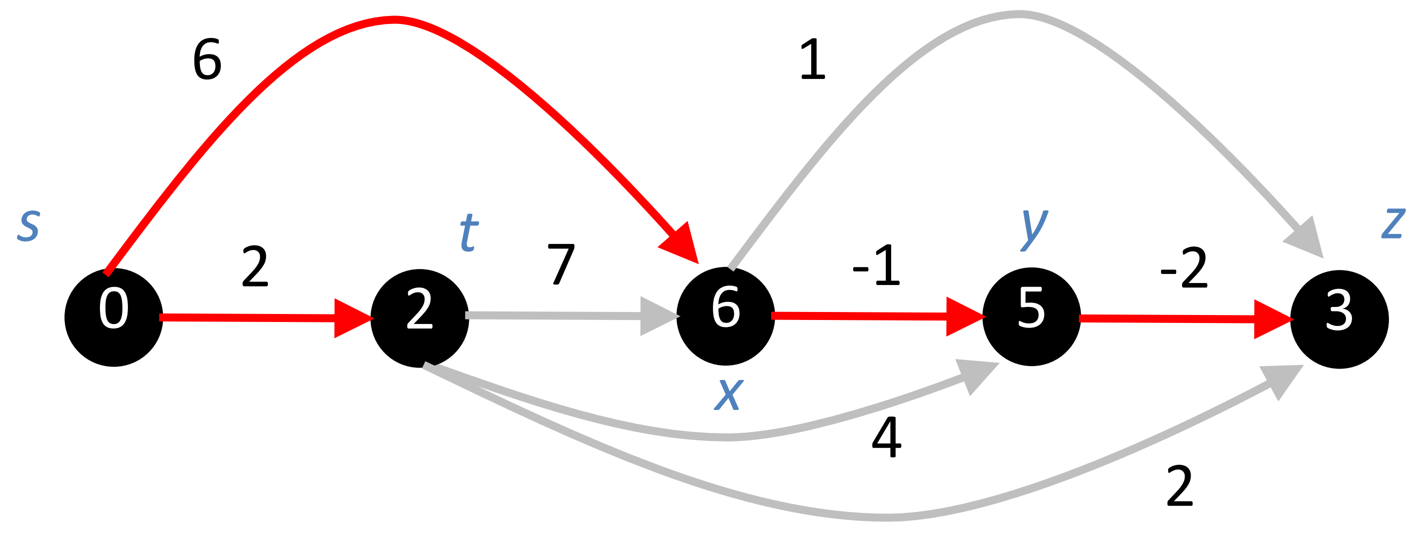


Example



Single-source shortest paths in a DAG

```
DAG-SHORTEST-PATHS(V,E,w,s)
topologically sort the vertices
INIT-SINGLE-SOURCE(V,s)
for each vertex u in topological order do
   for each vertex v ∈ Adj[u] do
      RELAX(u,v,w)
```

Time: $O(V + E)$.

No edge leads you to a vertex already processed.

Correctness:

Because we process vertices in topologically sorted order, edges of **any** path must be relaxed in order of appearance in the path.

⇒ Edges on any shortest path are relaxed in order.

⇒ By the path-relaxation property, the result is correct.

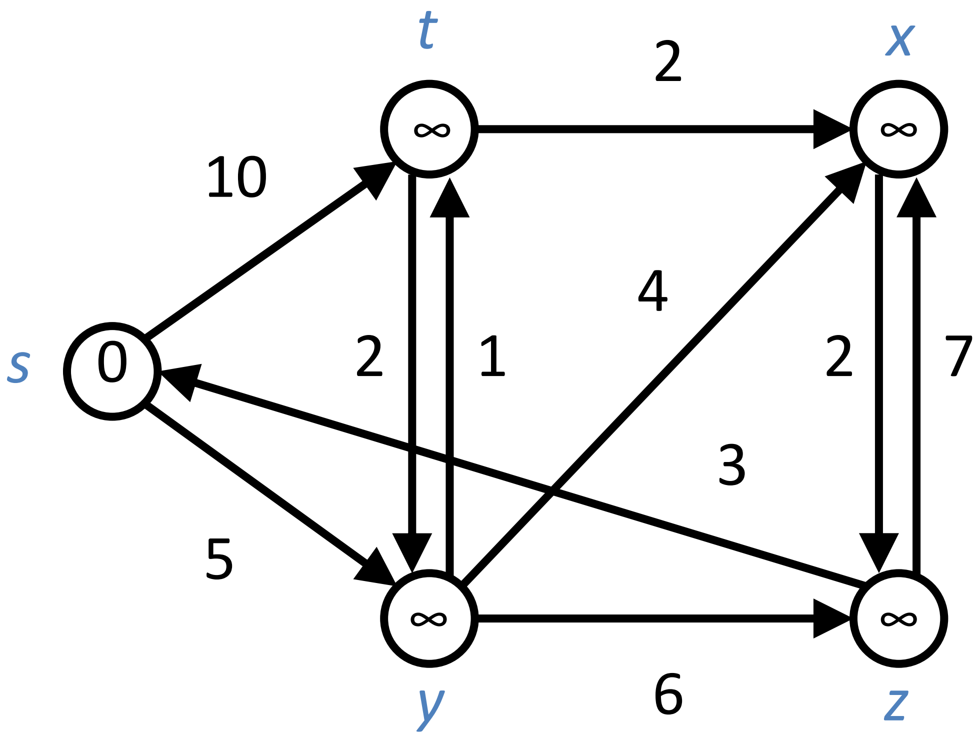
Dijkstra's algorithm

- No negative-weight edges.
- Weighted version of BFS:
  - **Instead of a FIFO queue, uses a priority queue**.
  - Keys are shortest-path weights ($d[v]$).
- Have two sets of vertices:
  - S = vertices whose final shortest-path weights are determined,
  - Q = priority queue = $V \setminus S$.
- Similar Prim's algorithm, but computing $d[v]$, and using shortest-path weights as keys.
- Greedy choice: At each step we choose the light edge.

Dijkstra's algorithm

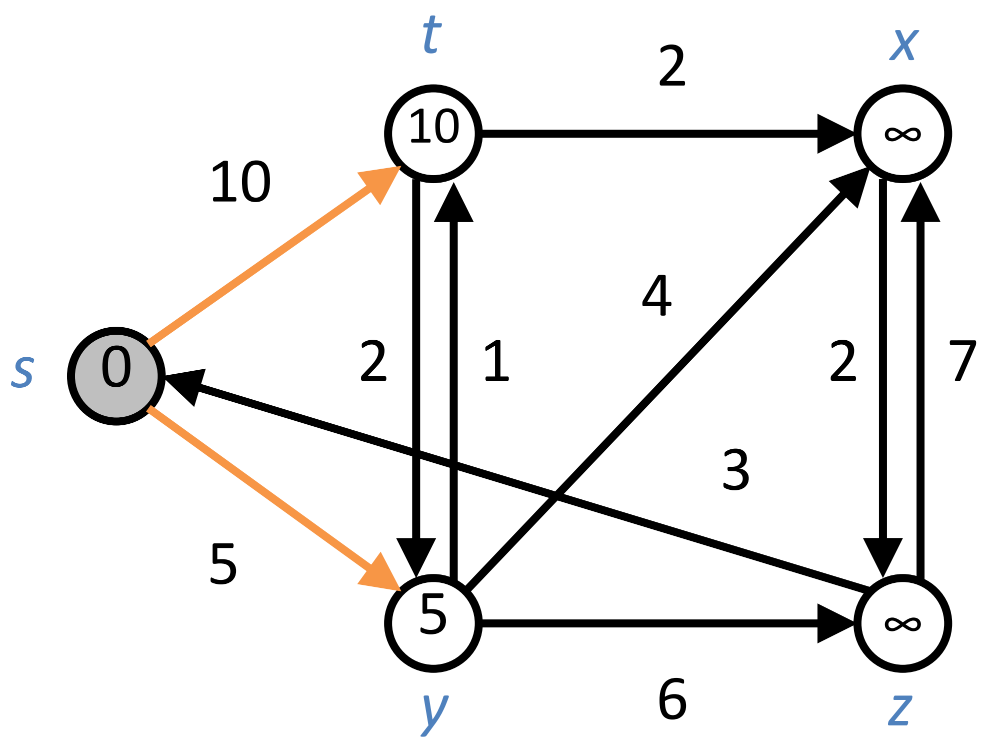
```
DIJKSTRA(V, E,w,s)
INIT-SINGLE-SOURCE(V,s)
S ← ∅
Q ← V
while Q ≠ ∅ do
   u ← EXTRACT-MIN(Q)
   S ← S ∪ {u}
   for each vertex v ∈ Adj[u] do
      RELAX(u,v,w)
```

Example



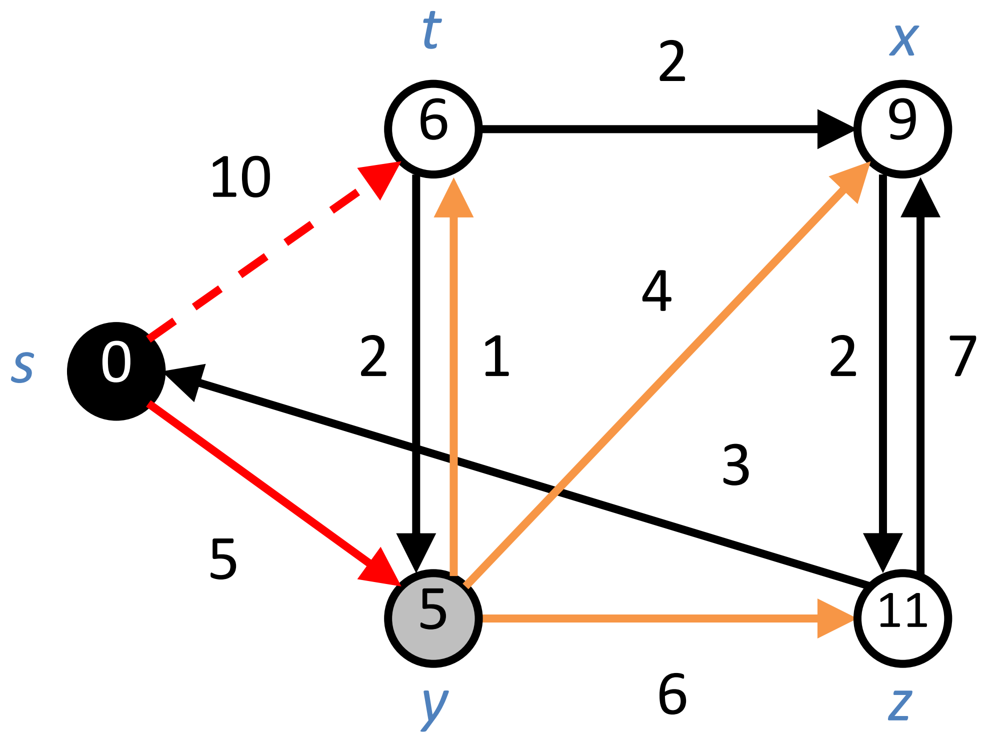
Q					
	s	t	y	x	z

Example



Q	y	t	x	z	

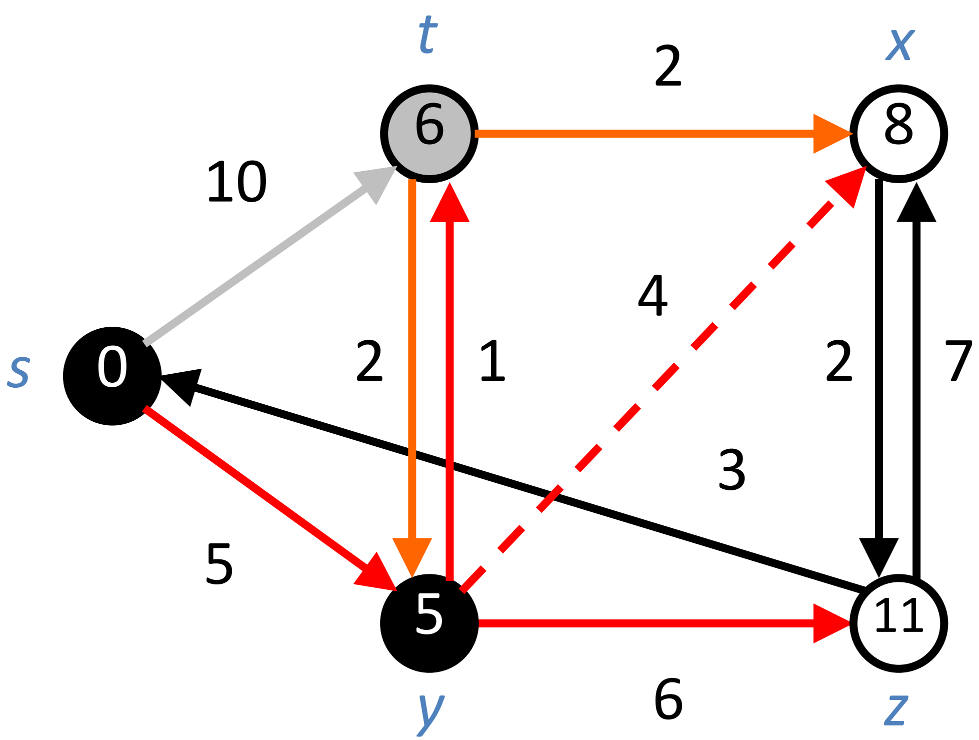
Example



Q

t	x	z		

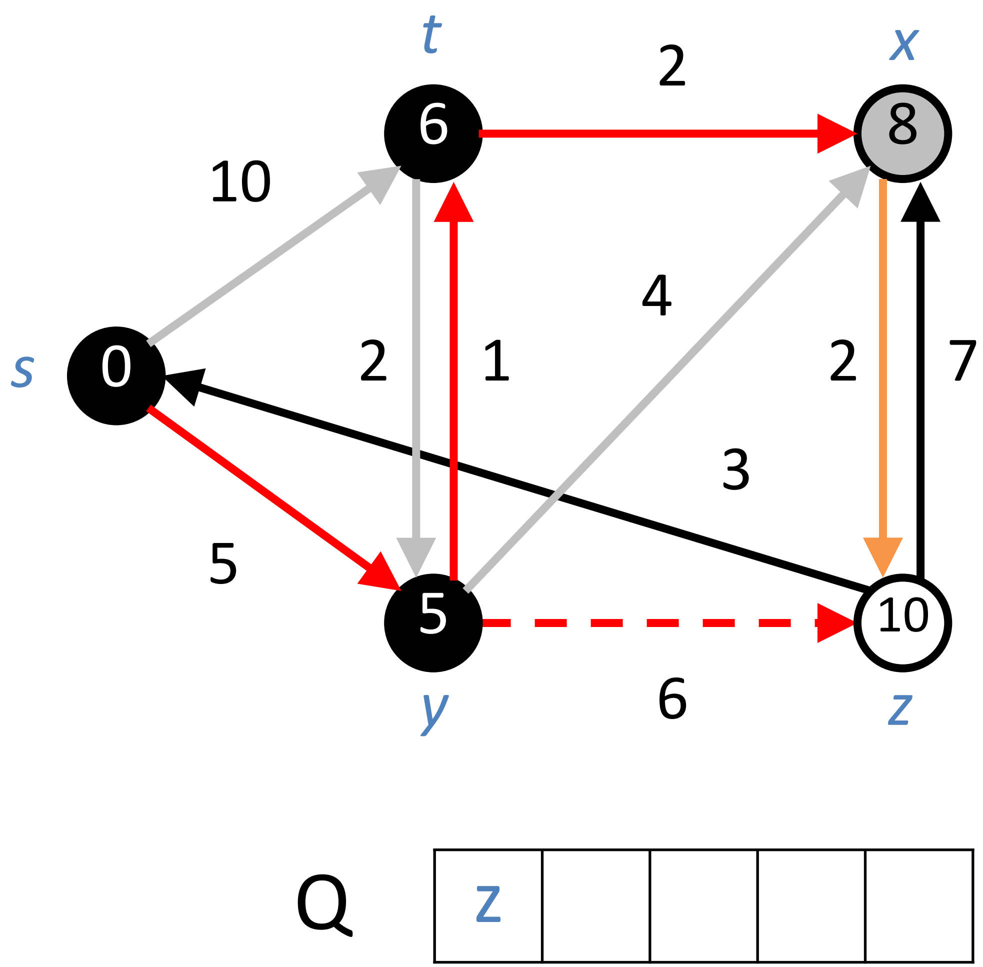
Example



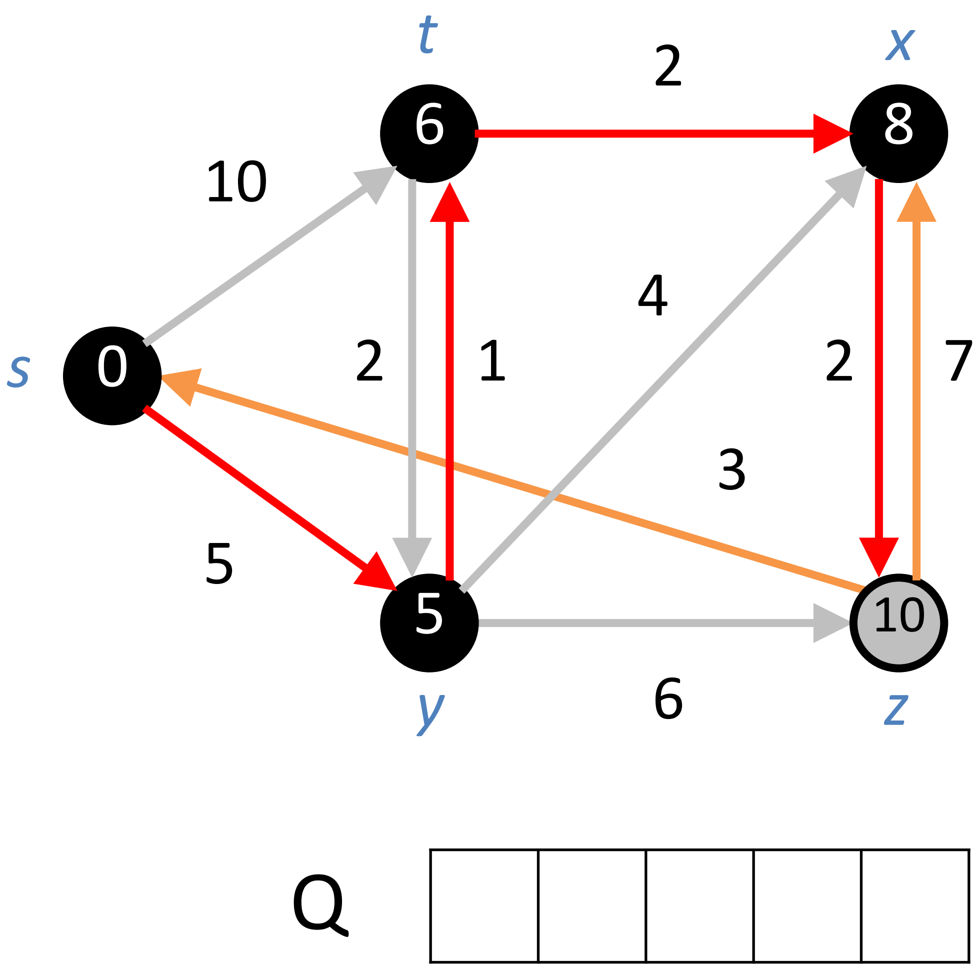
Q

x	z			

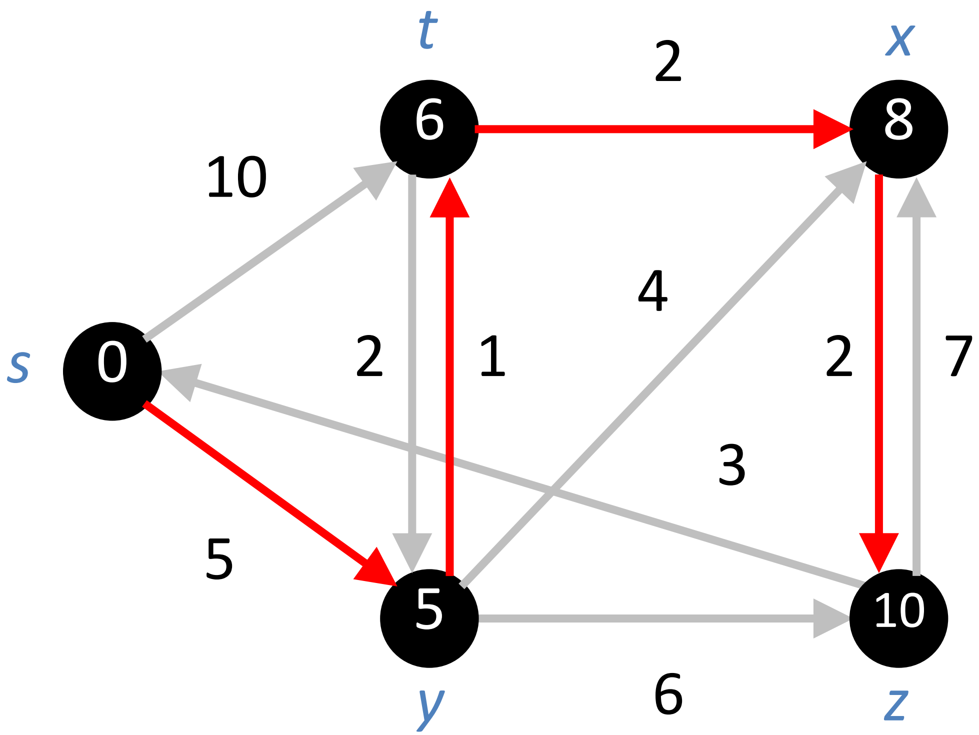
Example



Example



Example



Correctness

Loop invariant property:
At the end of each iteration of the while loop, $d[v] = \delta(s, v)$ for all $v \in S$.

Initialization:
Initially, $S = \emptyset$, so trivially true.

Termination:
At each iteration, we remove one vertex from Q and never add one in. Thus, the algorithm terminates after $|V|$ iterations.

Maintenance:
Show that $d[u] = \delta(s, u)$ when u is added to S in each Iteration.

Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Suppose there exists u such that $d[u] \neq \delta(s,u)$.

Let u be the first vertex for which $d[u] \neq \delta(s,u)$ when u is added to S.

- $u \neq s$, since $d[s] = \delta(s,s) = 0$.
- Therefore, $s \in S$ *and* $S \neq \emptyset$.

Now, we can discuss the general case:

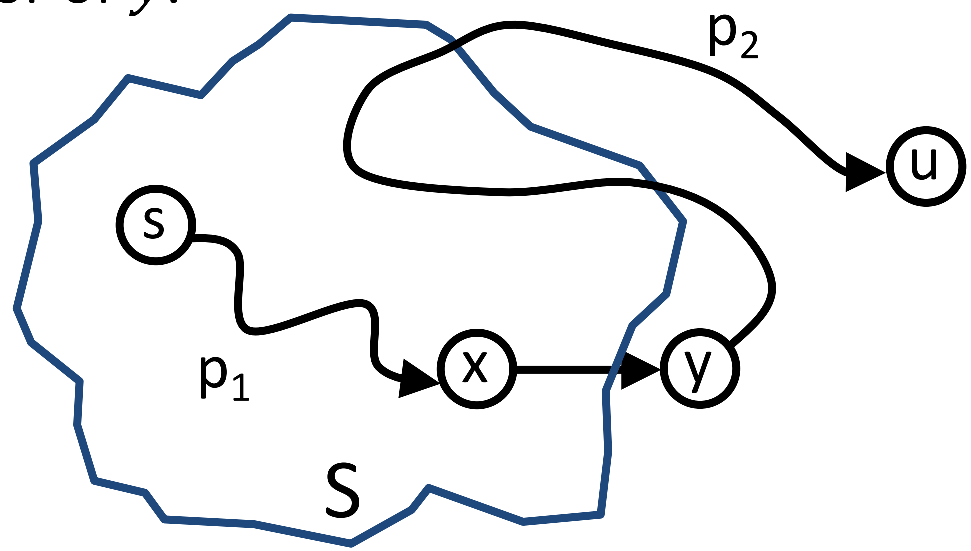
- There must be some path $s \rightsquigarrow u$. Otherwise $d[u] = \delta(s,u) = \infty$ by no-path property.
- Since there is a path $s \rightsquigarrow u$, there is also **a shortest *p* path $\boldsymbol{s \rightsquigarrow u}$**.

Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Just before u is added to S, the shorted path p connects a vertex in S (i.e., s) to a vertex in Q(i.e., u).

Let y be the first vertex along p that is in Q and let $x \in S$ be the predecessor of y.



Note: if $y = u$, then $d[u] = \delta(s,u)$ by the convergence property.

Decompose p into $s \overset{p_1}{\rightsquigarrow} x \to y \overset{p_2}{\rightsquigarrow} u$.

Correctness (cont'd)

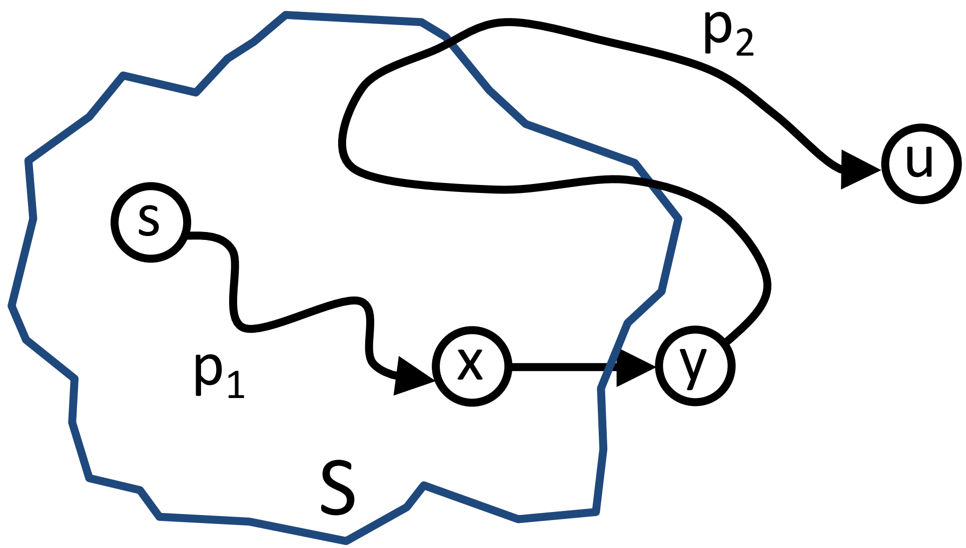
Claim 1: $d[y] = \delta(s, y)$ when u is added to S.

We will need this claim to complete the proof

Proof:

$x \in S$ and u is the first vertex such that $d[u] \neq \delta(s, u)$ when u is added to $S \Rightarrow d[x] = \delta(s, x)$ when x is added to S. But when x is added we relax the edge (x, y), so by the *convergence property*, $d[y] = \delta(s, y)$.

We relax all outgoing edges of a vertex before adding it into S, and y is on the shortest path.



Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Now, we can get a contradiction to $d[u] \neq \delta(s,u)$:

y is on shortest path p ($s \rightsquigarrow u$), and all edge weights are nonnegative.

$\Rightarrow \delta(s,y) \leq \delta(s,u)$

Now, by Claim 1 we have $d[y] = \delta(s,y)$

$\leq \delta(s,u)$ (See inequality above)

$\leq d[u]$ (upper-bound property)

In addition, $\boldsymbol{y}$ **and** $\boldsymbol{u}$ **were in** $\boldsymbol{Q}$ **when we chose** $\boldsymbol{u}$**, *thus*** $\boldsymbol{d[u] \leq d[y]}$**.**

We can now conclude:

We have $d[y] \leq d[u]$ & $d[u] \leq d[y] \Rightarrow d[u] = d[y]$.

This is why we use the priority queue!

Therefore, $d[y] = \delta(s,y) \leq \delta(s,u) \leq d[u] = d[y]$

Contradicts assumption that $d[u] \neq \delta(s,u)$. ■

Correctness

Loop invariant property:
At the end of each iteration of the while loop, $d[v] = \delta(s, v)$ for all $v \in S$.

Initialization: ✓

Termination: ✓

Maintenance: ✓

Conclude: At the end, $Q = \emptyset \Rightarrow S = V \Rightarrow d[v] = \delta(s, v)$ for all $v \in V$.

Analysis

It depends on implementation of priority queue.

If binary heap, each operation takes $O(\lg V)$ time
$\Rightarrow O(E \lg V)$.

Note: We can achieve $O(V \lg V + E)$ with Fibonacci heaps.