COMP251: Topological Sort & Strongly Connected Components

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Based on (Cormen *et al.*, 2002)

Based on slides from D. Plaisted (UNC)

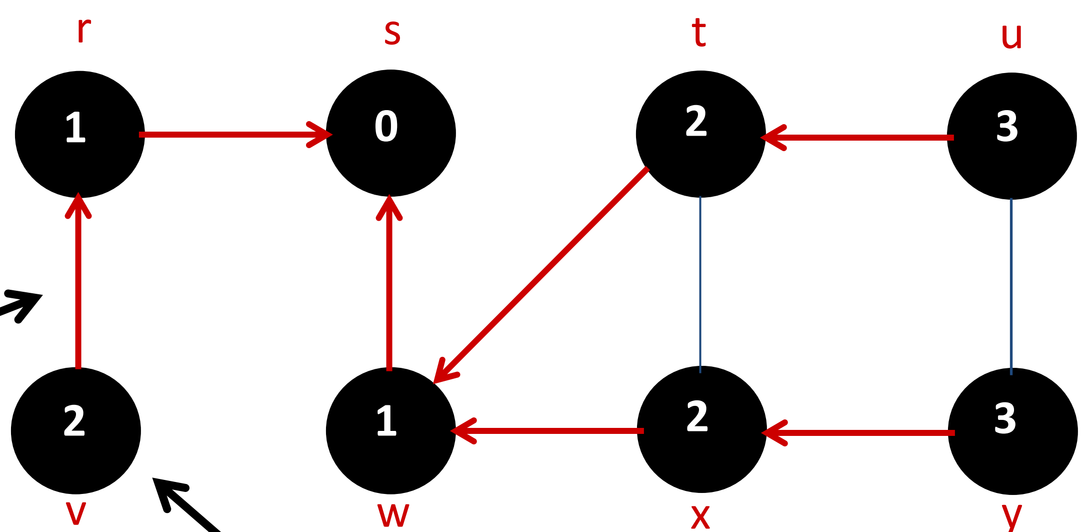
Outline

- Recap: DFS & BFS
- Background material
  - Parenthesis theorem
  - White-Path theorem
  - Edge classification
- Direct Acyclic Graphs (DAGs)
  - Definition
  - Topological Sort
- Strongly Connected Components

Recap: Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Recap: BFS Example



Predecessor
$\pi[v]$

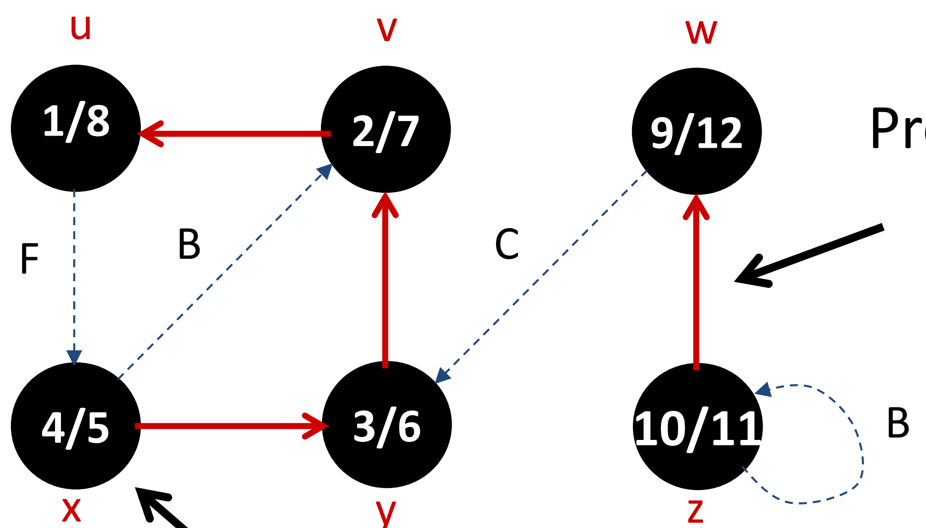
Distance from source $d[v]$

Recap: Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given.
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and $2|V|$.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f\ [v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u' s adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Recap: DFS Example

Vertices on the DFS path are descendants of their predecessors (e.g., x is a descendant of v)



Predecessor
$\pi[z]$

Discovery time
$d[x]$

Finishing time
$f[x]$

Note: The direction of the edges on the DFS path have been reversed to represent the predecessor.

Recap: Parenthesis Theorem

Theorem 1:

For all u, v in a depth-first-search forest, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:

```
OK: ( { } ) [ ]          Not OK: ( { ) }
    1 2 3 4 5 6                  1 2 3 4
```

Corollary

v is a proper descendant of u *if and only if* $d[u] < d[v] < f[v] < f[u]$.

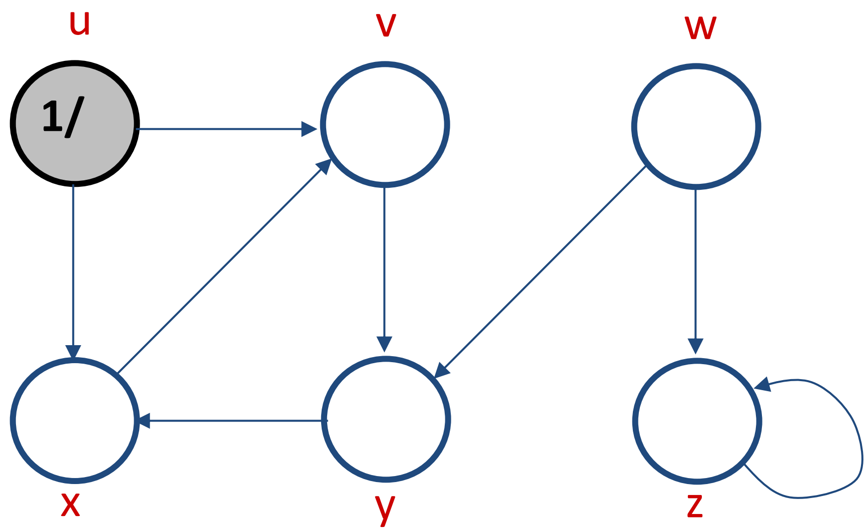
White-path Theorem

Theorem *2*

v is a descendant of u *if and only if* at time $d[u]$, there is a path $u \leadsto v$ consisting of only white vertices (Except for u, which was *just* colored gray).

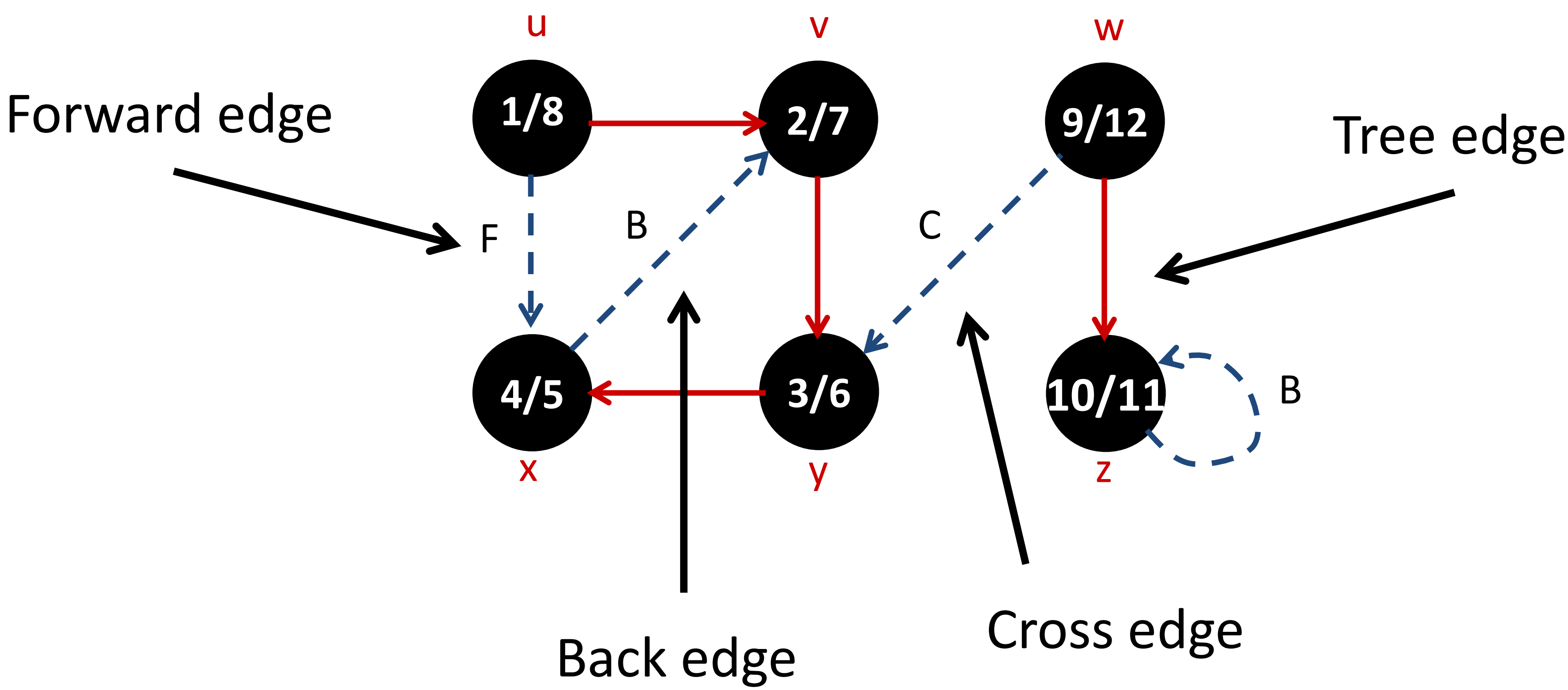
Notation: the arrow $\leadsto$ represents a path of any length (i.e., sequence of one or more consecutive edges).

Example (white-path theorem)



v, y, and x are descendants of u.

Edge classification with DFS



The red edges show the edges used by the DFS algorithm (i.e., tree edges)

Classification of Edges

- **Tree edge:** (u, v) in the depth-first forest. v is a descendant of u and the edge was used by DFS.
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees. It's a (u, v) such that the subtrees rooted at u and v are distinct

Theorem 3

In DFS of a connected undirected graph, we get only tree and back edges. No forward or cross edges.

Proof left as an exercise…

Identification of Edges

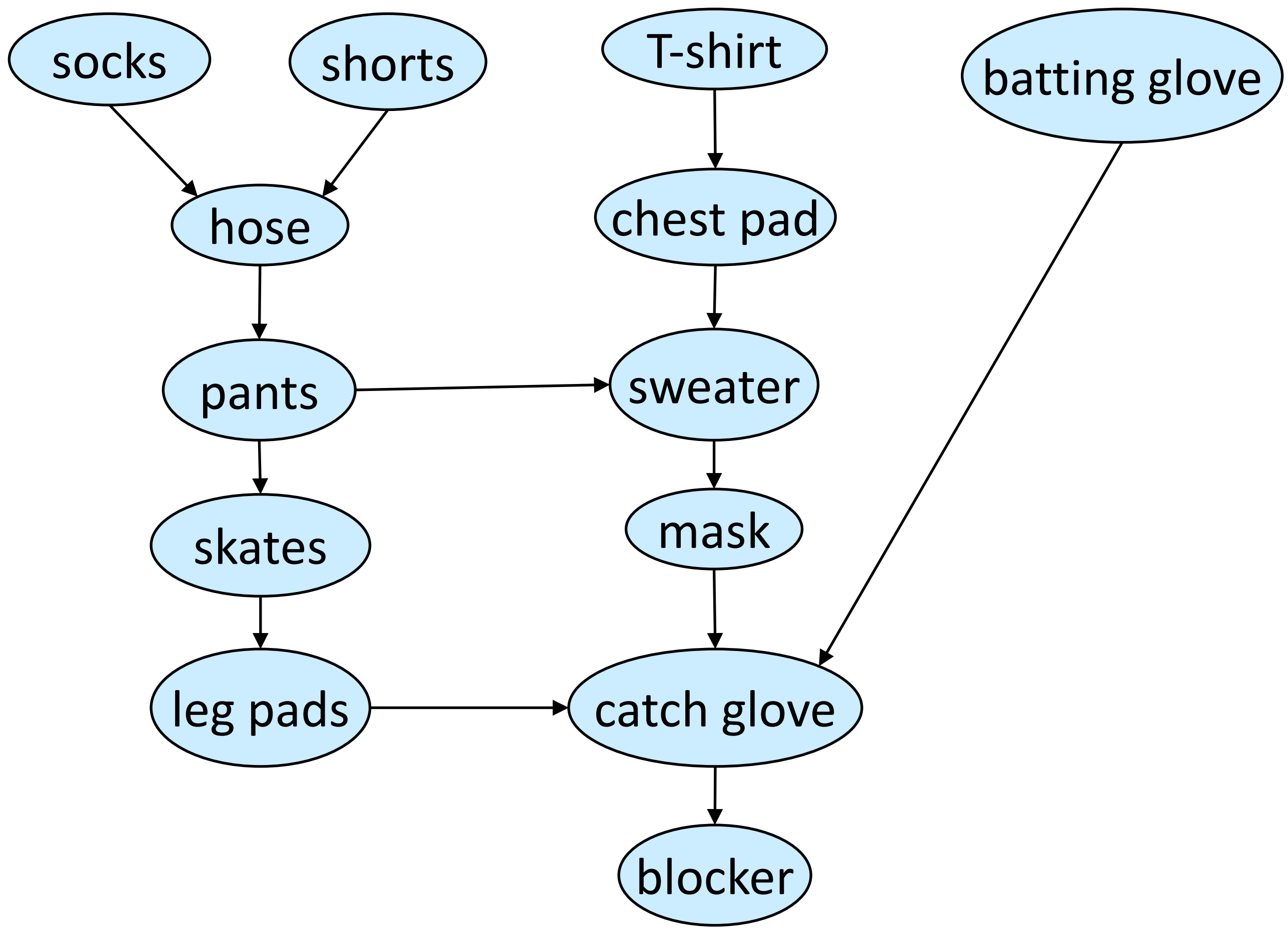
- Edge type for edge (u, v) can be identified when it is first explored by DFS.
- Identification is based on the color of v.
  - White – tree edge.
  - Gray – back edge.
  - Black – forward or cross edge.

Directed Acyclic Graph

- DAG – Directed graph with no cycles.
- Good for modeling processes and structures that have a **partial order:**
  - $a > b$ and $b > c \Rightarrow a > c$.
  - But may have a and b such that neither $a > b$ nor $b > a$.
- Can always make a **total order** (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order.

Example

DAG of dependencies for putting on goalie equipment.

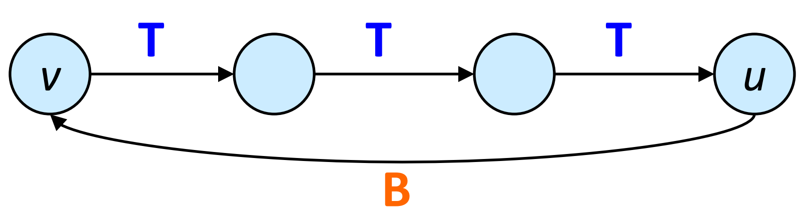


Characterizing a DAG

Lemma 1
A directed graph G is acyclic *iff* a DFS of G yields no back edges.

Proof:

- ($\Rightarrow$) Show that back edge $\Rightarrow$ cycle.
  - Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest (by definition of a back edge).
  - Therefore, there is a path $v \rightsquigarrow u$, so $v \rightsquigarrow u \rightsquigarrow v$ is a cycle.



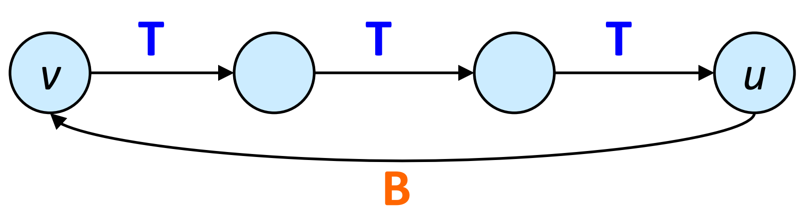
Characterizing a DAG

Lemma 1

A directed graph G is acyclic *iff* a DFS of G yields no back edges.

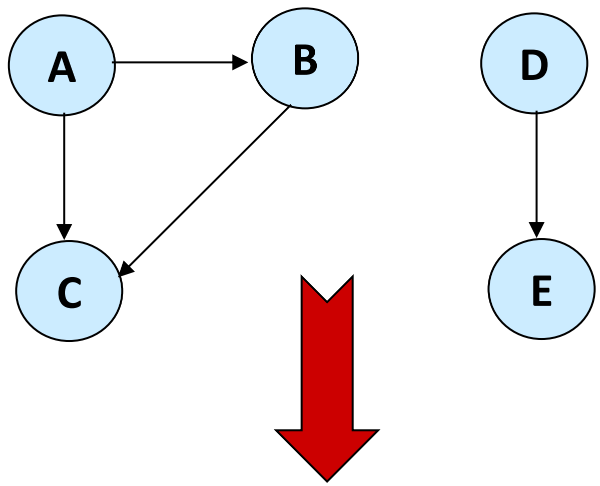
Proof (Contd.):

- ($\Leftarrow$) Show that a cycle implies a back edge.
  - c : cycle in G; v : first vertex discovered in c; (u, v) : preceding edge in c.
  - At time $d[v]$, vertices of c form a white path $v \rightsquigarrow u$.
  - By **white-path theorem**, u is a descendent of v in depth-first forest.
  - Therefore, (u, v) is a back edge.

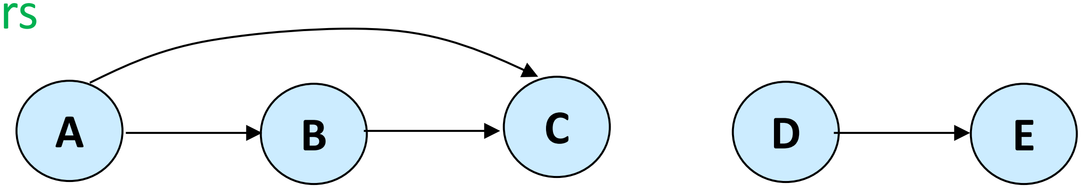


Topological Sort

Want to “sort” a directed acyclic graph (DAG).



You may have several valid total orders



Think of original DAG as a **partial order**.

We want a **total order** that extends this partial order.

Topological Sort

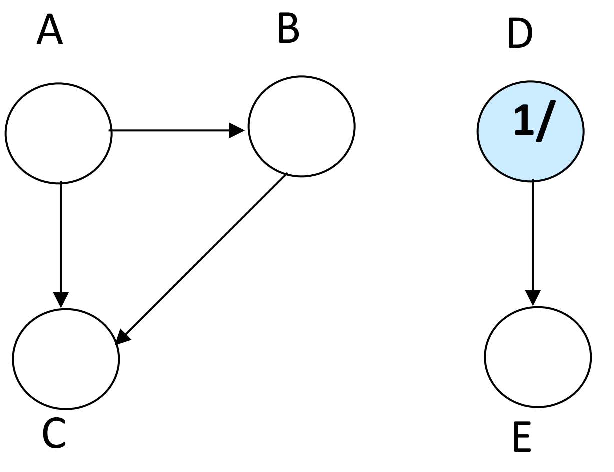
- **Performed on a DAG**.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Topological-Sort (G)

1. call $DFS(G)$ to compute finishing times $f\ [v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices

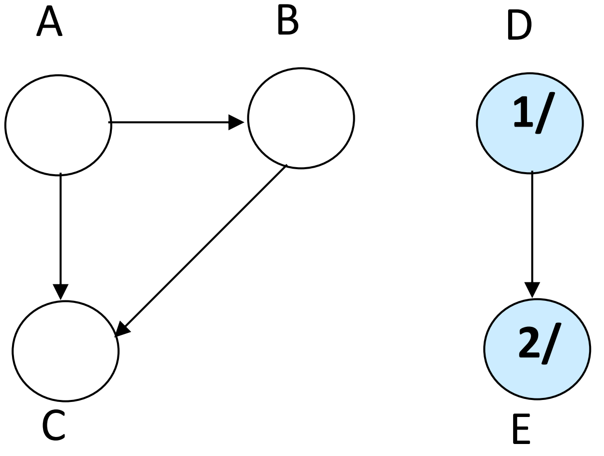
Time: $\Theta(V\ +\ E)$.

Example 1



Linked List:

Example 1



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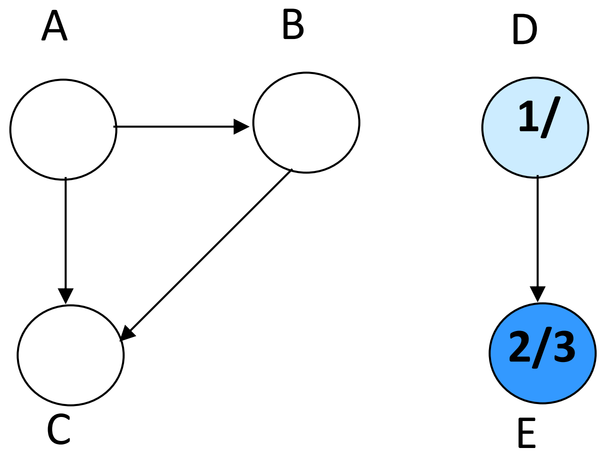

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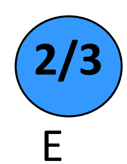
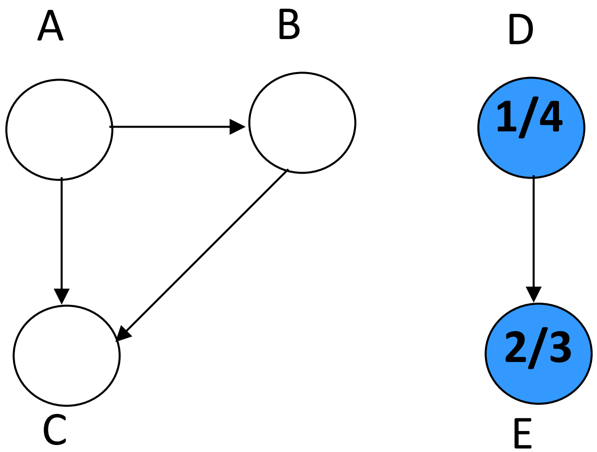


Example 1



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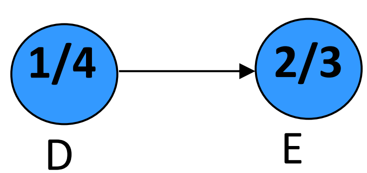

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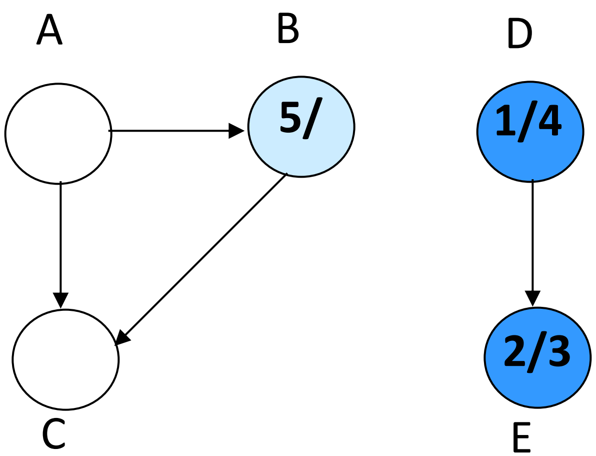

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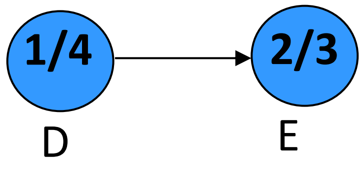

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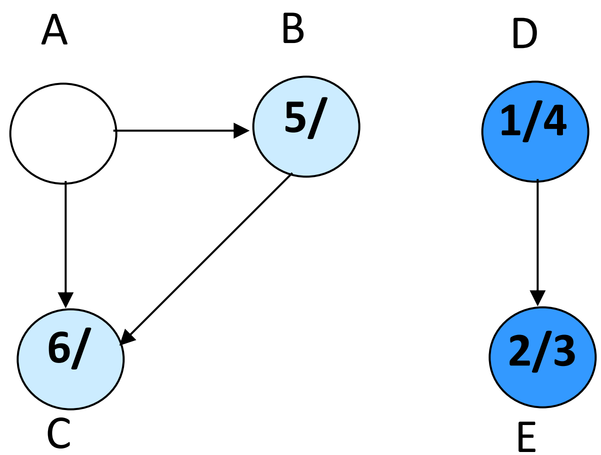

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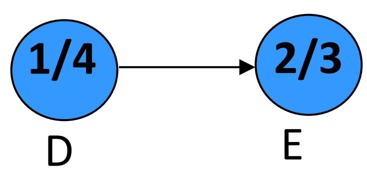

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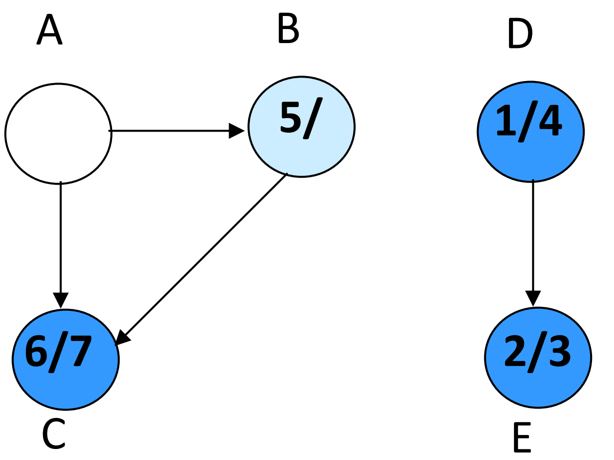

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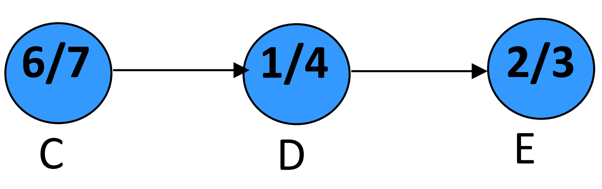
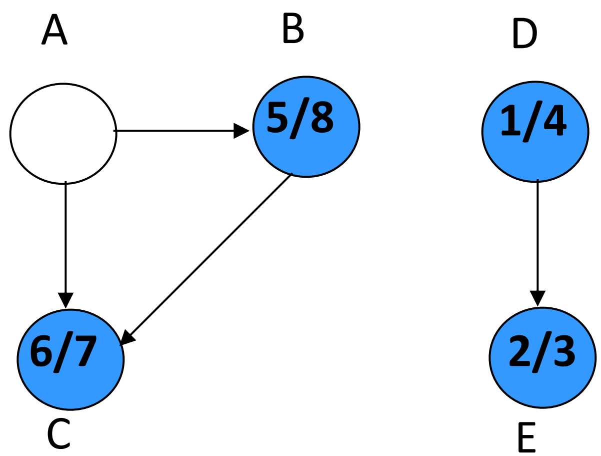


Example 1



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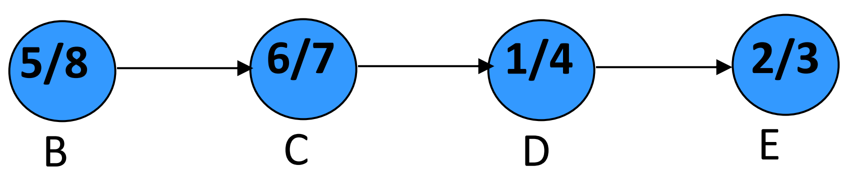
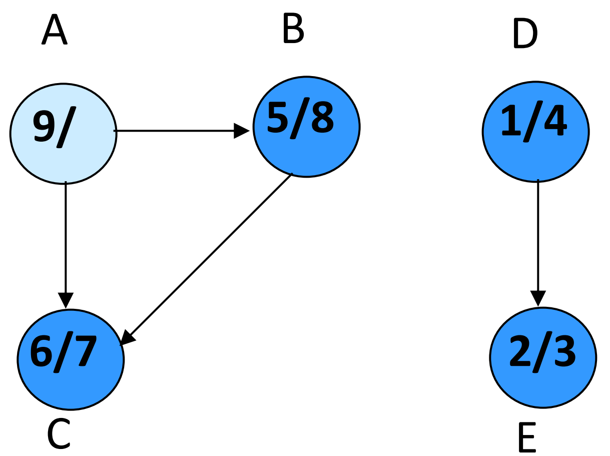


Example 1



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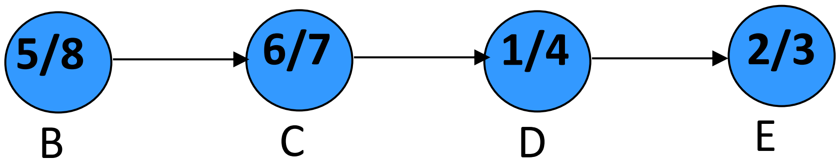
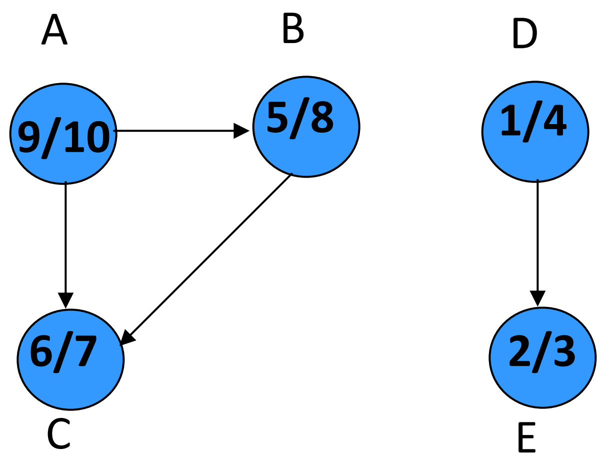


Example 1



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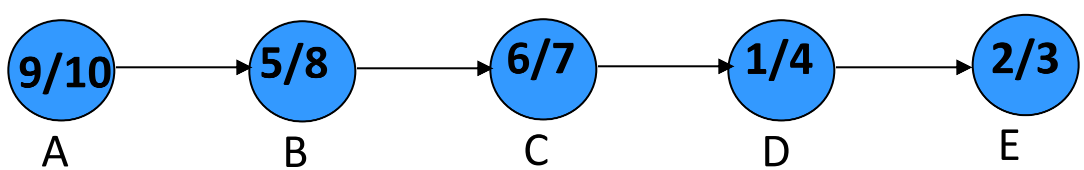
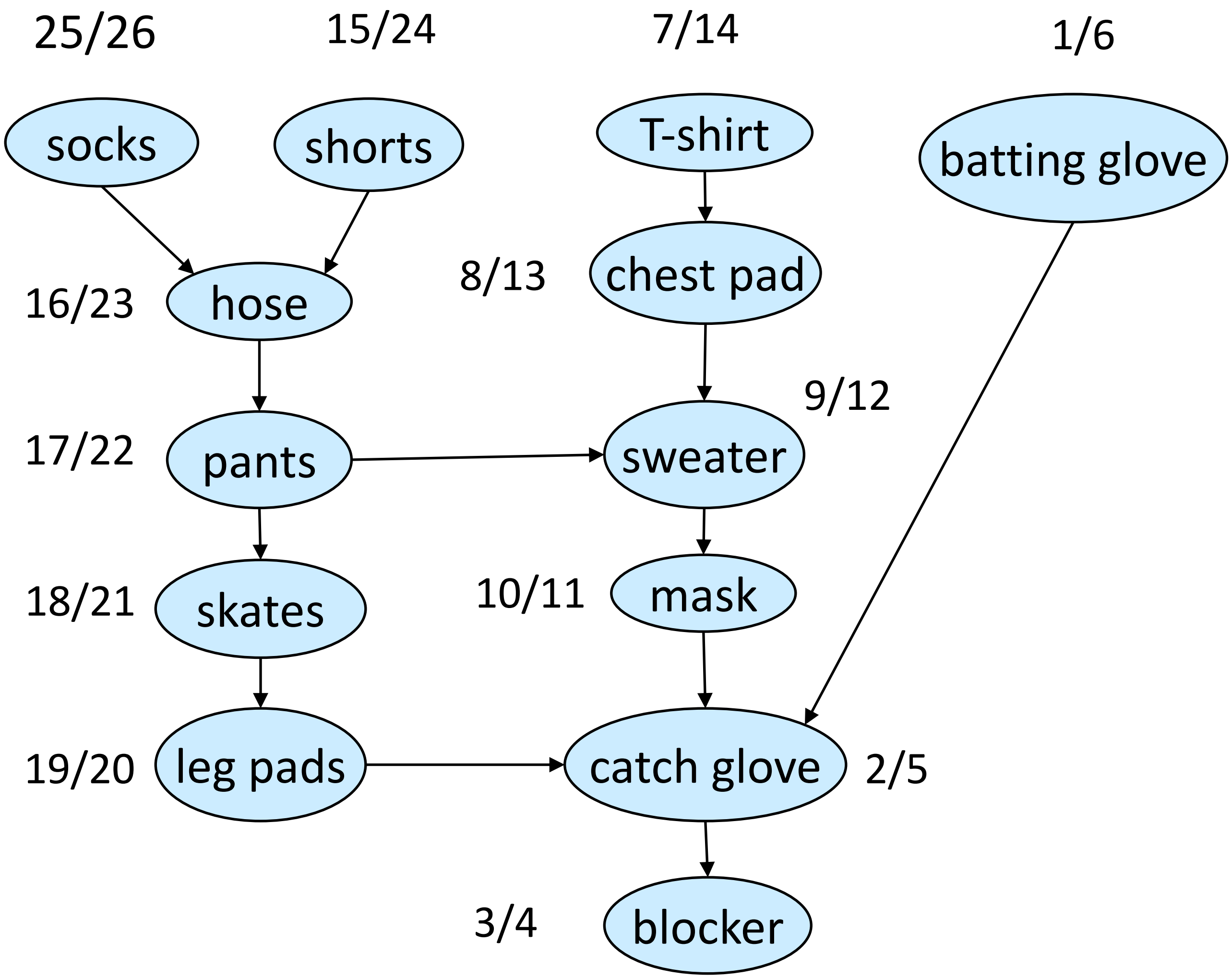


Note: The output may change if the choices of vertices is different, but the result remains valid.

Example 2



26 socks
24 shorts
23 hose
22 pants
21 skates
20 leg pads
14 t-shirt
13 chest pad
12 sweater
11 mask
6 batting glove
5 catch glove
4 blocker

In the sequence of vertices given by the total order.

Correctness (1)

We want to prove: "Linear ordering of the vertices of *G* such that if $(u, v) \in E$, then u appears somewhere before v."

Vertices are inserted at the head of the list as soon as they are finished.

⇒ We need to show **if** $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}$**, then** $\boldsymbol{f}[\boldsymbol{v}] < \boldsymbol{f}[\boldsymbol{u}]$**.**

When we explore (u, v), what are the colors of u and v?

Assume we just discovered u, which is thus gray.

Then, what are the possible colors of v ?

- Can v be gray?
- Can v be white?
- Can v be black?

Correctness (2)

When we explore (u, v), what are the colors of u and v?

- Assume *u* is gray (by hypothesis, we just discovered it).
- Is v gray, too?

  No, because then v would be ancestor of u.

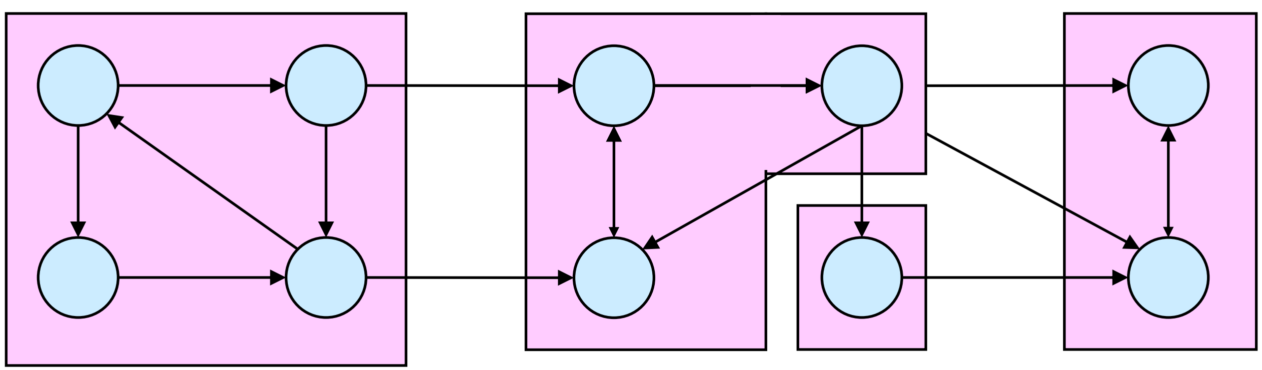
  $\Rightarrow (u, v)$ is a back edge (by definition of a back edge).

  $\Rightarrow$ contradiction of **Lemma 1** (DAG has no back edges).
- Is v white?
  - Then becomes descendant of u.
  - By **parenthesis theorem**, $d[u] < d[v] < \boldsymbol{f}\,[\boldsymbol{v}] < \boldsymbol{f}\,[\boldsymbol{u}]$.
- Is v black?
  - Then v is already finished.
  - Since we are exploring (u, v), we have not yet finished u.
  - Therefore, $f\,[v] < f\,[u]$.

Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.

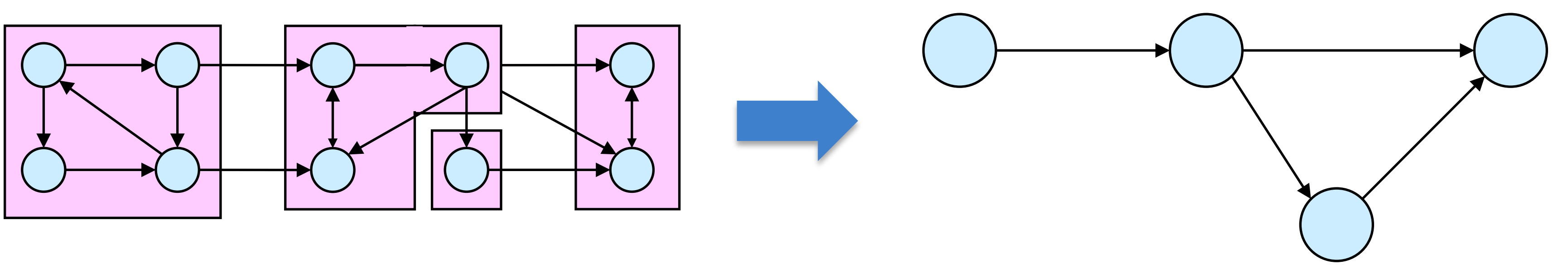
- A **strongly connected component** (***SCC***) of G is a maximal set of vertices $C \in V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$ exist.



Component Graph

- $G^{SCC} = (VSCC, ESCC)$.
- V^{SCC} has one vertex for each SCC in G.
- E^{SCC} has an edge if there is an edge between the corresponding SCC's in G.

Example:



G^{SCC} is a DAG

Lemma 2
Let C and C' be distinct SCC's in G, let $u, v \in C$ & $u', v' \in C'$, and suppose there is a path $u \rightsquigarrow u'$ in G. Then there cannot also be a path $v' \rightsquigarrow v$ in G.

Proof (by contradiction)**:**

- Assume there is a path $v' \rightsquigarrow v$ in G.
- Then, there are paths $u \rightsquigarrow u' \rightsquigarrow v'$ and $v' \rightsquigarrow v \rightsquigarrow u$ in G.
- Therefore, u and v' are reachable from each other, so they are not in separate SCC' s.

Transpose of a Directed Graph

- G^T = **transpose** of directed G.
  - $G^T = (V, ET), E^T = \{(u, v) \colon (v, u) \in E\}$.
  - G^T is G with all edges reversed.

- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.

- G and G^T have the *same* SCC' s. (u and v are reachable from each other in G *if and only if* they are reachable from each other in G^T.)

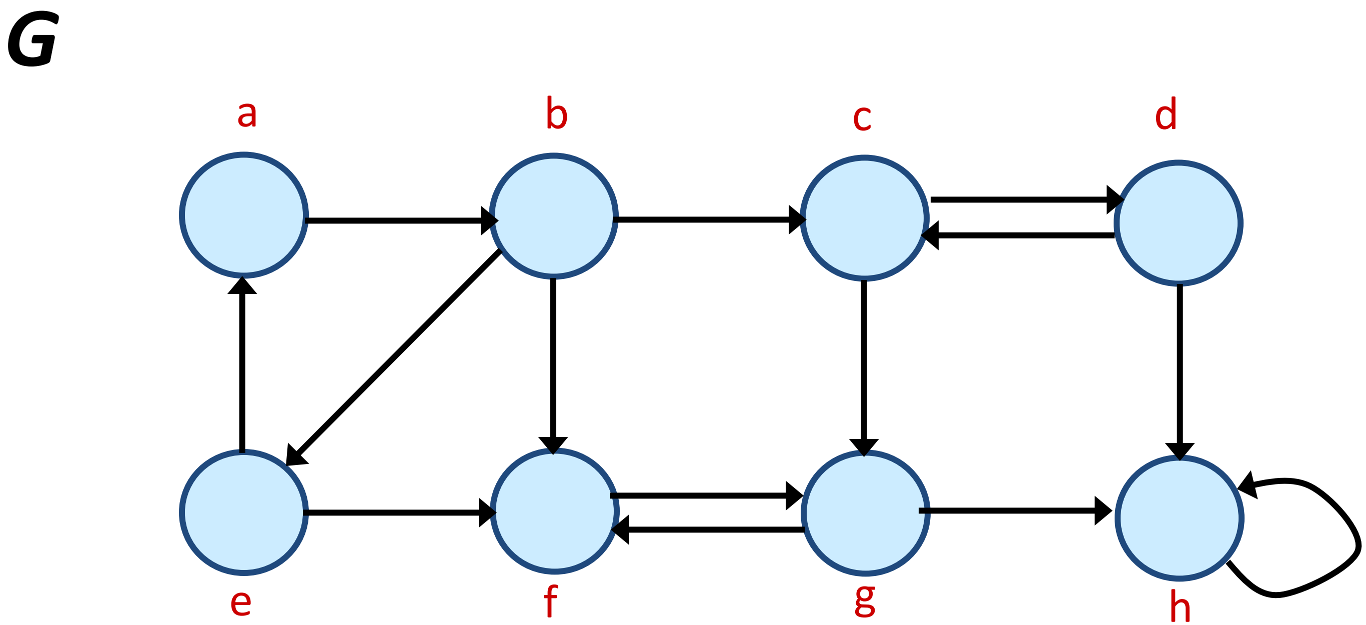
Algorithm to determine SCCs

SCC(*G*)

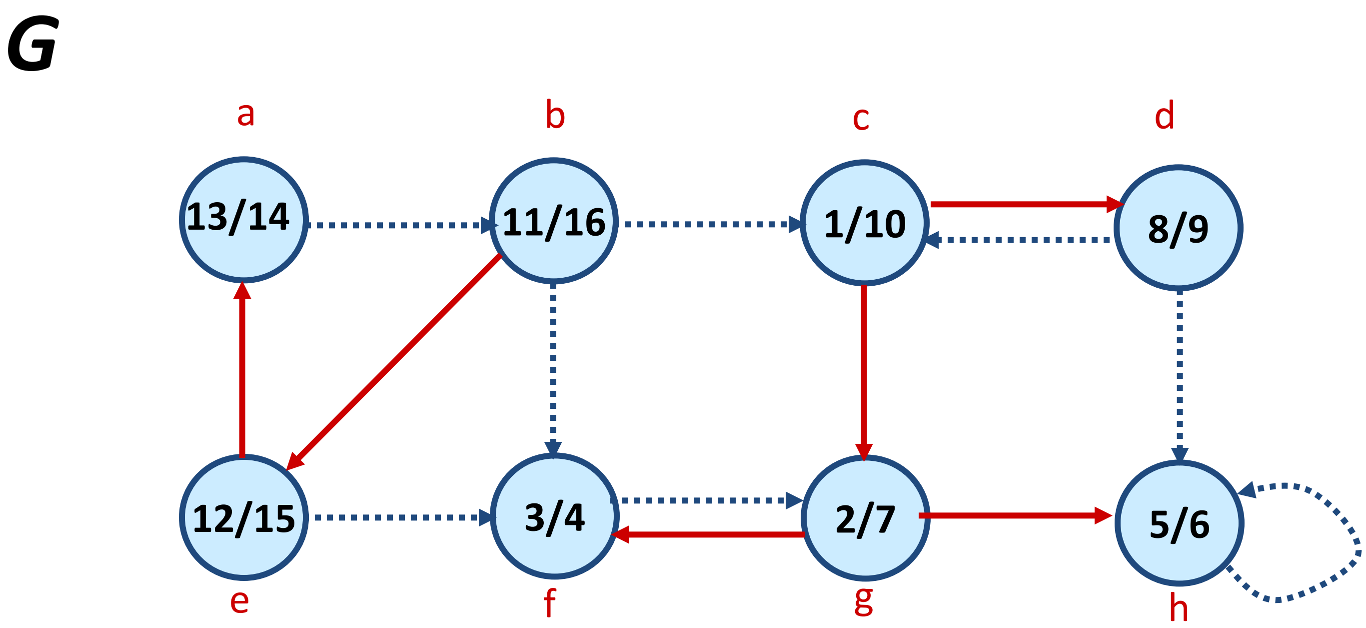
1. call $DFS(G)$ to compute finishing times $f[u]$ for all u
2. compute G^T
3. call $DFS(GT)$, but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

Time: $\Theta(V + E)$.

Example



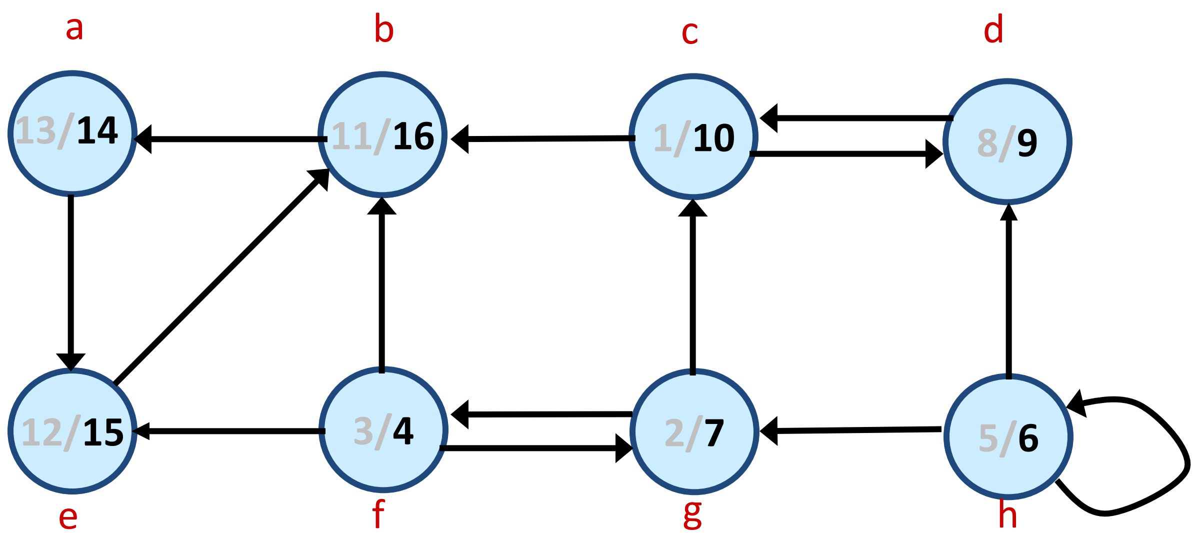
Example



After the first DFS. We computed all finishing times in G.

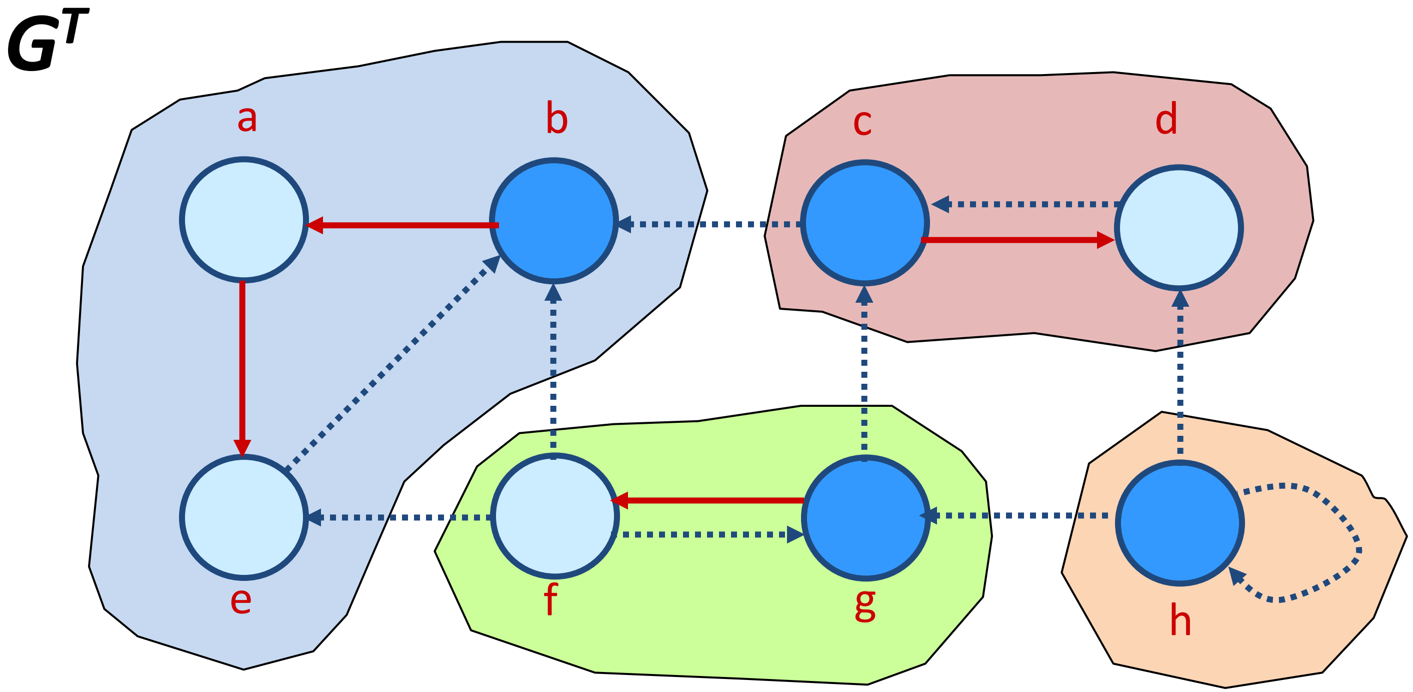
Example

G^T



Then, we compute the transpose G^T of G and sort the vertices with the finishing time calculated in G.

Example



(b (a (e e) a) b) (c (d d) c) (g (f f) g) (h)

How does it work?

- **Idea:**
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the **component graph** in topologically sorted order.
  - Because we are running DFS on G^T, we will not be visiting any v from a u, where v and u are in different components.

Recall: the component graph is a DAG!

- **Notation:**
  - $d[u]$ and $f\ [u]$ always refer to ***first* DFS**.
  - Extend notation for d and f to sets of vertices $U\ \subseteq\ V$:
  - $d(U)\ =\ min_{u \in U}\{d[u]\}$ (earliest discovery time)
  - $f\ (U)\ =\ max_{u \in U}\{\ f\ [u]\}$ (latest finishing time)

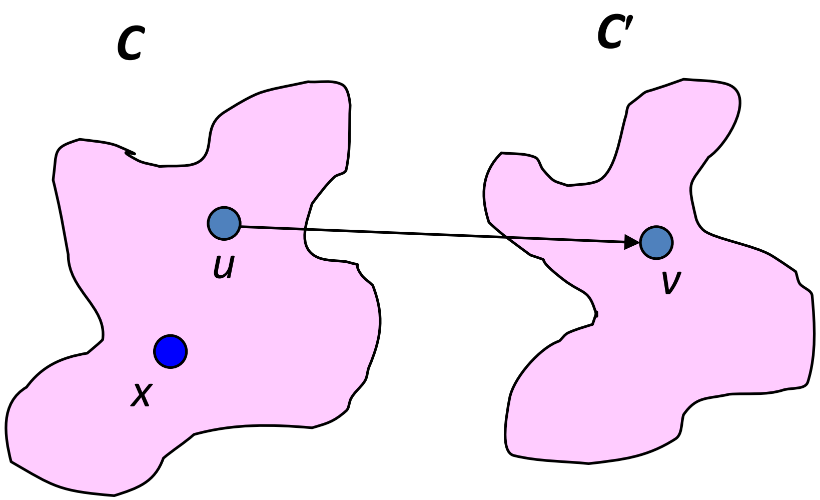
SCCs and DFS finishing times

Lemma 3

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then, $f(C) > f(C')$.

Proof:

- Case 1: $d(C) < d(C')$
  - Let *x* be the first vertex discovered in *C*.
  - At time $d[x]$, all vertices in C and C' are white. Thus, there exist paths of white vertices from x to all vertices in C and C'.
  - By the white-path theorem, all vertices in C and C' are descendants of x in depth-first tree.
  - By the parenthesis theorem, $f[x] = f(C) > f(C')$.



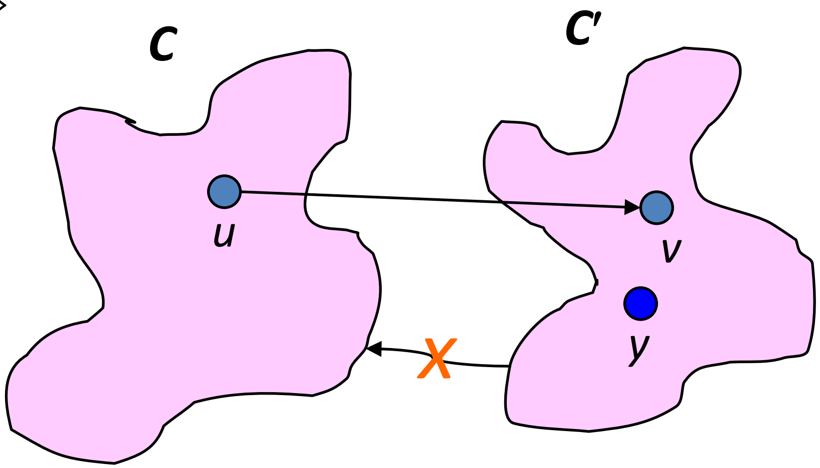
SCCs and DFS finishing times

Lemma 3

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then, $f(C) > f(C')$.

Proof:

- Case 2: $d(C) > d(C')$
  - Let y be the first vertex discovered in C'.
  - At $d[y]$, all vertices in C' are white and there is a white path from y to each vertex in $C' \Rightarrow$ all vertices in C' become descendants of y. Again, $f[y] = f(C')$.
  - At $d[y]$, all vertices in C are also white.
  - By lemma 2, since there is an edge (u, v), we cannot have a path from C' to C.
  - So, no vertex in C is reachable from y.
  - Therefore, at time $f[y]$, all vertices in C are still white.
  - Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.



SCCs and DFS finishing times

Corollary 1
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then, $f(C) < f(C')$.

Proof:

- $(u, v) \in E^T \Rightarrow (v, u) \in E$.
- Since SCC's of G and G^T are the same, $f(C') > f(C)$, by Lemma 3.

Correctness of SCC

1) At beginning, DFS visits only vertices in the first SCC

- When we do the second DFS on G^T, we start with the SCC C such that $f(C)$ is maximum.
- This second DFS starts from some $x \in C$, and it visits all vertices in C.
- Corollary 1 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
- Therefore, **DFS will visit *only* vertices in *C***.
- Which means that the depth-first tree rooted at x contains *exactly* the vertices of C.

Correctness of SCC

2) DFS does not visit more than one new SCC at the time

- The next root in the second DFS is in SCC C' such that $f\,(C')$ is maximum over all SCC's other than C.
  - DFS visits all vertices in C', but **the only edges out of C' go to *C, which we have already visited***.
  - Therefore, the only tree edges will be to vertices in C'.
- Iterate the process.
- Each time we choose a root, it can reach only:
  - vertices in its SCC—get tree edges to these,
  - vertices in SCC's *already visited* in second DFS—get *no* tree edges to these.