COMP251: Elementary graph algorithms

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Based on (Cormen et al., 2002)

Based on slides from D. Plaisted (UNC)
Announces

• Assignment 1: Due on Oct. 18 at 11h59pm (no extension)
• Assignment 2: Released today due on Oct. 29.
• Midterm: November 2 at 10am on CrowdMark.
• Final: Dec. 13 at 9am. **In Person!**
Outline

- Vocabulary, definition, and properties of graphs
- Exploring graphs:
  - Breadth First Search (BFS)
  - Depth First Search (DFS)

Why?
We will cover many algorithms on graphs based on these techniques.
Graphs

• **Graph G = (V, E)**
  - $V$ = set of vertices
  - $E$ = set of edges $\subseteq (V \times V)$

• Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all $v$, $(v, v) \notin E$ (No self loops.)
  - Directed: $(u, v)$ is edge from $u$ to $v$, denoted as $u \to v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \to \mathbb{R}$.
  - Dense: $|E| \approx |V|^2$.
  - Sparse: $|E| \ll |V|^2$.

• $|E| = O(|V|^2)$
Properties

• If \((u, v) \in E\), then vertex \(v\) is adjacent to vertex \(u\).

• Adjacency relationship is:
  – Symmetric if \(G\) is undirected.
  – Not necessarily so if \(G\) is directed.

• If \(G\) is connected:
  – There is a path between every pair of vertices.
  – \(|E| \geq |V| - 1\).
  – Furthermore, if \(|E| = |V| - 1\), then \(G\) is a tree.
Ingoing edges of $u$: $\{ (v,u) \in E \}$ (e.g. $\text{in}(e) = \{ (b,e), (d,e) \}$)
Outgoing edges of $u$: $\{ (u,v) \in E \}$ (e.g. $\text{out}(d) = \{ (d,e) \}$)
In-degree($u$): $| \text{in}(u) |$
Out-degree($u$): $| \text{out}(u) |$
Representation of Graphs

• Two standard ways.
  – Adjacency Lists.

  a
  ├── b
  │    ├── c
  │    └── d
  └── c
      ├── b
      │    └── d
      └── d

  a
  ├── b
  │    └── d
  └── c
      ├── d
      └── a

  a
  ├── b
  │    └── c
  └── d

  a
  ├── b
  ├── c
  └── d

– Adjacency Matrix.

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Adjacency Lists

- Consists of an array $Adj$ of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to $u$.

Note: If weighted, store weights also in adjacency lists.
Storage Requirement

• For directed graphs:
  – Sum of lengths of all adj. lists is
    \[ \sum_{v \in V} \text{out-degree}(v) = |E| \]
  – Total storage: \( \Theta(V+E) \)

• For undirected graphs:
  – Sum of lengths of all adj. lists is
    \[ \sum_{v \in V} \text{degree}(v) = 2|E| \]
  – Total storage: \( \Theta(V+E) \)
Pros and Cons: adj list

• Pros
  – Space-efficient, when a graph is sparse.
  – Can be modified to support many graph variants.

• Cons
  – Determining if an edge \((u,v) \in E\) is not efficient.
    • Have to search in \(u\)'s adjacency list. \(\Theta(\text{degree}(u))\) time.
    • \(\Theta(V)\) in the worst case.
Adjacency Matrix

• $|V| \times |V|$ matrix $A$.

• Number vertices from 1 to $|V|$ in some arbitrary manner.

• $A$ is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$A = A^T$$ for undirected graphs.
Space and Time

• **Space:** $\Theta(V^2)$.
  - Not memory efficient for large sparse graphs.

• **Time:** to list all vertices adjacent to $u$: $\Theta(V)$.

• **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.

• Can store weights instead of bits for weighted graph.
Graph-searching Algorithms (COMP250)

- Searching a graph:
  - Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).
Breadth-first Search

• Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  – A vertex is “discovered” the first time it is encountered during the search.
  – A vertex is “finished” if all vertices adjacent to it have been discovered.

• Colors the vertices to keep track of progress.
  – White – Undiscovered.
  – Gray – Discovered but not finished.
  – Black – Finished.
  • Colors are required only to reason about the algorithm. Can be implemented without colors.
Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and source vertex $s \in V$.

- **Output:**
  - $d[v] =$ distance (smallest # of edges, or shortest path) from $s$ to $v$, for all $v \in V$. $d[v] = \infty$ if $v$ is not reachable from $s$.
  - $\pi[v] = u$ such that $(u, v)$ is last edge on shortest path $s \leadsto v$.
    - $u$ is $v$'s predecessor.
  - Builds breadth-first tree with root $s$ that contains all reachable vertices.
We use a priority queue to determine the next vertices to visit. The first vertex we add in the queue is the source.
Color code:
- White: not visited yet
- Gray: visited but neighborhood not fully explored
- Black: Complete

We store in the queue the vertices in the neighborhood of the current vertex.

The vertices are indexed with the number of edges from the source.
Example (BFS)

Q: r t x
   1 2 2
Example (BFS)

Q: t x v
    2 2 2
Example (BFS)

Q: x v u
   2 2 3
Example (BFS)

Q: v u y
2 3 3
Example (BFS)

Q: u y 3 3
Example (BFS)
Example (BFS)

Q: Ø
Example (BFS)

The index of each vertex stores the length of the shortest path to reach them (Note: unweighted graph!).
Analysis of BFS

• Initialization takes $O(V)$.

• Traversal Loop
  – After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.
  – The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$.

• Summing up over all vertices $\Rightarrow$ total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph.
Depth-first Search (DFS)

• Explore edges out of the most recently discovered vertex \( v \).
• When all edges of \( v \) have been explored, backtrack to explore other edges leaving the vertex from which \( v \) was discovered (its predecessor).
• “Search as deep as possible first.”
• Continue until all vertices reachable from the original source are discovered.
• If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.
Depth-first Search

• **Input:** $G = (V, E)$, directed or undirected. No source vertex given.

• **Output:**
  
  – 2 **timestamps** on each vertex. Integers between 1 and 2$|V|$.
    
    * $d[v] = \textit{discovery time}$ ($v$ turns from white to gray)
    * $f[v] = \textit{finishing time}$ ($v$ turns from gray to black)

  – $\pi[v]$ : predecessor of $v = u$, such that $v$ was discovered during the scan of $u$’s adjacency list.

• Uses the same coloring scheme for vertices as BFS.
Pseudo-code

**DFS(G)**
1. **for** each vertex $u \in V[G]$
2. \hspace{1em} **do** $\text{color}[u] \leftarrow \text{white}$
3. \hspace{1em} $\pi[u] \leftarrow \text{NIL}$
4. $\text{time} \leftarrow 0$
5. **for** each vertex $u \in V[G]$
6. \hspace{1em} **do** if $\text{color}[u] = \text{white}$
7. \hspace{1em} \hspace{1em} then DFS-Visit($u$)

**DFS-Visit($u$)**
1. $\text{color}[u] \leftarrow \text{GRAY}$  \# White vertex $u$ has been discovered
2. $\text{time} \leftarrow \text{time} + 1$
3. $d[u] \leftarrow \text{time}$
4. **for** each $v \in \text{Adj}[u]$
5. \hspace{1em} **do** if $\text{color}[v] = \text{WHITE}$
6. \hspace{1em} \hspace{1em} then $\pi[v] \leftarrow u$
7. \hspace{1em} DFS-Visit($v$)
8. $\text{color}[u] \leftarrow \text{BLACK}$  \# Blacken $u$; it is finished.
9. $f[u] \leftarrow \text{time} \leftarrow \text{time} + 1$

Uses a global timestamp \textit{time}. 
Example (DFS)

Starting time $d(x)$

Diagram:

- Node 1/ (grey)
- Nodes $u$, $v$, $w$, $x$, $y$, $z$
- Arrows indicating connections between nodes
Example (DFS)
Example (DFS)
Example (DFS)
Example (DFS)
Example (DFS)

Starting time \( d(x) \)

Finishing time \( f(x) \)
Example (DFS)
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Example (DFS)
Analysis of DFS

• Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.

• DFS-Visit is called once for each white vertex $v \in V$ when it’s painted gray the first time. Lines 3-6 of DFS-Visit is executed $|\text{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\text{Adj}[v]| = \Theta(E)$

• Total running time of DFS is $\Theta(V+E)$. 
Example (DFS)

Starting time $d(x)$

Finishing time $f(x)$
Parenthesis Theorem

**Theorem 1:**
For all $u$, $v$, exactly one of the following holds:

2. $d[u] < d[v] < f[v] < f[u]$ and $v$ is a descendant of $u$.

- Like parentheses:
  - OK: ( ) [ ] ( [ ] ) [ ( ) ]
  - Not OK: ( [ ) ] [ ( ] )

**Corollary**
$v$ is a proper descendant of $u$ if and only if $d[u] < d[v] < f[v] < f[u]$. 
Example (Parenthesis Theorem)

\[(s \ (z \ (y \ (x \ x) \ y) \ (w \ w) \ z) \ s) \ (t \ (v \ v) \ (u \ u) \ t)\]