COMP251: Single source shortest paths

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Based on (Cormen et al., 2002)
Problem

What is the shortest road to go from one city to another?

Example: Which road should I take to go from Montréal to Boston (MA)?

Variants:
• What is the fastest road?
• What is the cheapest road?
Modeling as graphs

Input:
- Directed graph $G = (V,E)$
- Weight function $w: E \rightarrow \mathbb{R}$

Weight of path $p = \langle v_0, v_1, \ldots, v_k \rangle$

\[
= \sum_{k=1}^{n} w(v_{k-1}, v_k)
\]

= sum of edges weights on path $p$

Shortest-path weight $u$ to $v$:

\[
\delta(u,v) = \begin{cases} 
\min \left\{ w(p) : u \rightarrow v \right\} & \text{If there exists a path } u \sim v.
\infty & \text{Otherwise.}
\end{cases}
\]

Shortest path $u$ to $v$ is any path $p$ such that $w(p) = \delta(u,v)$

Generalization of breadth-first search to weighted graphs
Shortest path from s?
Shortest paths are organized as a tree. Vertices store the length of the shortest path from $s$. 
Example

Shortest paths are not necessarily unique!
Variants

• **Single-source:** Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.

• **Single-destination:** Find shortest paths to a given destination vertex.

• **Single-pair:** Find shortest path from $u$ to $v$.

  *Note: No way to known that is better in worst case than solving the single-source problem!*

• **All-pairs:** Find shortest path from $u$ to $v$ for all $u, v \in V$. 

Negative weight edges can create issues.

**Why?** If we have a negative-weight cycle, we can just keep going around it, and get $w(s, v) = -\infty$ for all $v$ on the cycle.

**When?** If they are reachable from the source (Corollary: It is OK to have a negative-weight cycles if it is not reachable from the source).

**What?** Some algorithms work only if there are no negative-weight edges in the graph. We must specify when they are allowed and not.
Cycles

Shortest paths cannot contain cycles:

• Negative-weight: Already ruled out.
• Positive-weight: we can get a shorter path by omitting the cycle.
• Zero-weight: no reason to use them $\Rightarrow$ assume that our solutions will not use them.
**Optimal substructure**

**Lemma**
Any subpath of a shortest path is a shortest path.

**Proof:** (cut and paste)

Suppose this path \( p \) is a shortest path from \( u \) to \( v \).

Then \( \delta(u,v) = w(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv}) \).
Optimal substructure

**Lemma**
Any subpath of a shortest path is a shortest path.

**Proof:** (cont’d)

Now suppose there exists a shorter path $x \sim y$.

Then $w(p'_{xy}) < w(p_{xy})$.

$$w(p') = w(p_{ux}) + w(p'_{xy}) + w(p_{yv}) < w(p_{ux}) + w(p_{xy}) + w(p_{yv}) = w(p).$$

*Contradiction of the hypothesis that $p$ is the shortest path!*
Customized breadth-first search

Vertices count the number of edges used to reach them.
Customized breadth-first search
Customized breadth-first search
Customized breadth-first search
Can we generalize BFS to use edge weights?
Output of single-source shortest-path algorithm

For each vertex $v \in V$:

- $d[v] = \delta(s,v)$.
  - Initially, $d[v] = \infty$.
  - Reduces as algorithms progress, but always maintain $d[v] \geq \delta(s,v)$.
  - Call $d[v]$ a **shortest-path estimate**.

- $\pi[v] =$ predecessor of $v$ on a shortest path from $s$.
  - If no predecessor, $\pi[v] = \text{NIL}$.
  - $\pi$ induces a tree - **shortest-path tree** (see proof in textbook).
Algorithm structure

1. Initialization
2. Scan vertices and relax edges

The algorithms differ in the order and how many times they relax each edge.
Initialization

\[ \text{INIT-SINGLE-SOURCE}(V, s) \]

\[ \text{for each } v \in V \text{ do} \]
\[ d[v] \leftarrow \infty \]
\[ \pi[v] \leftarrow \text{NIL} \]
\[ d[s] \leftarrow 0 \]
Relaxing an edge

\texttt{RELAX}(u,v,w) \\
\textbf{if} \ d[v] > d[u] + w(u,v) \ \textbf{then} \\
\quad d[v] \leftarrow d[u] + w(u,v) \\
\quad \pi[v] \leftarrow u
Triangle inequality

For all \((u,v) \in E\), we have \(\delta(u,v) \leq \delta(u,x) + \delta(x,v)\).

Proof:

Weight of shortest path \(u \rightsquigarrow v\) is \(\leq\) weight of any path \(u \rightsquigarrow v\).

Path \(u \rightsquigarrow x \rightsquigarrow v\) is a path \(u \rightsquigarrow v\), and if we use a shortest path \(u \rightsquigarrow x\) and \(x \rightsquigarrow v\), its weight is \(\delta(u, x) + \delta(x, v)\).
Upper bound property

Always have $\delta(s, v) \leq d[v]$ for all $v$. Once $d[v] = \delta(s, v)$, it never changes.

Proof:

• Initially true.

• Then, assume it exists a vertex $v$ such that $d[v] < \delta(s, v)$. WLOG, $v$ is the first vertex for which this happens.

Let $u$ be the vertex that causes $d[v]$ to change.

Then $d[v] = d[u] + \delta(u, v)$. But, we also have:

$$d[v] < \delta(s,v) \leq \delta(s, u) + \delta(u, v) \leq d[u] + \delta(u, v)$$

(triangle inequality) \quad (v \text{ is first violation})

$\Rightarrow d[v] < d[u] + \delta(u, v)$.

Contradicts $d[v] = d[u] + \delta(u, v)$. 
No-path property

If \( \delta(s, v) = \infty \), then \( d[v] = \infty \) always.

**Proof:** \( d[v] \geq \delta(s,v) = \infty \Rightarrow d[v] = \infty \).
Convergence property

If:
1. \( s \mapsto u \rightarrow v \) is a shortest path,
2. \( d[u] = \delta(s,u) \),
3. we call \( \text{RELAX}(u,v,w) \),
then \( d[v] = \delta(s,v) \) afterward.

Proof:

After relaxation:
\[
d[v] \leq d[u] + w(u,v) \quad \text{(RELAX code)}
\]
\[
= \delta(s,u) + w(u,v) \quad \text{(} d[u] = \delta(s,u) \text{)}
\]
\[
= \delta(s,v) \quad \text{(} s \mapsto u \rightarrow v \text{ is a shortest path \& lemma sub-optimal structure)}
\]

Since \( d[v] \geq \delta(s,v) \), must have \( d[v] = \delta(s,v) \).
Path-relaxation property

Let \( p = \langle v_0, v_1, ..., v_k \rangle \) be a shortest path from \( s = v_0 \) to \( v_k \).
If we relax, in order, \( (v_0, v_1) \), \( (v_1, v_2) \), ..., \( (v_{k-1}, v_k) \), even intermixed with other relaxations, then \( d[v_k] = \delta(s, v_k) \).

Proof:

Induction to show that \( d[v_i] = \delta(s, v_i) \) after \( (v_{i-1}, v_i) \) is relaxed.

Basis: \( i = 0 \). Initially, \( d[v_0] = 0 = \delta(s, v_0) = \delta(s, s) \).

Inductive step: Assume \( d[v_{i-1}] = \delta(s, v_{i-1}) \). Relax \( (v_{i-1}, v_i) \). By convergence property, \( d[v_i] = \delta(s, v_i) \) afterward and \( d[v_i] \) never changes.
Single-source shortest paths in a DAG

DAG ⇒ no negative-weight cycles.

DAG-SHORTEST-PATHS( V, E, w, s)

topologically sort the vertices

INIT-SINGLE-SOURCE( V, s)

for each vertex u in topological order do
    for each vertex v∈Adj[u] do
        RELAX( u, v, w)
Example
Example
Example
Example
Example
Single-source shortest paths in a DAG

DAG-SHORTEST-PATHS($V, E, w, s$)
topologically sort the vertices
INIT-SINGLE-SOURCE($V, s$)
for each vertex $u$ in topological order do
  for each vertex $v \in Adj[u]$ do
    RELAX($u, v, w$)

Time: $(V + E)$.

Correctness:
Because we process vertices in topologically sorted order, edges of any path must be relaxed in order of appearance in the path.
⇒ Edges on any shortest path are relaxed in order.
⇒ By path-relaxation property, correct.
Dijkstra’s algorithm

- No negative-weight edges.
- Weighted version of BFS:
  - Instead of a FIFO queue, uses a priority queue.
  - Keys are shortest-path weights ($d[v]$).
- Have two sets of vertices:
  - $S$ = vertices whose final shortest-path weights are determined,
  - $Q$ = priority queue = $V - S$.
- Similar Prim’s algorithm, but computing $d[v]$, and using shortest-path weights as keys.
- Greedy choice: At each step we choose the light edge.
Dijkstra’s algorithm

DIJKSTRA(\(V, E, w, s\))

INIT-SINGLE-SOURCE(\(V, s\))

\(S \leftarrow \emptyset\)

\(Q \leftarrow V\)

while \(Q \neq \emptyset\) do

\(u \leftarrow\) EXTRACT-MIN(\(Q\))

\(S \leftarrow S \cup \{u\}\)

for each vertex \(v \in Adj[u]\) do

\(\text{RELAX}(u,v,w)\)
Example

\[
\begin{align*}
Q & = \begin{bmatrix}
    s & t & y & x & z \\
\end{bmatrix} \\
\end{align*}
\]
Example

Q

y t x z
Example

```
Q
```

```
t  x  z
```
Example
Example
Example
Correctness

Loop invariant:
At the start of each iteration of the while loop,
\( d[v] = \delta(s,v) \) for all \( v \in S \).

Initialization:
Initially, \( S = \emptyset \), so trivially true.

Termination:
At end, \( Q=\emptyset \Rightarrow S = V \Rightarrow d[v] = \delta(s,v) \) for all \( v \in V \).

Maintenance:
Show that \( d[u] = \delta(s,u) \) when \( u \) is added to \( S \) in each iteration.
Correctness (cont’d)

Show that \( d[u] = \delta(s,u) \) when \( u \) is added to \( S \) in each iteration.

Suppose there exists \( u \) such that \( d[u] \neq \delta(s,u) \).

Let \( u \) be the first vertex for which \( d[u] \neq \delta(s,u) \) when \( u \) is added to \( S \).

- \( u \neq s \), since \( d[s] = \delta(s,s) = 0 \).
- Therefore, \( s \in S \), so \( S \neq \emptyset \).
- There must be some path \( s \rightsquigarrow u \). Otherwise \( d[u] = \delta(s,u) = \infty \) by no-path property.
- So, there is a path \( s \rightsquigarrow u \). Thus, there is a shortest \( p \) path \( s \rightsquigarrow u \).
Correctness (cont’d)

Show that $d[u] = \delta(s,u)$ when $u$ is added to $S$ in each iteration.

Just before $u$ is added to $S$, the path $p$ connects a vertex in $S$ (i.e., $s$) to a vertex in $V - S$ (i.e., $u$).

Let $y$ be the first vertex along $p$ that is in $V - S$, and let $x \in S$ be $y$’s predecessor.

Decompose $p$ into $s \sim x \rightarrow y \sim u$. 
Correctness (cont’d)

Claim 1: \( d[y] = \delta(s, y) \) when \( u \) is added to \( S \).

Proof:
\( x \in S \) and \( u \) is the first vertex such that \( d[u] \neq \delta(s, u) \) when \( u \) is added to \( S \) \( \Rightarrow \) \( d[x] = \delta(s, x) \) when \( x \) is added to \( S \).
But when \( x \) is added we relax the edge \( (x, y) \), so by the convergence property, \( d[y] = \delta(s, y) \).
Correctness (cont’d)

Show that $d[u] = \delta(s, u)$ when $u$ is added to $S$ in each iteration.

Now, we can get a contradiction to $d[u] \neq \delta(s, u)$:

$y$ is on shortest path $p$ ($s \leadsto u$), and all edge weights are nonnegative. 
$\Rightarrow \delta(s, y) \leq \delta(s, u)$

Then by Claim 1, we have $d[y] = \delta(s, y)$

\[ \leq \delta(s, u) \]

\[ \leq d[u] \quad \text{(upper-bound property)} \]

In addition, $y$ and $u$ were in $Q$ when we chose $u$, thus $d[u] \leq d[y]$.

We have $d[y] \leq d[u] \& d[u] \leq d[y] \Rightarrow d[u] = d[y]$.

Therefore, $d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u] = d[y]$

Contradicts assumption that $d[u] \neq \delta(s, u)$. ■
Analysis

Like Prim’s algorithm, it depends on implementation of priority queue.

If binary heap, each operation takes $O(lg V)$ time
$\Rightarrow O(E \ lg \ V)$.

Note: We can achieve $O(V \ lg \ V + E)$ with Fibonacci heaps.