COMP251: Greedy algorithms

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Based on (Cormen et al., 2002)

Based on slides from D. Plaisted (UNC) & (goodrich & Tamassia, 2009)
Overview

• Algorithm design technique to solve optimization problems.

• Problems exhibit optimal substructure.

• Idea (the greedy choice):
  – When we have a choice to make, make the one that looks best right now.
  – Make a locally optimal choice in hope of getting a globally optimal solution.
Greedy Strategy

The choice that seems best at the moment is the one we go with.

– Prove that when there is a choice to make, one of the optimal choices is the greedy choice. Therefore, it is always safe to make the greedy choice.

– Show that all but one of the sub-problems resulting from the greedy choice are empty.
Activity-selection Problem

- **Input:** Set $S$ of $n$ activities, $a_1, a_2, \ldots, a_n$.
  - $s_i =$ start time of activity $i$.
  - $f_i =$ finish time of activity $i$.

- **Output:** Subset $A$ of maximum number of compatible activities.
  - 2 activities are compatible, if their intervals do not overlap.

Example:
Activity-selection Problem

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
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</table>

Activities sorted by finishing time.

Optimal compatible set: \{ $a_1$, $a_3$, $a_5$ \}
Optimal Substructure

• Assume activities are sorted by finishing times.

• Suppose an optimal solution includes activity $a_k$. This solution is obtained from:
  
  – An optimal selection of $a_1, \ldots, a_{k-1}$ activities compatible with one another, and that finish before $a_k$ starts.

  – An optimal solution of $a_{k+1}, \ldots, a_n$ activities compatible with one another, and that start after $a_k$ finishes.
Optimal Substructure

- Let $S_{ij} =$ subset of activities in $S$ that start after $a_i$ finishes and finish before $a_j$ starts.

$$S_{ij} = \left\{ a_k \in S : \forall i, j \quad f_i \leq s_k < f_k \leq s_j \right\}$$

- $A_{ij} =$ optimal solution to $S_{ij}$

- $A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$
Recursive Solution

• Subproblem: Selecting the maximum number of mutually compatible activities from $S_{ij}$.
• Let $c[i, j] =$ size of maximum-size subset of mutually compatible activities in $S_{ij}$.

Recursive solution: $c[i, j] = \begin{cases} 
0 & \text{if } S_{ij} = \emptyset \\
\max_{k < i < j \text{ and } a_k \in S_{ij}} \{c[i, k] + c[k, j] + 1\} & \text{if } S_{ij} \neq \emptyset 
\end{cases}$

Note: We do not know (yet) which k to use for the optimal solution.
Greedy choice

Theorem:
Let $S_{ij} \neq \emptyset$, and let $a_m$ be the activity in $S_{ij}$ with the earliest finish time $f_m = \min\{f_k : a_k \in S_{ij}\}$. Then:

1. $a_m$ is used in some maximum-size subset of mutually compatible activities of $S_{ij}$.
2. $S_{im} = \emptyset$, so that choosing $a_m$ leaves $S_{mj}$ as the only nonempty subproblem.
Greedy choice

Proof:

(1) \(a_m\) is used in some maximum-size subset of mutually compatible activities of \(S_{ij}\).

- Let \(A_{ij}\) be a maximum-size subset of mutually compatible activities in \(S_{ij}\) (i.e. an optimal solution of \(S_{ij}\)).
- Order activities in \(A_{ij}\) in monotonically increasing order of finish time, and let \(a_k\) be the first activity in \(A_{ij}\).
- If \(a_k = a_m\) ⇒ done.
- Otherwise, let \(A'_{ij} = A_{ij} - \{a_k\} U \{a_m\}\)
- \(A'_{ij}\) is valid because \(a_m\) finishes before \(a_k\)
- Since \(|A_{ij}| = |A'_{ij}|\) and \(A_{ij}\) maximal ⇒ \(A'_{ij}\) maximal too.
Greedy choice

Proof:

(2) $S_{im} = \emptyset$, so that choosing $a_m$ leaves $S_{mj}$ as the only nonempty subproblem.

If there is $a_k \in S_{im}$ then $f_i \leq s_k < f_k \leq s_m < f_m \Rightarrow f_k < f_m$ which contradicts the hypothesis that $a_m$ has the earliest finishing time.
# Greedy choice

<table>
<thead>
<tr>
<th></th>
<th>Before theorem</th>
<th>After theorem</th>
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<tbody>
<tr>
<td># subproblems in</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>optimal solution</td>
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<tr>
<td># choices to consider</td>
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We can now solve the problem $S_{ij}$ top-down:

- Choose $a_m \in S_{ij}$ with the earliest finish time (greedy choice).
- Solve $S_{mj}$. 
Activity-selection Problem

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Activities sorted by finishing time.
### Activity-selection Problem

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Activities sorted by finishing time.
Recursive Algorithm

Recursive-Activity-Selector \((s, f, i, n)\)

1. \(m \leftarrow i+1\)
2. \(\text{while } m \leq n \text{ and } s_m < f_i \quad // \text{Find first activity in } S_{i,n+1}\)
3. \(\text{do } m \leftarrow m+1\)
4. \(\text{if } m \leq n\)
5. \(\text{then return } \{a_m\} \cup \) Recursive-Activity-Selector\((s, f, m, n)\)
6. \(\text{else return } \emptyset\)

Initial Call: Recursive-Activity-Selector \((s, f, 0, n+1)\)

Complexity: \(\Theta(n)\)

Note 1: We assume activities are already ordered by finishing time.
Note 2: Straightforward to convert the algorithm to an iterative one.
Typical Steps

• Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.
• Prove that there is always an optimal solution that makes the greedy choice (greedy choice is safe).
• Show that greedy choice and optimal solution to subproblem $\Rightarrow$ optimal solution to the problem.
• Make the greedy choice and solve top-down.
• You may have to preprocess input to put it into greedy order (e.g. sorting activities by finish time).
Elements of Greedy Algorithms

No general way to tell if a greedy algorithm is optimal, but two key ingredients are:

• Greedy-choice Property.
  – We can build a globally optimal solution by making a locally optimal (greedy) choice.

• Optimal Substructure.
Text Compression

• Given a string X, efficiently encode X into a smaller string Y (Saves memory and/or bandwidth)

  A → 0; B → 10; C → 110; D → 1110
  DDCB → 1110 1110 110 10 (13 bits)

  A → 1110; B → 110; C → 10; D → 0
  DDCA → 0 0 10 110 (7 bits)

• A good approach: **Huffman encoding**
  – Compute frequency f(c) for each character c.
  – Encode high-frequency characters with short code words
  – No code word is a prefix for another code
  – Use an optimal encoding tree to determine the code words
Encoding Tree Example

• A **code** is a mapping of each character of an alphabet to a binary code-word

• A **prefix code** is a binary code such that no code-word is the prefix of another code-word

• An **encoding tree** represents a prefix code
  – Each external node (leaf) stores a character
  – The code word of a character is given by the path from the root to the external node storing the character (0 for a left child and 1 for a right child)
Encoding Example

Initial string: $X = \text{acda}$

Encoded string: $Y = 00 \ 011 \ 10 \ 00$
Encoding Tree Optimization

• Given a text string $X$, we want to find a prefix code for the characters of $X$ that yields a small encoding for $X$
  – Rare characters should have long code-words
  – Frequent characters should have short code-words

• Example
  – $X =$ abracadabra
  – $T_1$ encodes $X$ into 29 bits
  – $T_2$ encodes $X$ into 24 bits
Example

$X = \text{abracadabra}$

Frequencies

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

```
\begin{array}{cccc}
a & b & c & d & r \\
5 & 2 & 1 & 1 & 2 \\
\end{array}
```
Extended Huffman Tree Example

String: a fast runner need never be afraid of the dark

<table>
<thead>
<tr>
<th>Character</th>
<th>a</th>
<th>b</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>h</th>
<th>i</th>
<th>k</th>
<th>n</th>
<th>o</th>
<th>r</th>
<th>s</th>
<th>t</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>1</td>
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Huffman’s Algorithm

• Given a string $X$, Huffman’s algorithm construct a prefix code that minimizes the size of the encoding of $X$.

• It runs in time $O(n + d \log d)$, where $n$ is the size of $X$ and $d$ is the number of distinct characters of $X$.

• A heap-based priority queue is used as an auxiliary structure.

Algorithm $\text{HuffmanEncoding}(X)$

Input string $X$ of size $n$

Output optimal encoding trie for $X$

$C \leftarrow \text{distinctCharacters}(X)$

$\text{computeFrequencies}(C, X)$

$Q \leftarrow$ new empty heap

for all $c \in C$

$T \leftarrow$ new single-node tree storing $c$

$Q.\text{insert}(\text{getFrequency}(c), T)$

while $Q.\text{size}() > 1$

$f_1 \leftarrow Q.\text{minKey}()$

$T_1 \leftarrow Q.\text{removeMin}()$

$f_2 \leftarrow Q.\text{minKey}()$

$T_2 \leftarrow Q.\text{removeMin}()$

$T \leftarrow \text{join}(T_1, T_2)$

$Q.\text{insert}(f_1 + f_2, T)$

return $Q.\text{removeMin}()$