COMP251: Randomized Algorithms

Jérôme Waldispühl
School of Computer Science
McGill University

Based on (Kleinberg & Tardos, 2006)
Algorithm Design Techniques

• Greedy Algorithms
• Dynamic Programming
• Divide-and-Conquer
• Network Flows
• Randomization
Randomization

**Principle:** Allow a fair coin flip in unit time.

**Why?** Can lead to simplest, fastest, or only known algorithm for a particular problem.

**Examples:**
- Quicksort
- Graph Algorithms
- Hashing
- Monte-Carlo integration
- Cryptography
Global Min Cut

**Definition:** Given a connected, undirected graph $G = (V, E)$, find a cut with minimum cardinality.

**Applications:**
- Partitioning items in database
- Identify clusters of related documents
- Network reliability
- TSP solver

**Network solution:**
- Replace every edge $(u, v)$ with 2 antiparallel edges $(u, v)$ & $(v, u)$
- Pick some vertex $s$, and compute min $s$-$v$ cut for each other vertex $v$.

**False Intuition:** Global min-cut is harder than min $s$-$t$ cut!
Contraction algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge \( e = (u, v) \) uniformly at random.
- **Contract** edge \( e \).
  - replace \( u \) and \( v \) by single new super-node \( w \)
  - preserve edges, updating endpoints of \( u \) and \( v \) to \( w \)
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes \( u_1 \) and \( v_1 \).
- Return the cut (all nodes that were contracted to form \( v_1 \)).
**Contraction algorithm.** [Karger 1995]

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Contraction Algorithm

**Contraction(V,E):**
While $|V| > 2$ do

Choose $e \in E$ uniformly at random

$G \leftarrow G - \{e\}$ // contract G

return $\{ \text{the only cut in } G \}$
Contraction algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$. 
($n = \vert V \vert$)

Pf. Consider a global min-cut $(A^*, B^*)$ of $G$.

- Let $F^*$ be edges with one endpoint in $A^*$ and the other in $B^*$.
- Let $k = \vert F^* \vert$ = size of min cut.
- In first step, algorithm contracts an edge in $F^*$ probability $k/\vert E \vert$.
- Every node has degree $\geq k$ since otherwise $(A^*, B^*)$ would not be a min-cut $\Rightarrow \vert E \vert \geq \frac{1}{2} k n$. $\iff \frac{k}{\vert E \vert} \leq \frac{2}{n}$
- Thus, algorithm contracts an edge in $F^*$ with probability $\leq 2/n$. 

\[ n = \vert V \vert \]
\[ k \leq 2 \]
\[ \frac{k}{\vert E \vert} \leq \frac{2}{n} \]
Contraction algorithm

**Claim.** The contraction algorithm returns a min cut with prob $\geq 2 / n^2$.

**Pf.** Consider a global min-cut $(A^*, B^*)$ of $G$.
- Let $F^*$ be edges with one endpoint in $A^*$ and the other in $B^*$.
- Let $k = |F^*| = \text{size of min cut}$.
- Let $G'$ be graph after $j$ iterations. There are $n' = n - j$ supernodes.
- Suppose no edge in $F^*$ has been contracted. The min-cut in $G'$ is still $k$.
- Since value of min-cut is $k$, $|E'| \geq \frac{1}{2} kn'$.
- Thus, algorithm contracts an edge in $F^*$ with probability $\leq 2 / n'$.
- Let $E_j = \text{event that an edge in } F^* \text{ is not contracted in iteration } j$.

\[
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] = \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}]
\geq \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{3}\right)
= \left(\frac{n-2}{n}\right) \left(\frac{n-3}{n-1}\right) \cdots \left(\frac{2}{4}\right) \left(\frac{1}{3}\right)
= \frac{2}{n(n-1)}
\geq \frac{2}{n^2}
\]
Contraction algorithm

**Amplification.** To amplify the probability of success, run the contraction algorithm many times.

**Claim.** If we repeat the contraction algorithm $n^2 \ln n$ times, then the probability of failing to find the global min-cut is $\leq 1 / n^2$.

**Pf.** By independence, the probability of failure is at most

$$
(1 - \frac{2}{n^2})^{n^2 \ln n} = \left(1 - \frac{2}{n^2}\right)^{\frac{1}{2}n^2 \cdot 2 \ln n} \leq \left(e^{-1}\right)^{2 \ln n} = \frac{1}{n^2}
$$

\[ (1 - 1/x)^x \leq 1/e \]

with independent random choices,
Contraction algorithm: example execution

trial 1

trial 2

trial 3

trial 4

trial 5 (finds min cut)

trial 6

Reference: Thore Husfeldt
Global min cut: context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time. Where $m = |E|$. Overall complexity $O(n^2 m \log n)$

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm
SAT formula

Boolean variables: $x_i$ or $\overline{x_i}$

Clause: $C_i = x_{i_1} \lor x_{i_2} \lor x_{i_3} \lor \cdots \lor x_{i_k}$

Size of a clause = number of variables in the clause

SAT formula: $\land_i C_i$ (a formula is satisfied if all clauses are true)

Example (3 SAT formula):

$$(x_1 \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_4)$$

$x_1 = 1; x_2 = 1; x_3 = 1; x_4 = 1 \Rightarrow$ True

$x_1 = 1; x_2 = 0; x_3 = 1; x_4 = 0 \Rightarrow$ False
Maximum 3-satisfiability

**Maximum 3-satisfiability.** Given a 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

\[
\begin{align*}
C_1 &= x_2 \lor \overline{x}_3 \lor \overline{x}_4 \\
C_2 &= x_2 \lor x_3 \lor \overline{x}_4 \\
C_3 &= \overline{x}_1 \lor x_2 \lor x_4 \\
C_4 &= \overline{x}_1 \lor \overline{x}_2 \lor x_3 \\
C_5 &= x_1 \lor x_2 \lor \overline{x}_4
\end{align*}
\]

**Remark.** NP-hard search problem.

**Simple idea.** Flip a coin, and set each variable true with probability \( \frac{1}{2} \), independently for each variable.
Maximum 3-satisfiability: analysis

Claim. Given a 3-SAT formula with $k$ clauses, the expected number of clauses satisfied by a random assignment is $7k/8$.

Pf. Consider random variable $Z_j = \begin{cases} 
1 & \text{if clause } C_j \text{ is satisfied} \\
0 & \text{otherwise.}
\end{cases}
$

- Let $Z =$ weight of clauses satisfied by assignment $Z_j$.

\[
E[Z] = \sum_{j=1}^{k} E[Z_j]
\]

linearity of expectation

\[
= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}]
\]

\[
= \frac{7}{8} k
\]
The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a $7/8$ fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. □

Probabilistic method. [Paul Erdös] Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability!
Maximum 3-satisfiability: analysis

Q. Can we turn this idea into a 7/8-approximation algorithm?
A. Yes (but a random variable can almost always be below its mean).

Lemma. The probability that a random assignment satisfies $\geq 7k / 8$ clauses is at least $1 / (8k)$.

Pf. Let $p_j$ be probability that exactly $j$ clauses are satisfied; let $p$ be probability that $\geq 7k / 8$ clauses are satisfied.

\[
\frac{7}{8}k = E[Z] = \sum_{j \geq 0} j p_j
\]

\[
= \sum_{j < 7k/8} j p_j + \sum_{j \geq 7k/8} j p_j
\]

\[
\leq \left( \frac{7}{8} - \frac{1}{8} \right) \sum_{j < 7k/8} p_j + k \sum_{j \geq 7k/8} p_j
\]

\[
\leq \left( \frac{7}{8} k - \frac{1}{8} \right) \cdot 1 + k p
\]

Rearranging terms yields $p \geq 1 / (8k)$. □
Maximum 3-satisfiability: analysis

**Johnson's algorithm.** Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

**Theorem.** Johnson's algorithm is a $7/8$-approximation algorithm.

**Pf.** By previous lemma, each iteration succeeds with probability $\geq 1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. □
Monte Carlo vs. Las Vegas algorithms

**Monte Carlo.** Guaranteed to run in poly-time, likely to find correct answer.  
**Ex:** Contraction algorithm for global min cut.

**Las Vegas.** Guaranteed to find correct answer, likely to run in poly-time.  
**Ex:** Randomized quicksort, Johnson's MAX-3-SAT algorithm.

**Remark.** Can always convert a Las Vegas algorithm into Monte Carlo, 
but no known method (in general) to convert the other way.