COMP250: Priority queue ADT, Heaps

Lecture 23

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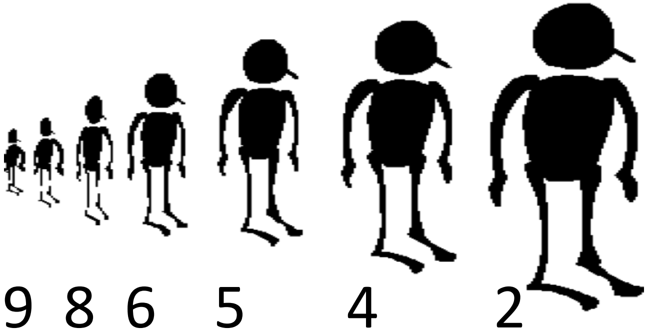
Priority queue ADT

- Like a dictionary, a priority queue stores a set of pairs (key, info)
- The rank of an object depends on its priority (key)



Rear of queue

Front of queue

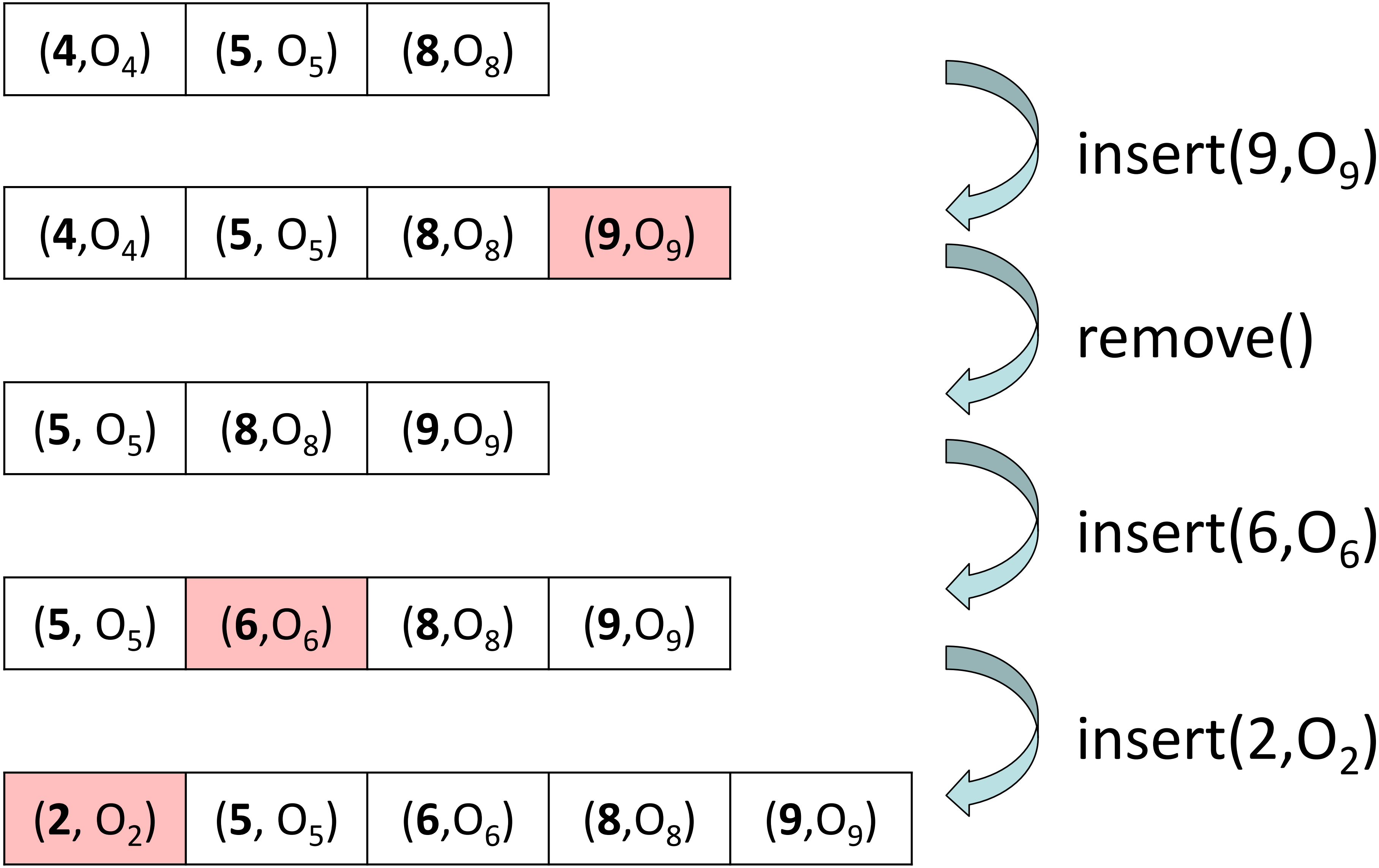
key: 9 8 6 5 4 2

- Allows only access to
  - Object findMin() //returns info of smallest key
  - Object removeMin() // removes smallest key
  - void insert(key k, info i) // inserts pair
- Applications: customers in line, Data compression, Graph searching, Artificial intelligence…

Outline

- Priority queues
- Heaps
- Operations on heaps
- Array-based implementation of heaps
- HeapSort

Priority queue ADT (as sorted array)



(**4**,O_4) | (**5**, O_5) | (**8**,O_8)

insert(9,O_9)

(**4**,O_4) | (**5**, O_5) | (**8**,O_8) | (**9**,O_9)

remove()

(**5**, O_5) | (**8**,O_8) | (**9**,O_9)

insert(6,O_6)

(**5**, O_5) | (**6**,O_6) | (**8**,O_8) | (**9**,O_9)

insert(2,O_2)

(**2**, O_2) | (**5**, O_5) | (**6**,O_6) | (**8**,O_8) | (**9**,O_9)

Array implementation

Unsorted array of pairs (key, info)

findMin(): Need to scan array O(n)

insert(key, info): Put new object at the end O(1)

removeMin(): First, findMin, then shift array O(n)

Sorted array of pairs (key, info)

findMin(): Return first element O(1)

insert(key, info):

Use binary-search to find position of insertion. O(log n)

Then shift array to make space. O(n)

removeMin(): findMin, then remove head O(1)

(Note: using rotating arrays)

Doubly-linked list implementation

Using a sorted doubly-linked list of pairs (key, info)

findMin(): Return first element O(1)

insert(key, info):

- First, find location of insertion.
- Binary Search?
- No. Too slow on linked list.
- Instead, we scan an array O(n)
- Then insertion is easy O(1)

removeMin(): Remove first element of list O(1)

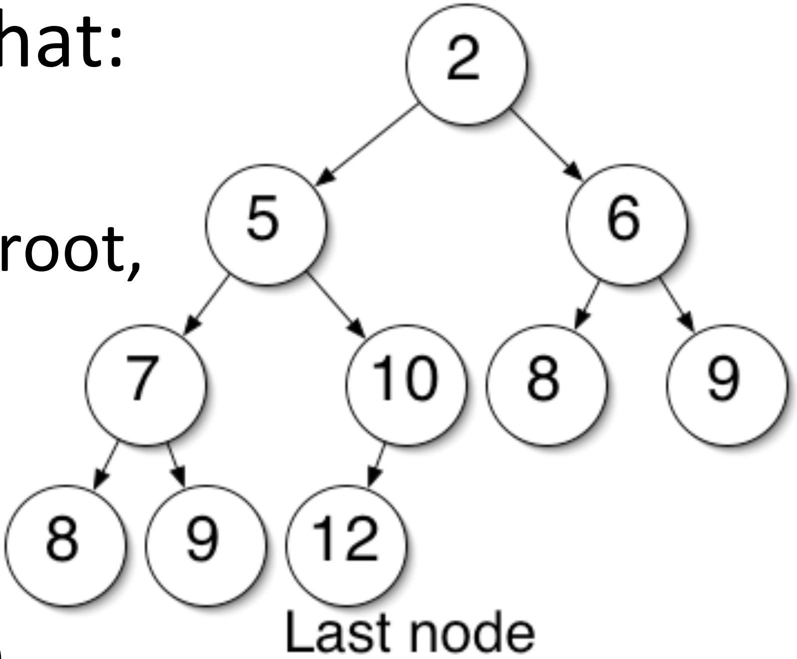
Heap data structure

- A heap is a data structure that implements a priority queue:
  - findMin(): O(1)
  - removeMin(): O(log n)
  - insert(key, info): O(log n)
- A heap is based on a binary tree, but with a different property than a binary search tree
- **heap ≠ binary *search* tree**

Heap - Definition

- A **heap** is a binary tree such that:
  - For any node *n* other than the root,

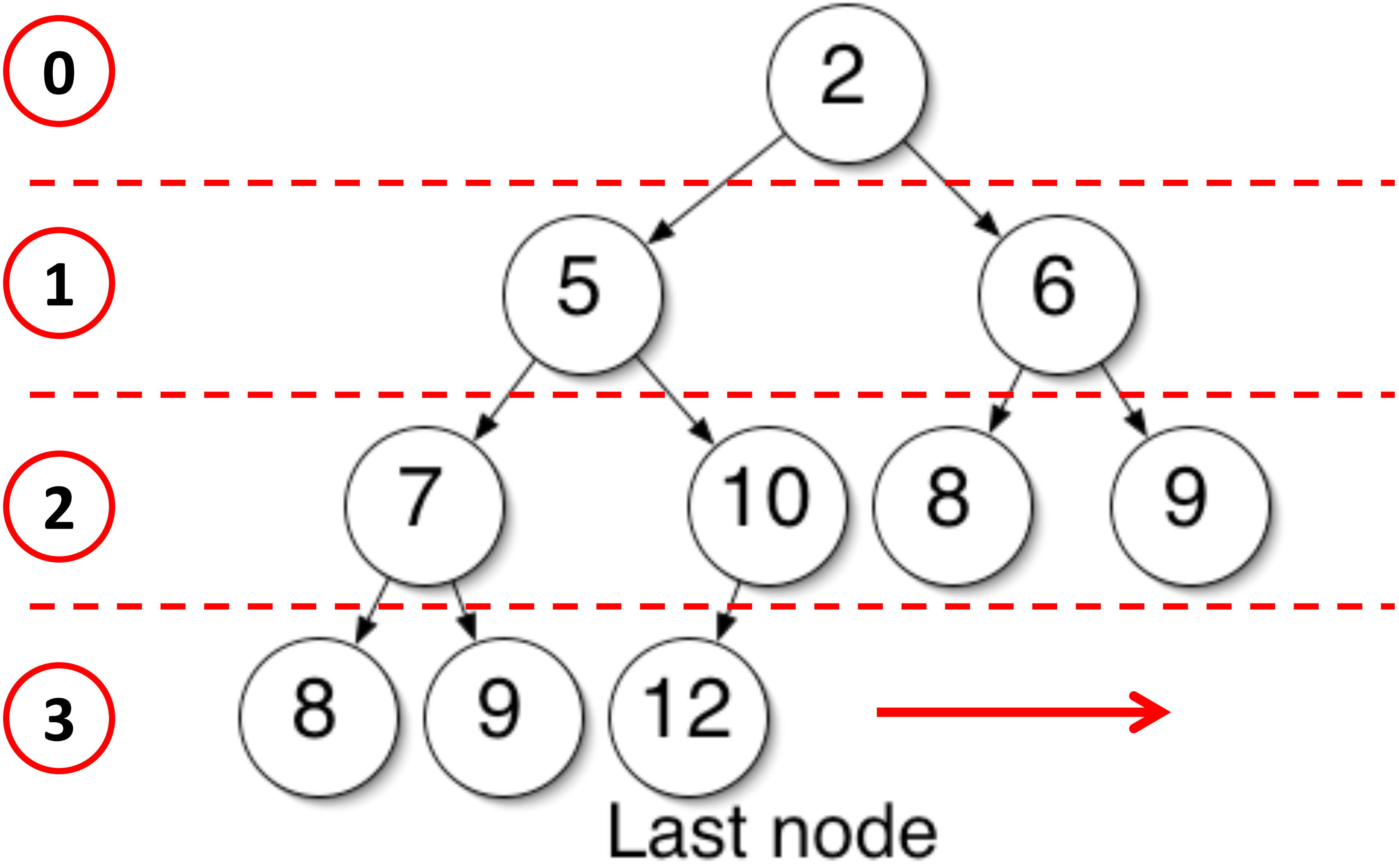
    key(n) $\geq$ key(parent(n))



  - Let h be the height of the heap
    - First h-1 levels are full:

      For i = 0,…,h-1, there are 2^i nodes of depth i
    - At depth h, the leaves are packed on the left side of the tree

Heap - Example



Height of a heap

What is the maximum number of nodes that fits in a heap of height h?

$$\sum_{k=0}^{h} 2^k = 2^{h+1} - 1$$

What is the minimum number?

$$(2^h - 1) + 1 = 2^h$$

Thus, the height of a heap with n nodes is:

$$\lfloor \log(n) \rfloor$$

Heaps: findMin()

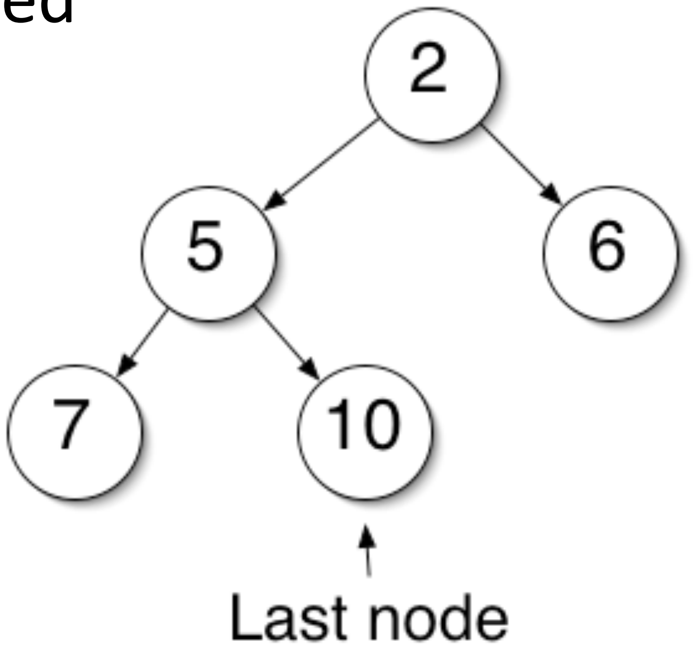
The minimum key is always at the root of the heap!

Heaps: Insert

Insert(key k, info i). Two steps:

1. Find the left-most unoccupied node and insert (k,i) there temporarily.

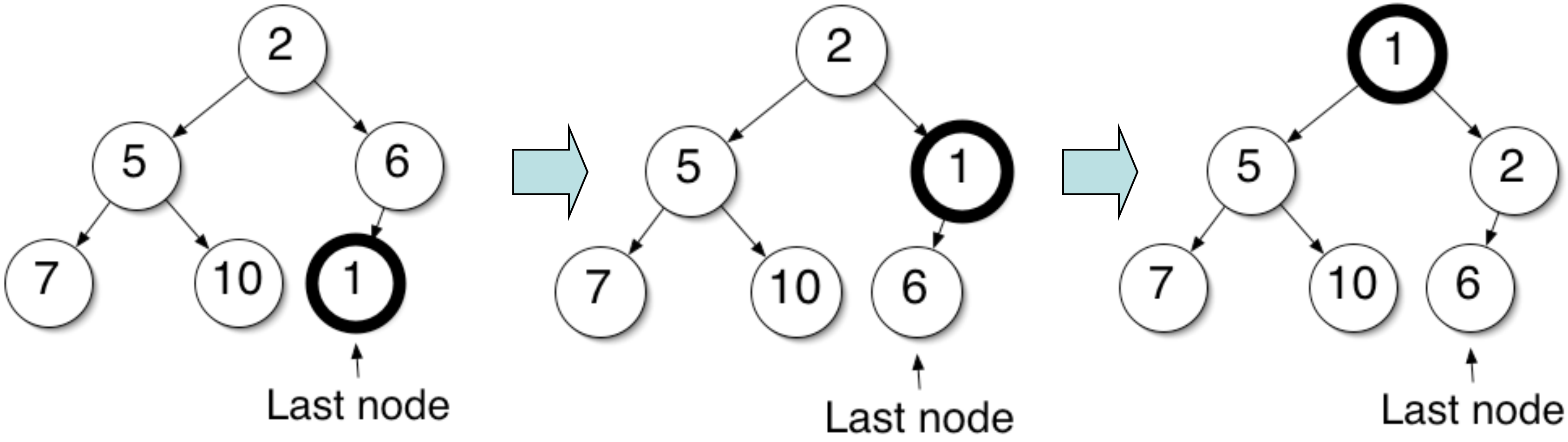
2. Restore the heap-order property (see next)



Heaps: Bubbling-up

Restoring the heap-order property:

- Keep swapping new node with its parent as long as its key is smaller than its parent's key



Running time? $O(h) = O(\log(n))$

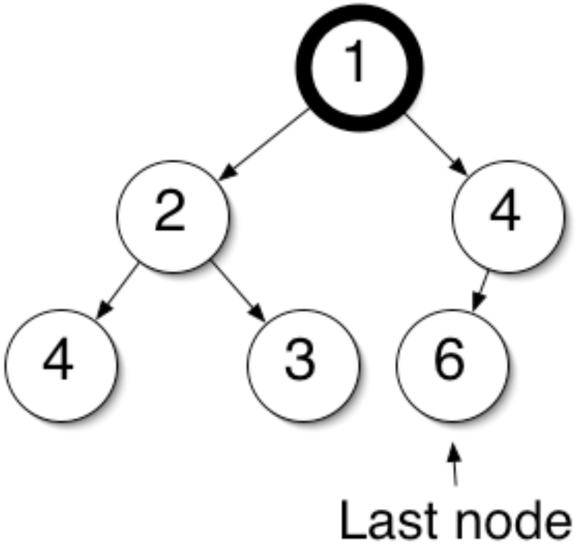
Insert pseudocode

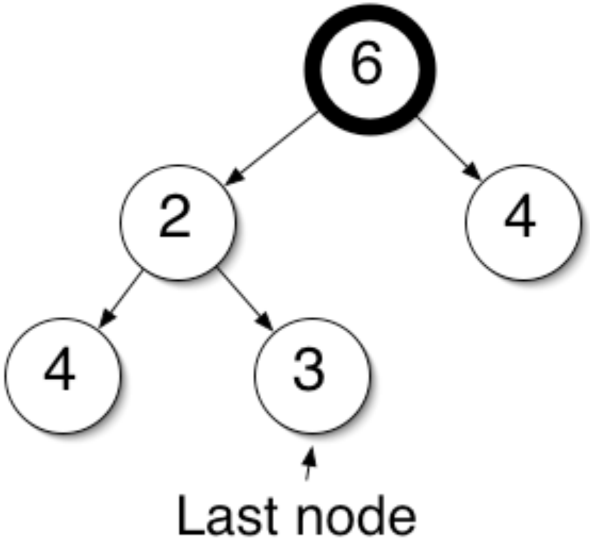
```
Algorithm insert(key k, info i)
Input: Key k and info i to add to the heap
Output: (k,i) is added

lastNode ← nextAvailableNode(lastNode)
lastNode.key ← k,
lastNode.info ← i
n ← lastnode
while (n.getParent()!=null and
    n.getParent().key > k) do
    swap (n.getParent(), n)
```

Heaps: RemoveMin()

- The minimum key is always at the root of the heap!
- Replace the root with last node



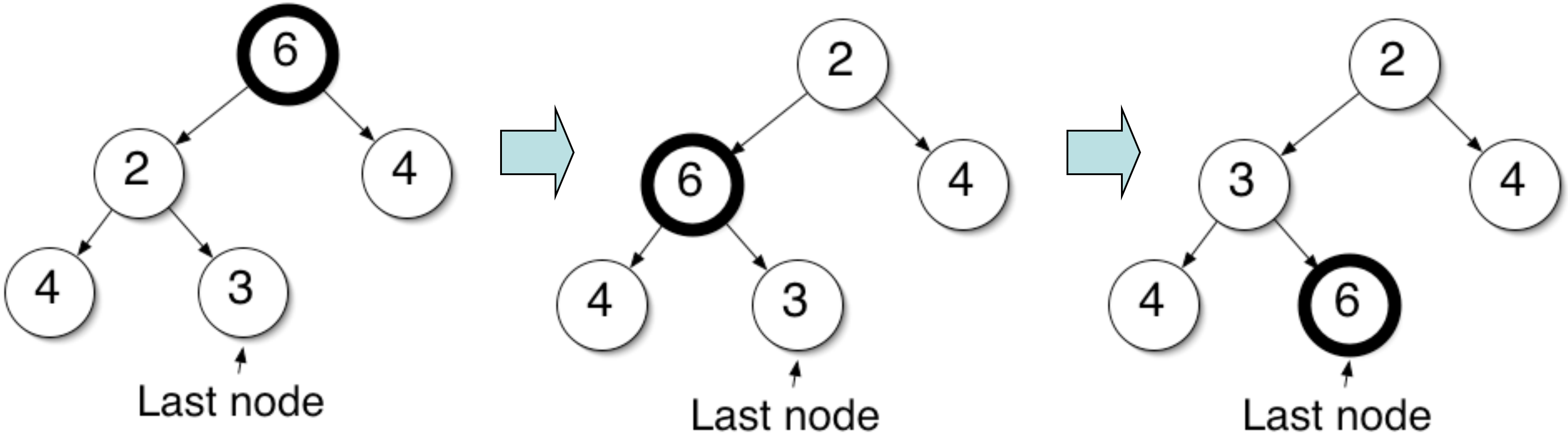


- Restore heap-order property (see next)

Heaps: Bubbling-down

Restoring the heap-order property:

- Keep swapping the node with its smallest child as long as the node's key is larger than its child's key



Running time? $O(h) = O(\log(n))$

removeMin pseudocode

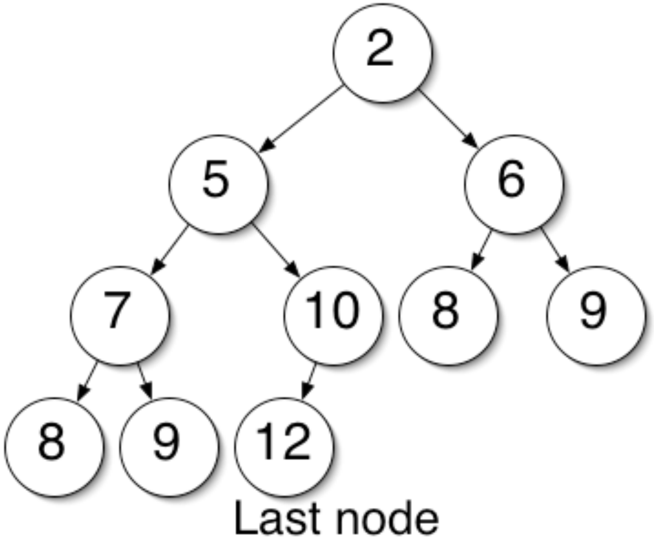
```
Algorithm removeMin()
Input: The heap
Output: A new heap where the node at the top of the
   input heap has been removed.

swap(lastNode, root)
Update lastNode
n ← root
while (n.key > min(n.getLeftChild().key,
                   n.getRightChild().key)) do
   if (n.getLeftChild().key <
   n.getRightChild().key)
   then
      swap(n, n.getLeftChild)
   else swap(n, n.getRightChild)
```

Array representation of heaps

- A heap with n keys can be stored in an array of length n+1:

0	1	2	3	4	5	6	7	8	9	10
-	2	5	6	7	10	8	9	8	9	12



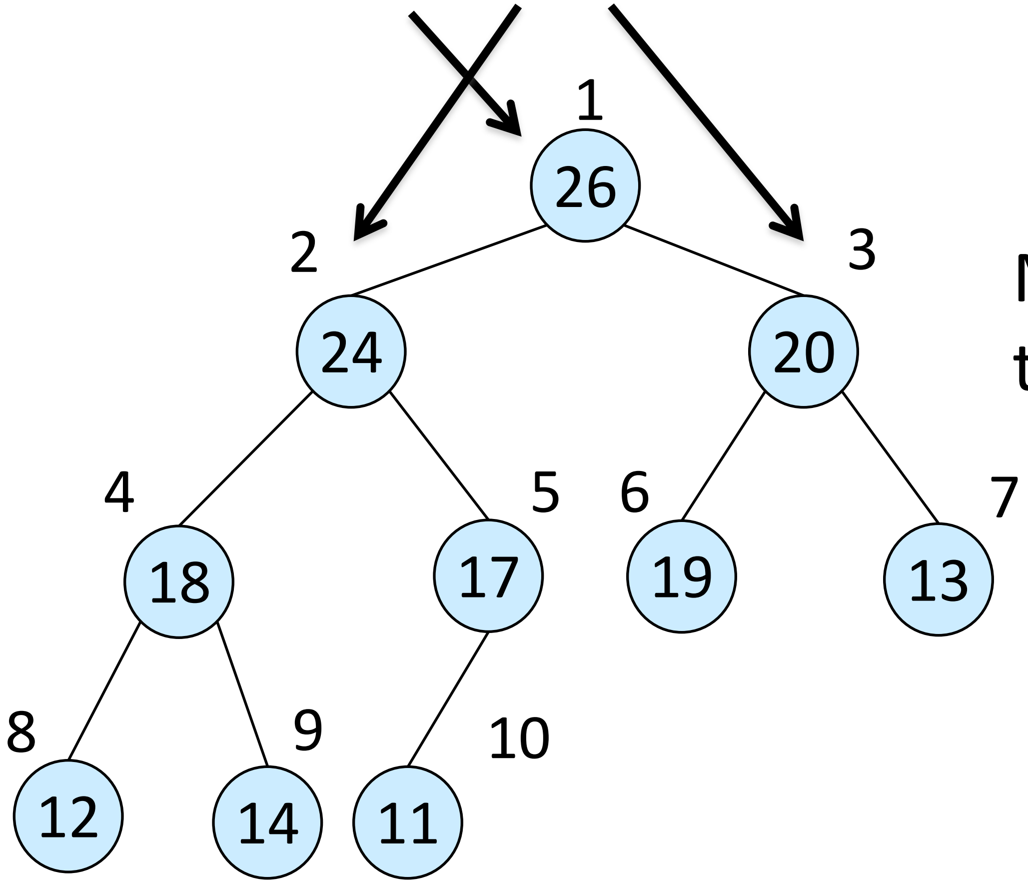
- For a node at index i,
  - The parent (if any) is at index $\lfloor i/2 \rfloor$
  - The left child is at index 2*i
  - The right child is at index 2*i + 1

- lastNode is the first empty cell of the array. To update it, either add or subtract one

Heaps as arrays

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

Max-heap as an array.



Map from array elements to tree nodes and vice versa

- Root – $A[1]$
- Left[i] – $A[2i]$
- Right[i] – $A[2i+1]$
- Parent[i] – $A[\lfloor i/2 \rfloor]$

HeapSort

```
Algorithm heapSort(array A[0…n-1])
Heap h ← new Heap()
for i=0 to n-1 do
  h.insert(A[i])
for i=0 to n-1 do
  A[i] ← h.removeMin()
```

Running time: O(n log n) in worst-case

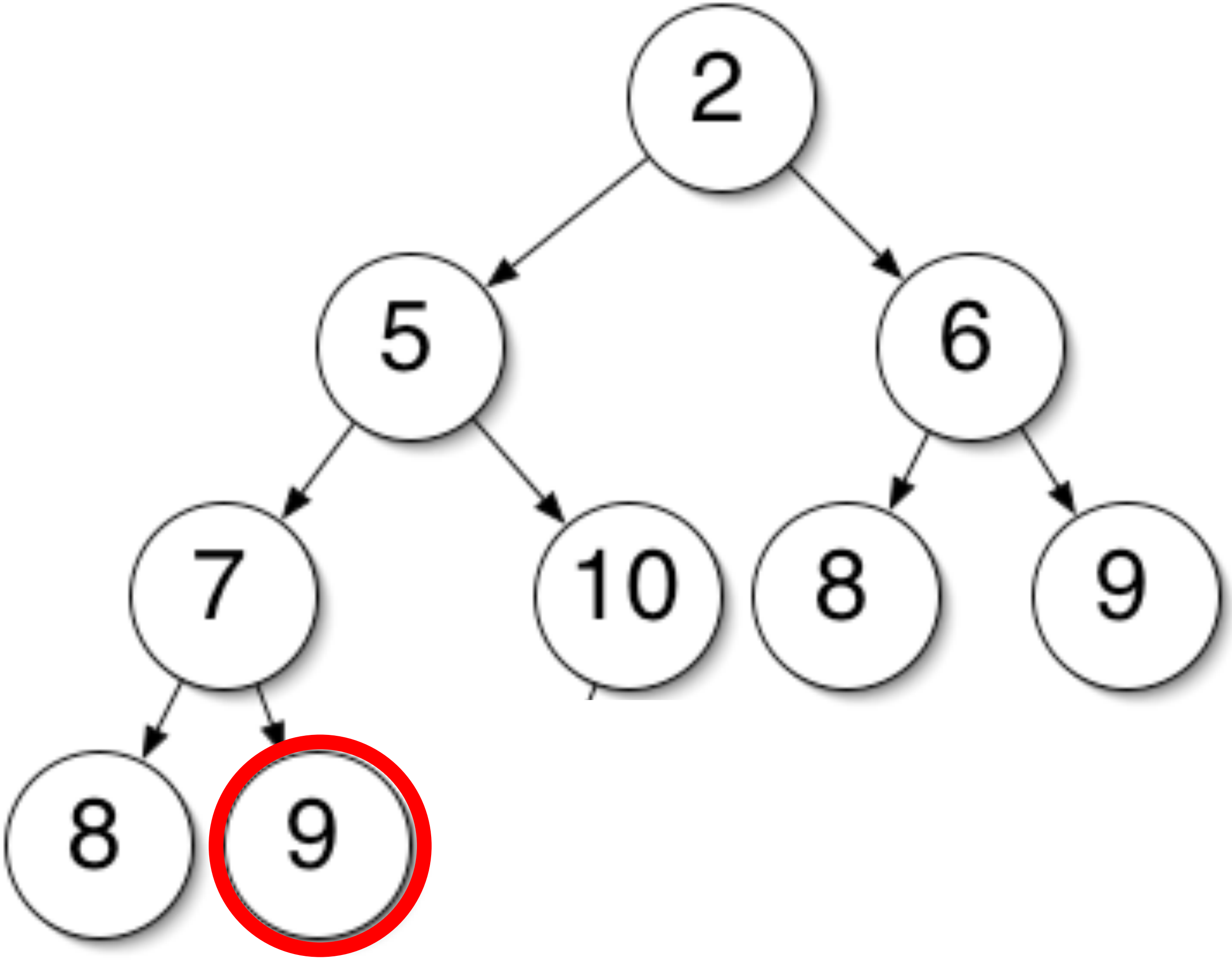
Easy to do in-place: Just use the array A to store the heap

Note: We can optimize the procedure to construct the initial heap (More in COMP251)

Supplement

Implementating nextAvailableNode

NextAvailableNode - Example



Finding nextAvailableNode

nextAvailableNode(lastNode) finds the location where the next node should be inserted. It runs in time O(n).

```
n = lastNode;
while (n==(n.parent).rightChild && n.parent!=null) do
      n = n.parent
if ( n.parent == null ) then
      return left child of the leftmost node of tree
else
      n = n.parent             // go up one more level
      if ( n has no right child) then
            return (right child of n)
      else
            n = n.rightChild   // go to right child
            while (n has a left child ) do
                  n = n.leftChild
            return (left child of n)
```