COMP250: Induction proofs.

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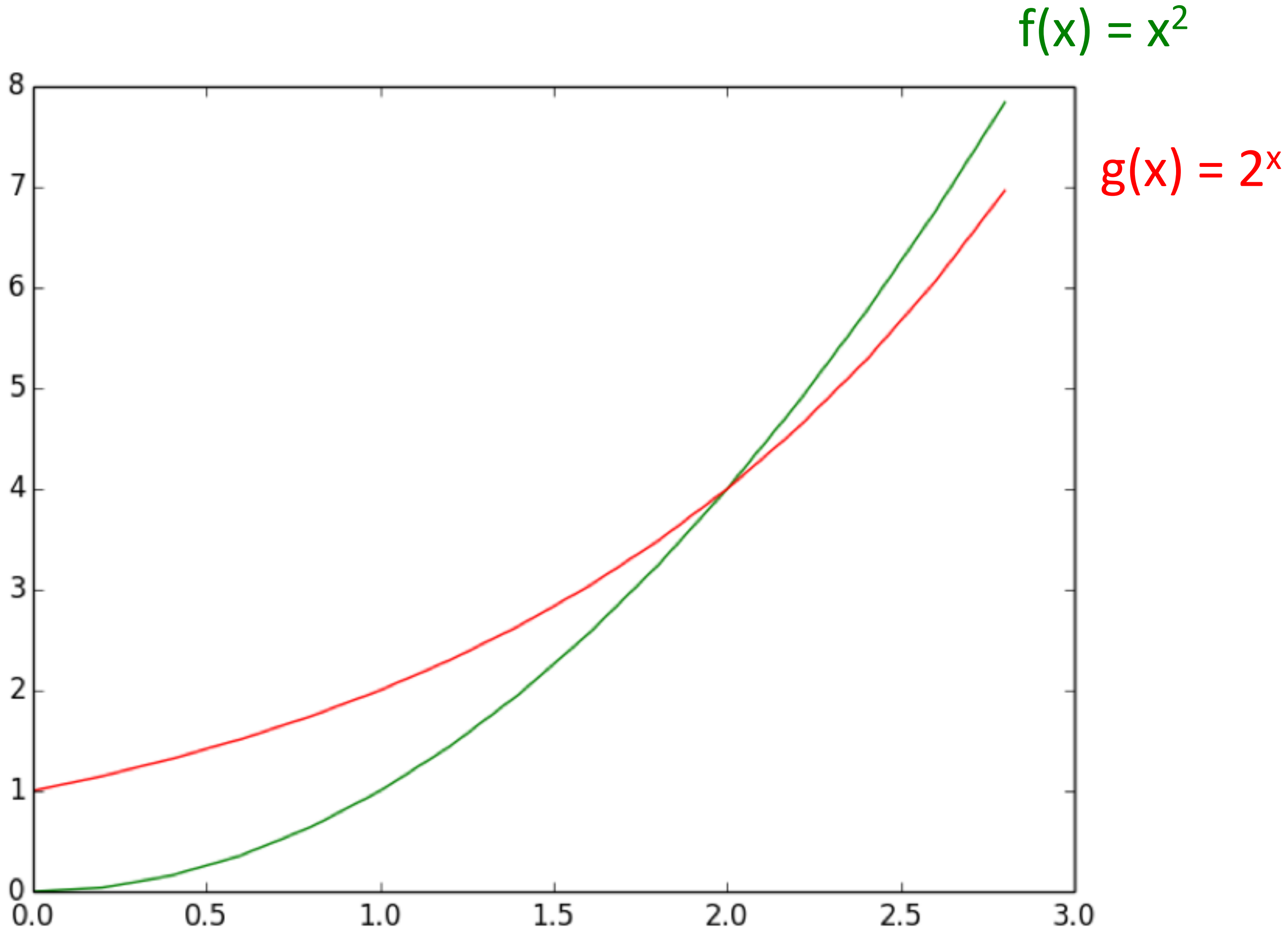
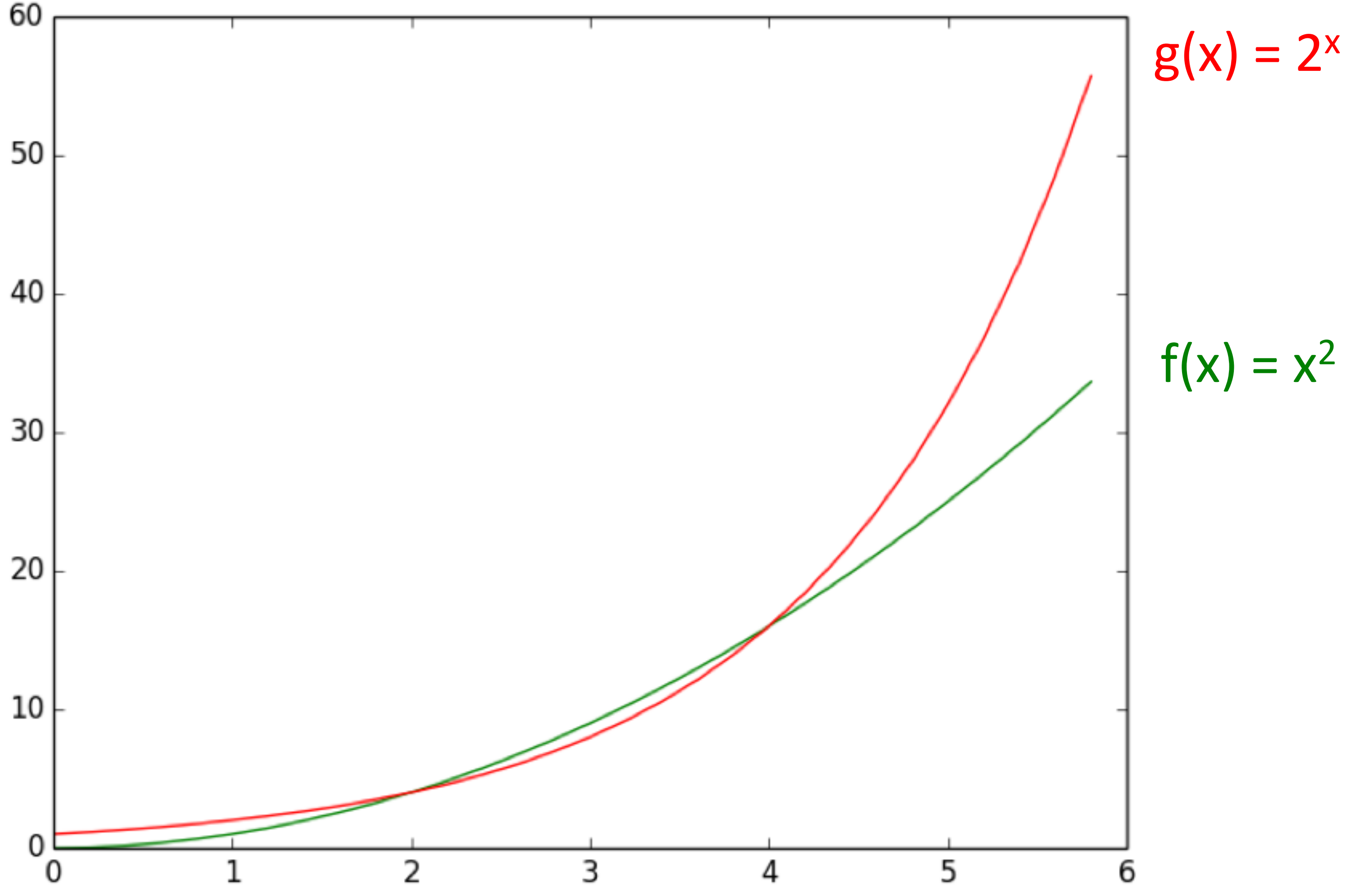
Based on slides from (Langer,2012)

$$for\ any \quad n \geq 2, \quad n^2 \geq 2^n \quad ?$$



$$for\ any \quad n \geq 5, \quad n^2 \leq 2^n \quad ?$$



Motivation

How to prove these?

$$for\, any\, n \geq 1, \quad 1 + 2 + 3 + 4 + \cdots + n = \frac{n \cdot (n+1)}{2}$$

$$for\, any\, n \geq 1, \quad 1 + 3 + 5 + 7 + \cdots + (2 \cdot n - 1) = n^2$$

$$for\, any\, n \geq 5, \quad n^2 \leq 2^n$$

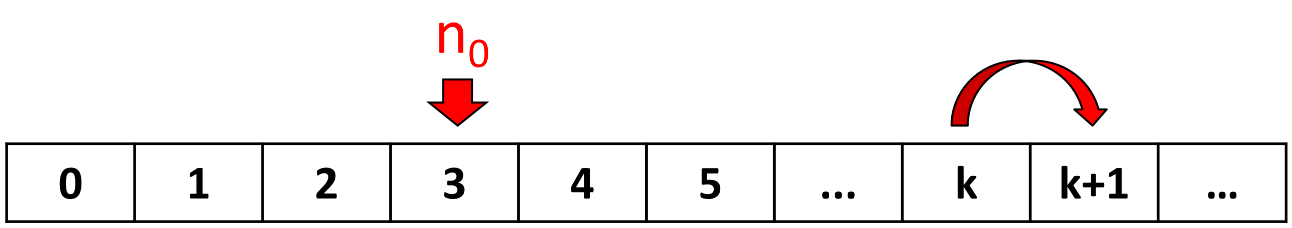
And in general, any statement of the form:
"for all $n \geq n_0$*, P(n)"* where *P(n) is some proposition.*

Mathematical induction

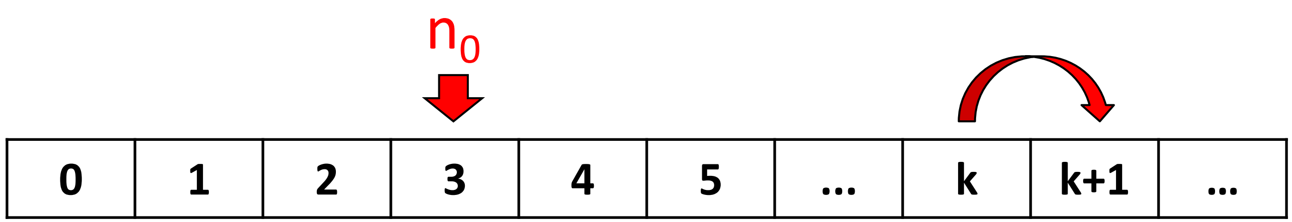
Many statement of the form *"for all* $n \geq n_0$, $P(n)$*"* can be proven with a logical argument call *mathematical induction.*

The proof has two components:

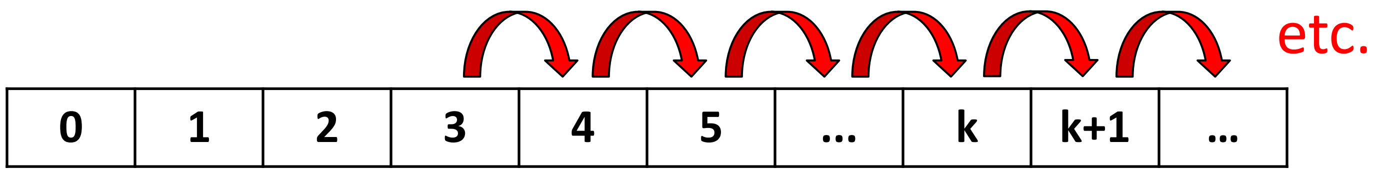
- **Base case**: $P(n_0)$
- **Induction step**: for any $n \geq n_0$, if $P(n)$ then $P(n+1)$



Principle



Implies



Example 1

Claim: $for\ any\ n \geq 1, \quad 1+2+3+4+\cdots+n = \frac{n \cdot (n+1)}{2}$

Proof:

- Base case: $n = 1 \qquad 1 = \frac{1 \cdot 2}{2}$ ✔
- Induction step:

$$for\ any\ k \geq 1, \quad if \quad 1+2+3+4+\cdots+k = \frac{k \cdot (k+1)}{2}$$

$$then \quad 1+2+3+4+\cdots+k+(k+1) = \frac{(k+1) \cdot (k+2)}{2}$$

Example 1

$$\textit{Assume} \quad 1+2+3+4+\cdots+k=\frac{k\cdot(k+1)}{2}$$

$$\textit{then} \quad 1+2+3+4+\cdots+k+(k+1)$$

$$=\frac{k\cdot(k+1)}{2}+(k+1)$$

Induction hypothesis

$$=\frac{k\cdot(k+1)+2\cdot(k+1)}{2}$$

$$=\frac{(k+2)\cdot(k+1)}{2}$$

Summary

Base case: P(1) ✔

Induction step: for any k≥1, if P(k) then P(k+1) ✔

Thus for all n≥1, P(n) □

Example 2

Claim: $for\ any\ n \geq 1, \quad 1+3+5+7+\cdots+(2 \cdot n-1)=n^2$

Proof:

- Base case: $n=1 \qquad 1=1^2$ ✔

- Induction step:

$$for\ any\ k \geq 1, \quad if \quad 1+3+5+7+\cdots+(2 \cdot k-1)=k^2$$

$$then \quad 1+3+5+7+\cdots+(2 \cdot (k+1)-1)=(k+1)^2$$

Example 2

$$\textit{Assume} \quad 1+3+5+7+\cdots+(2\cdot k-1)=k^2$$

$$\textit{then} \quad 1+3+5+7+\cdots+(2\cdot k-1)+(2\cdot(k+1)-1)$$

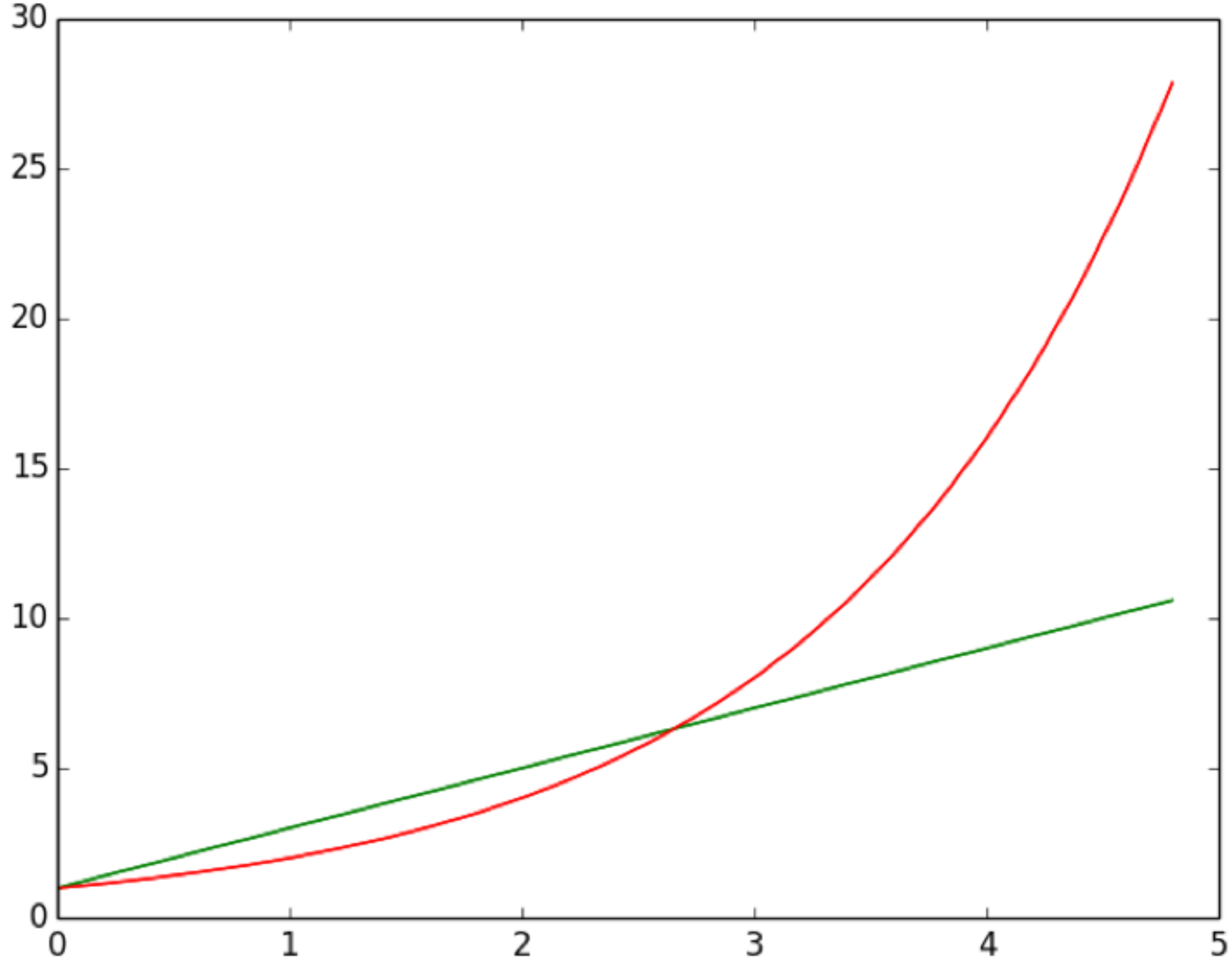
Induction hypothesis

$$=k^2+2\cdot(k+1)-1$$

$$=k^2+2\cdot k+1$$

$$=(k+1)^2$$

Example 3



$g(x) = 2^x$

$f(x) = 2x+1$

Example 3

Claim: $for\ any\ n \geq 3, \quad 2 \cdot n + 1 < 2^n$

Proof:

- Base case: $n = 3 \qquad 2 \cdot 3 + 1 = 7 < 2^3 = 8$ ✔
- Induction step:

$$for\ any\ k \geq 3, \quad if \quad 2 \cdot k + 1 < 2^k$$

$$then \quad 2 \cdot (k+1) + 1 < 2^{k+1}$$

Example 3

$$Assume \quad 2 \cdot k + 1 < 2^k$$

$$then \quad 2 \cdot (k+1) + 1$$

$$= 2 \cdot k + 2 + 1$$

$$< 2^k + 2$$

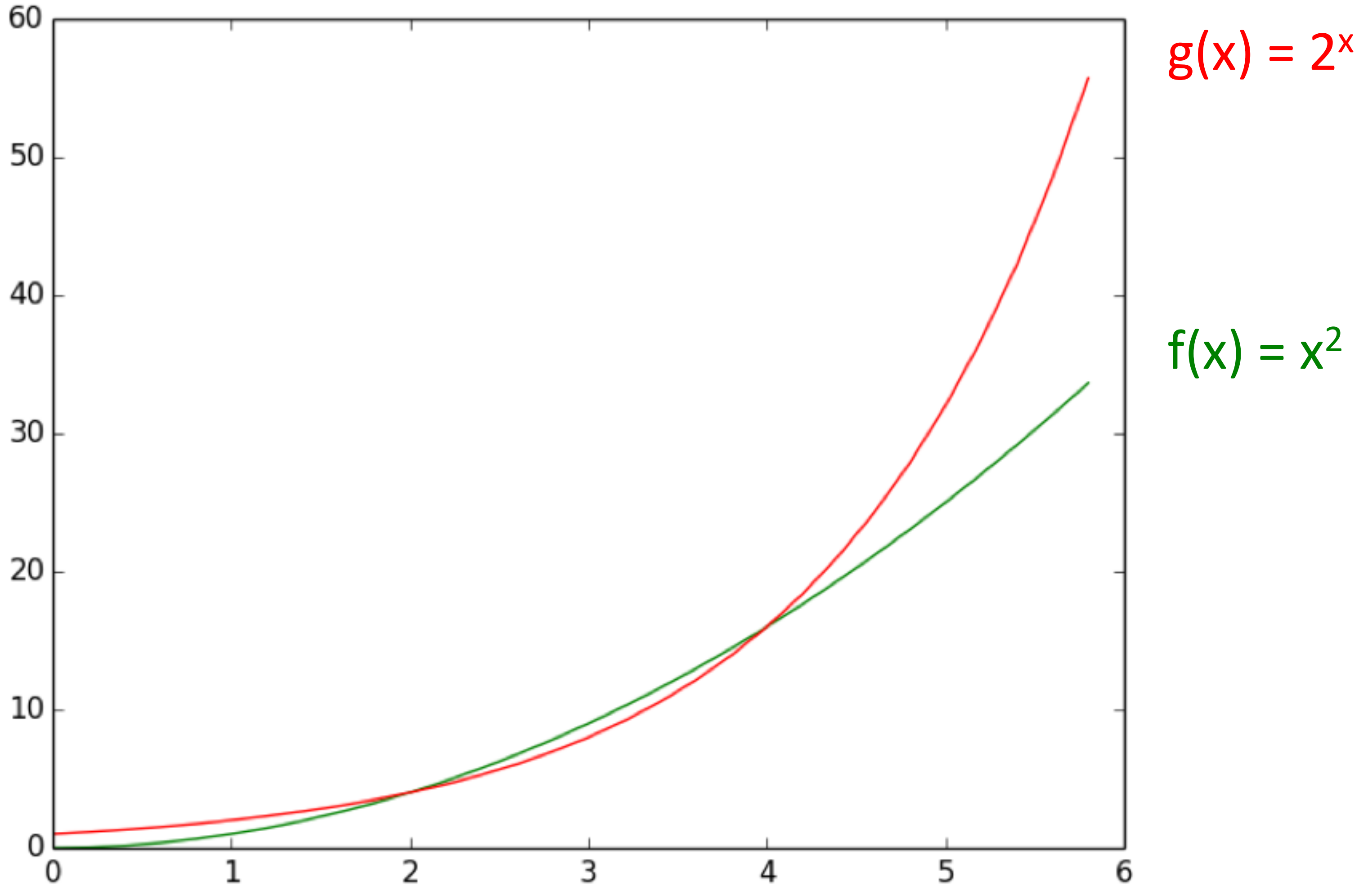
Induction hypothesis

$$\leq 2^k + 2^k \quad for\ k \geq 1$$

$$= 2^{k+1}$$

Stronger than we need, but that works!

Example 4



Example 4

Claim: $for\, any\, n \geq 5, \quad n^2 \leq 2^n$

Proof:

- Base case: $n = 5 \qquad 25 \leq 32$ ✔

- Induction step:

$$for\, any\, k \geq 5, \quad if \quad k^2 \leq 2^n$$
$$then \quad (k+1)^2 \leq 2^{k+1}$$

Example 4

$$Assume \quad k^2 \leq 2^k$$

$$then \quad (k+1)^2$$

$$= k^2 + 2 \cdot k + 1$$

Induction hypothesis

$$\leq 2^k + 2 \cdot k + 1$$

From previous example

$$\leq 2^k + 2^k$$

$$= 2^{k+1}$$

Example 5

Fibonacci sequence:

$Fib_0 = 0$ base case

$Fib_1 = 1$ base case

$Fib_n = Fib_{n-1} + Fib_{n-2}$ for $n > 1$ recursive case

Claim: For all $n \geq 0$, $Fib_n < 2^n$

Base case: $Fib_0 = 0 < 2^0 = 1$, $Fib_1 = 1 < 2^1 = 2$

Q: Why should we check both Fib_0 and Fib_1?

Induction step: for any $i \leq k$, if $Fib_i < 2^i$ then $Fib_{k+1} < 2^{k+1}$

Example 5

Assume that *for all $i \leq k$*, $Fib_i < 2^i$ (Note variation of induction hypothesis)

Then $Fib_{k+1} = Fib_k + Fib_{k-1}$

$< 2^k + 2^{k-1}$ Induction hypothesis (x2)

$\leq 2^k + 2^k$ for $k \geq 1$

$= 2^{k+1}$