COMP251: Bipartite graphs

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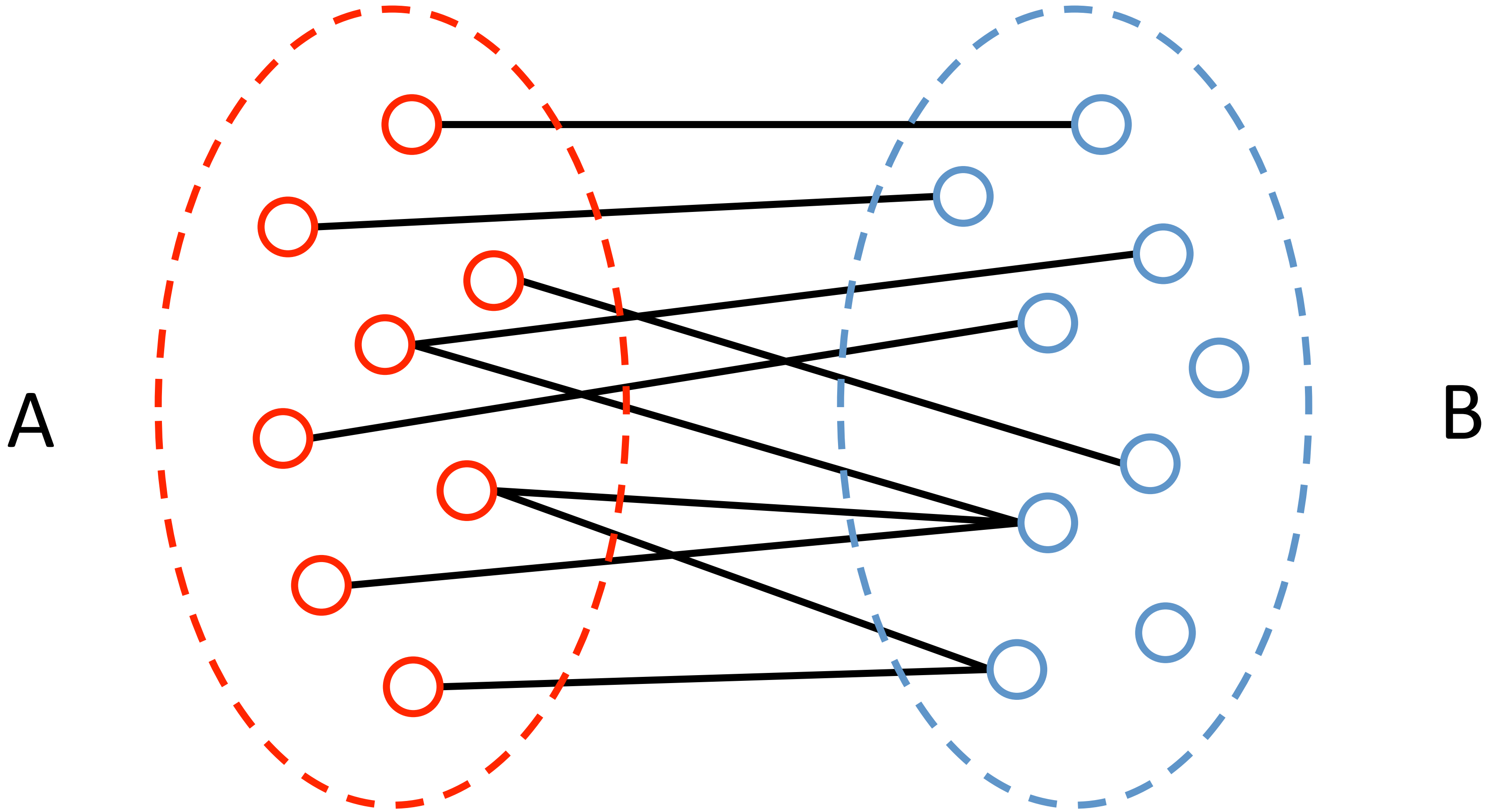
Based on slides fom M. Langer (McGill) & P. Beame (UofW)

Bipartite graphs

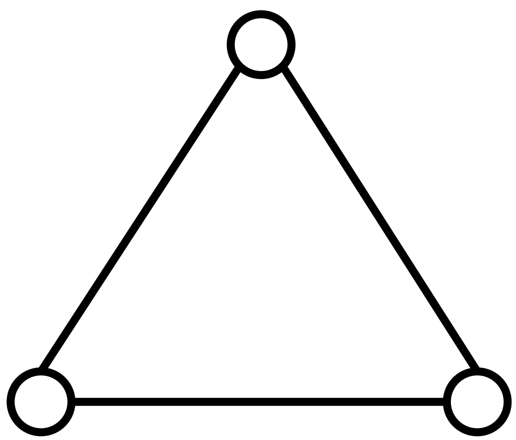


Vertices are partitioned into 2 sets.
All edges cross the sets.

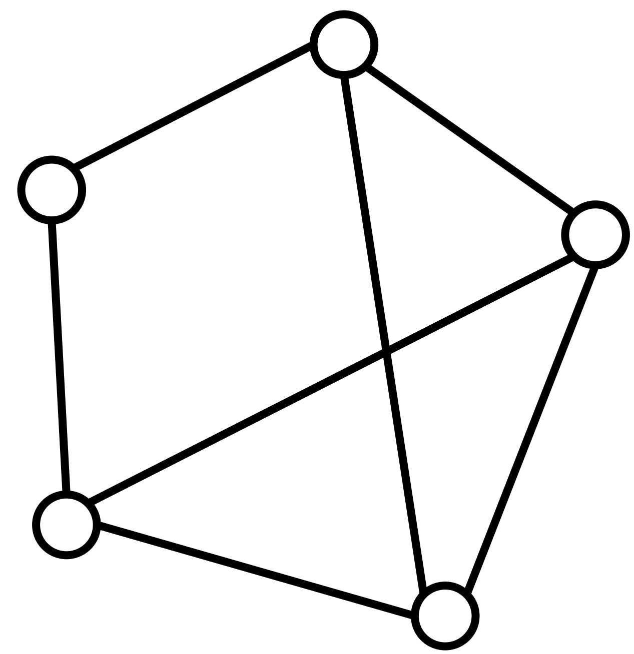
Examples

A		B
Women	Traditional marriage	Men
Students	registration	Courses
People	employment	Companies
People	Have read/seen	Books/Movies

Counter-examples



Easy to identify.



But not always...

Cycles

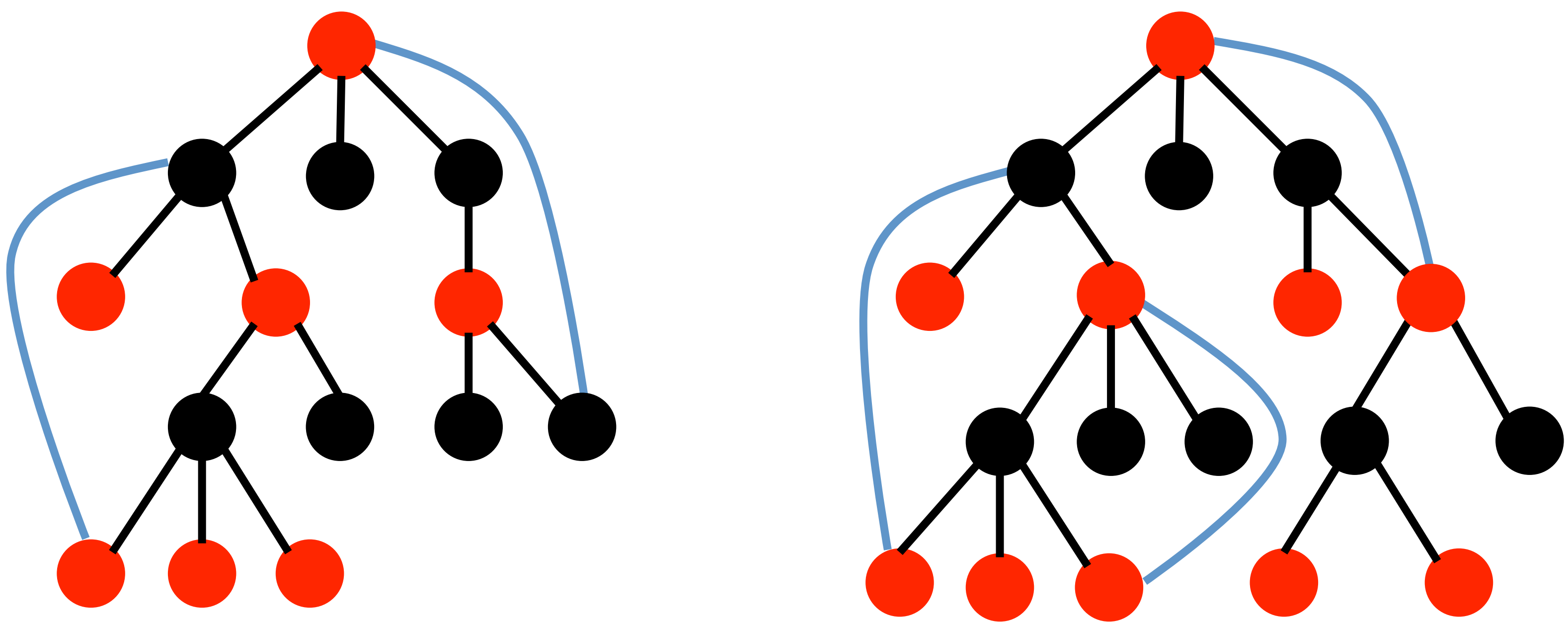
Claim: If a graph is bipartite if and only if does not contain an odd cycle.

Proof: Q5 of assignment 2.

Is it a bipartite graph?

Assuming G=(V,E) is an undirected connected graph.

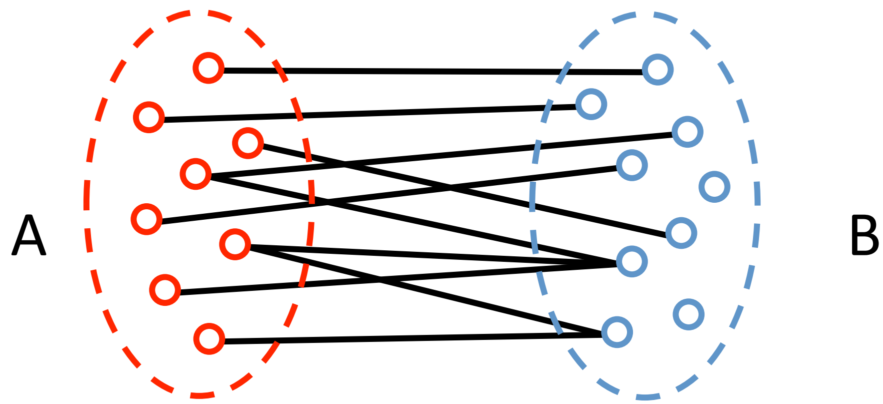
1. Run DFS and build a DFS tree.
2. Color vertices by layers (e,g. red & black)
3. If all non-tree edges join vertices of different color then the graph is bipartite.



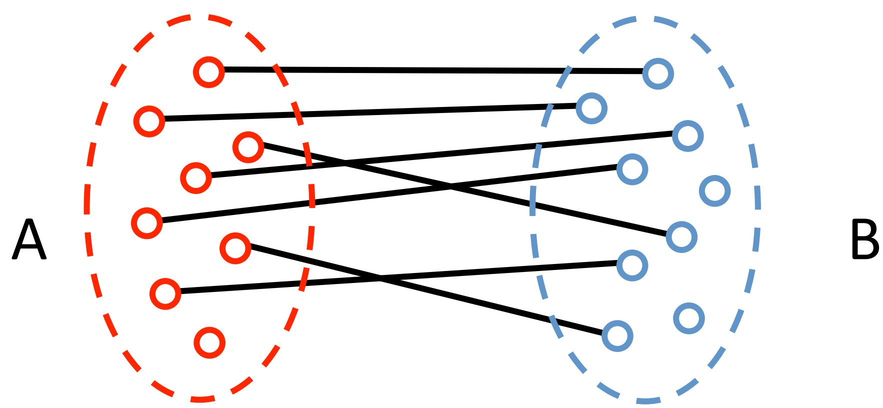
Non-tree edges in DFS tree cross 2 or more levels. Why?

Bipartite matching

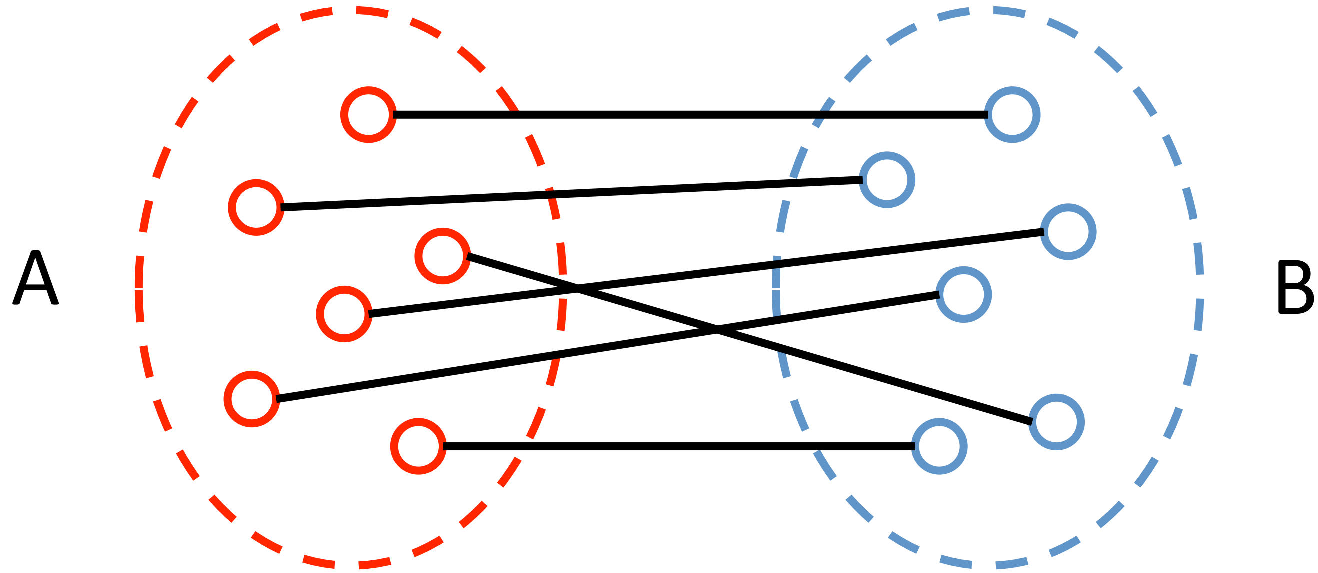
Consider an undirected bipartite graph.



A matching is a subset of the edges { (α, β) } such that no two edges share a vertex.



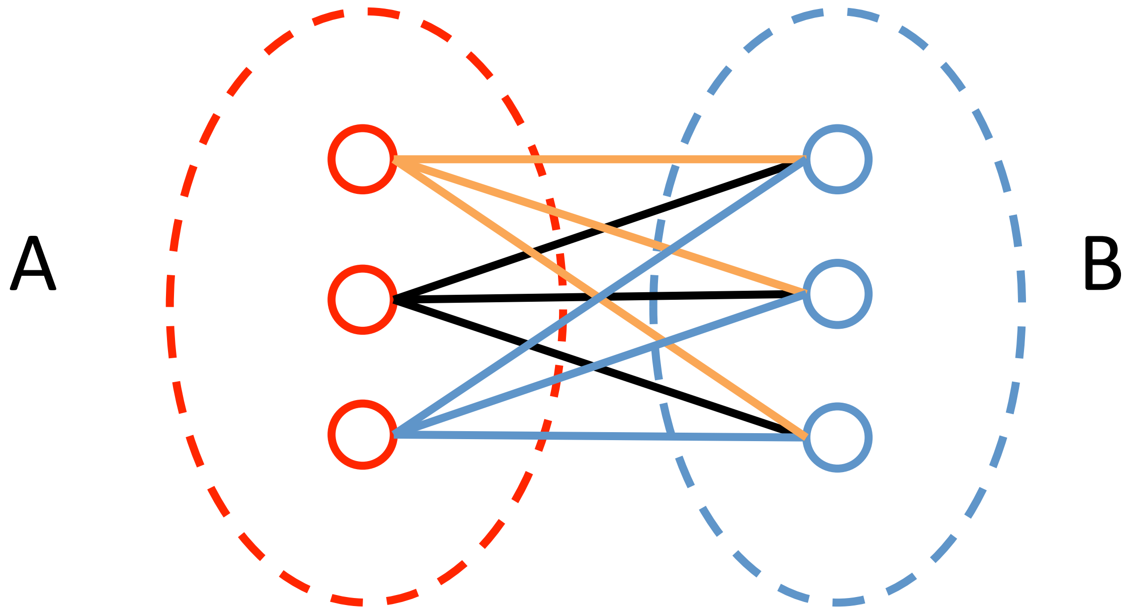
Perfect matching



Suppose we have a bipartite graph with *n* vertices in each A and B.
A **perfect matching** is a matching that has *n* edges.

Note: It is not always possible to find a perfect matching.

Complete bipartite graph



A complete bipartite graph is a bipartite graph that has an edge for every pair of vertice (α, β) such that α∈A, β∈B.

The algorithm of happiness

Resident matching program

- **Goal:** Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.
- **Unstable pair:** applicant **x** and hospital **y** are unstable if:
  - **x** prefers **y** to their assigned hospital.
  - **y** prefers **x** to one of its admitted students.
- **Stable assignment:** Assignment with no unstable pairs.
  - Natural and desirable condition.
  - Individual self-interest will prevent any applicant/hospital deal from being made.

Stable marriage problem

Goal: Given **n** men and **n** women, find a "suitable" matching.
Participants rate members of opposite sex.

- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

Men's preferences

	1st	2nd	3rd
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Women's preferences

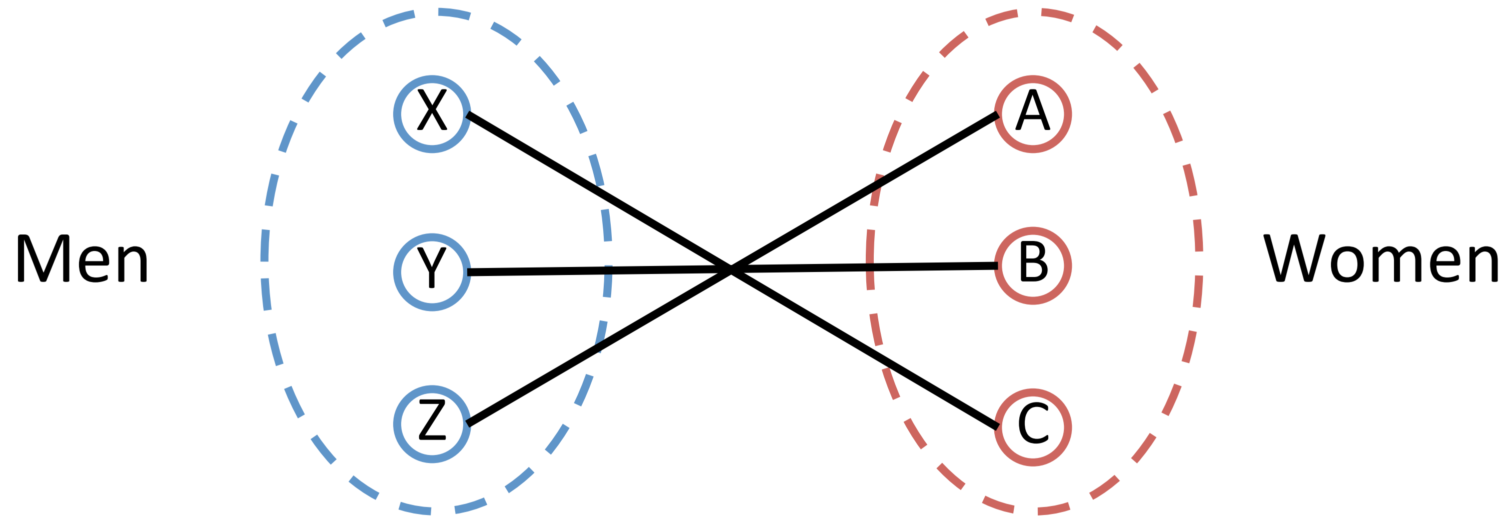
	1st	2nd	3rd
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Stable marriage problem

- **Perfect matching:** everyone is matched monogamously.
  - Each man gets exactly one woman.
  - Each woman gets exactly one man.
- **Stability:** no incentive for some pair of participants to undermine assignment by joint action.
  - In matching **M**, an unmatched pair **m**-**w** is unstable if man **m** and woman **w** prefer each other to current partners.
  - Unstable pair **m**-**w** could each improve by eloping.
- **Stable matching:** perfect matching with no unstable pairs.
- **Stable matching problem:** Given the preference lists of **n** men and **n** women, find a stable matching (if one exists).

Example

Q: Is X-C, Y-B, Z-A a good assignment?



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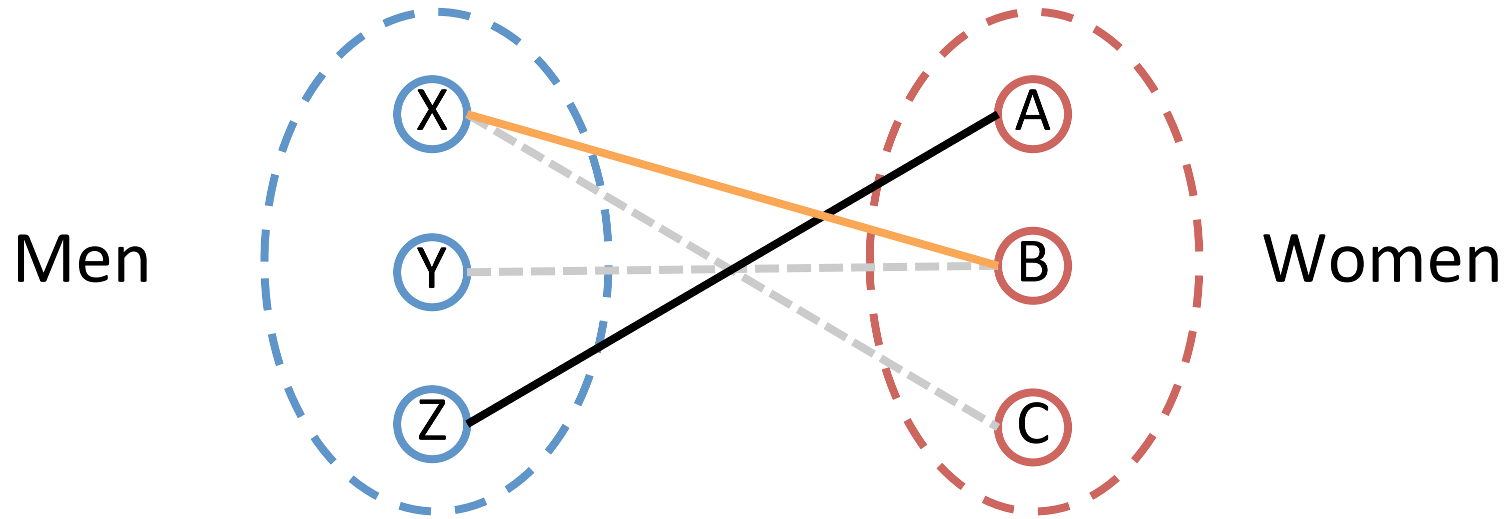
Example

Q: Is X-C, Y-B, Z-A a good assignment?

A: No! Brenda and Xavier will hook up…



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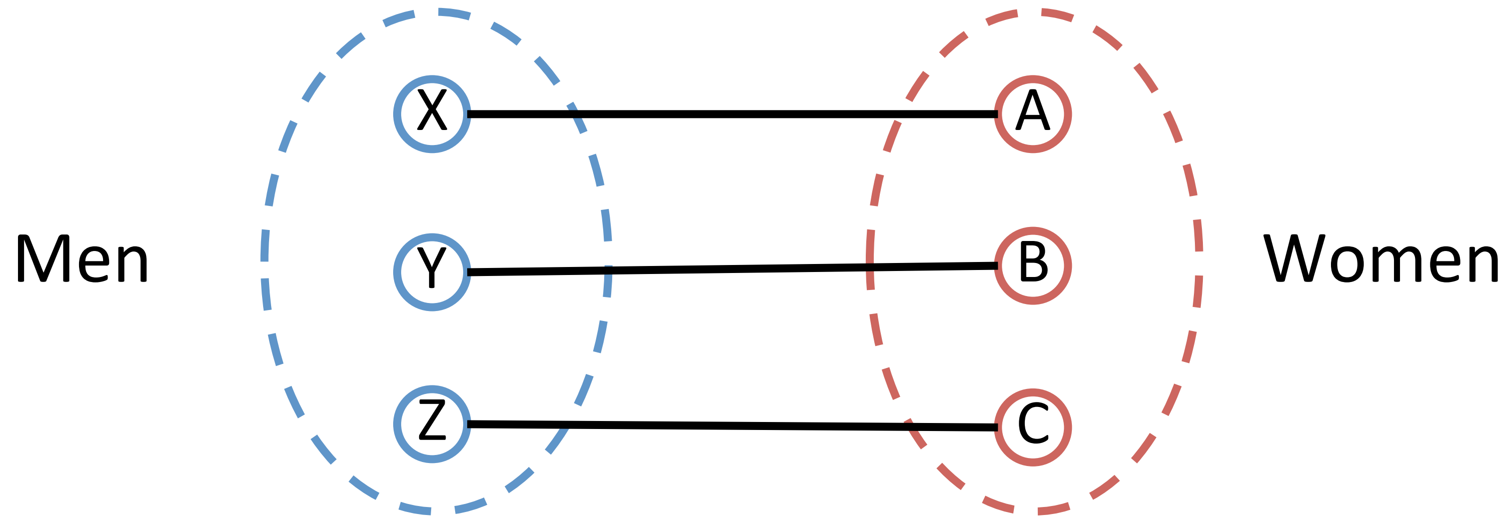
Example

Q: Is X-A, Y-B, Z-C a good assignment?

A: Yes!



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Stable marriage problem

Consider a complete bipartite graph such that |A|=|B|=n.

- Each member of A has a preference ordering of members of B.
- Each member of B has a preference ordering of members of A.

Algorithm for finding a matching:

- Each A member proposes to a B, in preference order.
- Each B member accepts the first proposal from an A, but then rejects that proposal if/when it receives a proposal from an A that it prefers more.

In our example: Men propose to women. Woman accept the first offer made to them, but women will drop their partner when/if a preferred man proposes to them.

Note the asymmetry between A and B.

Gale-Shapley algorithm

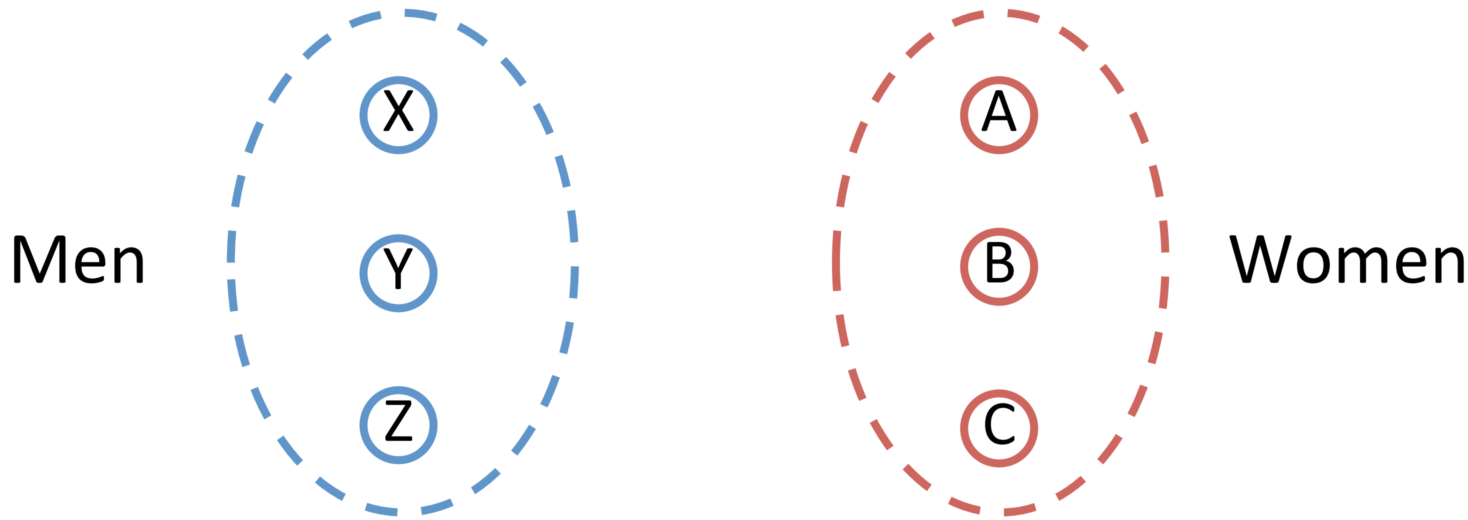
For each α∈A, let pref[α] be the ordering of its preferences in B

For each β∈B, let pref[β] be the ordering of its preferences in A

Let matching be a set of crossing edges between A and B

```
matching←∅
while there is α∈A not yet matched do
    β←pref[α].removeFirst()
    if β not yet matched then
        matching←matching∪{(α,β)}
    else
        γ←β's current match
        if β prefers α over γ then
            matching←matching-{(γ,β)}∪{(α,β)}
return matching
```

Example



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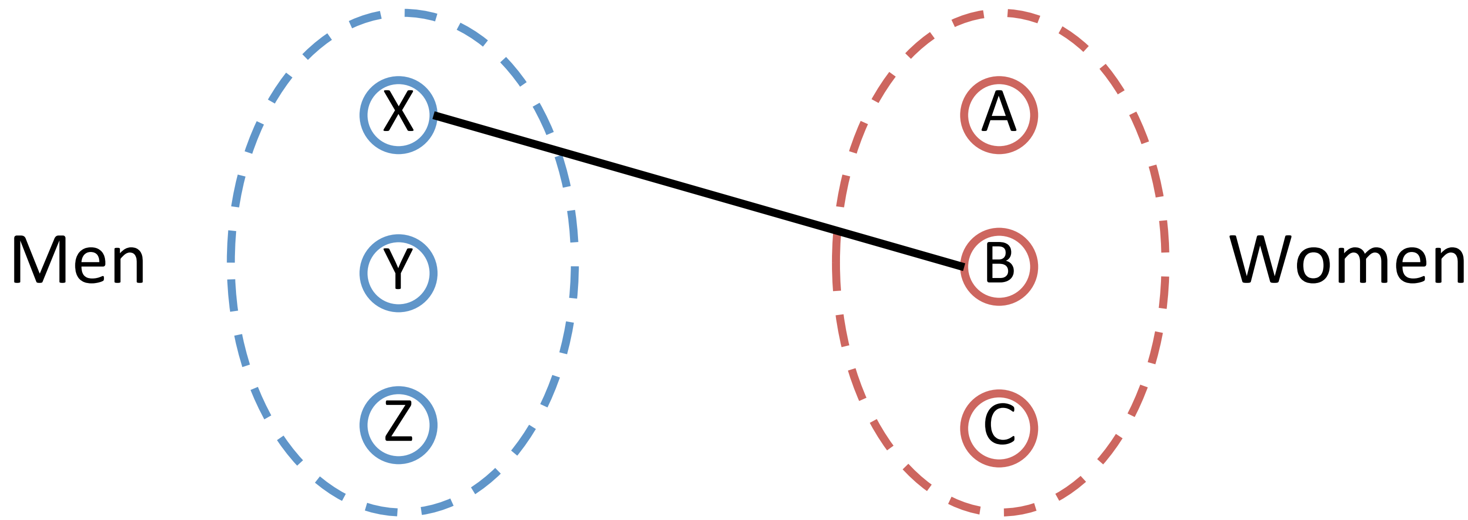

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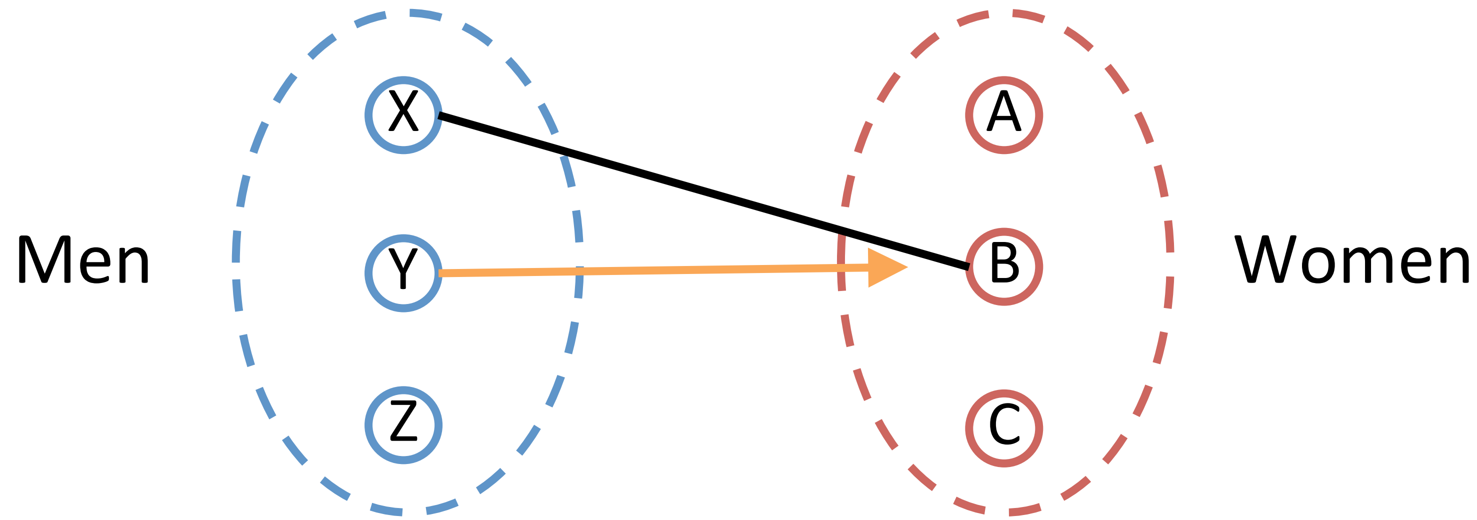

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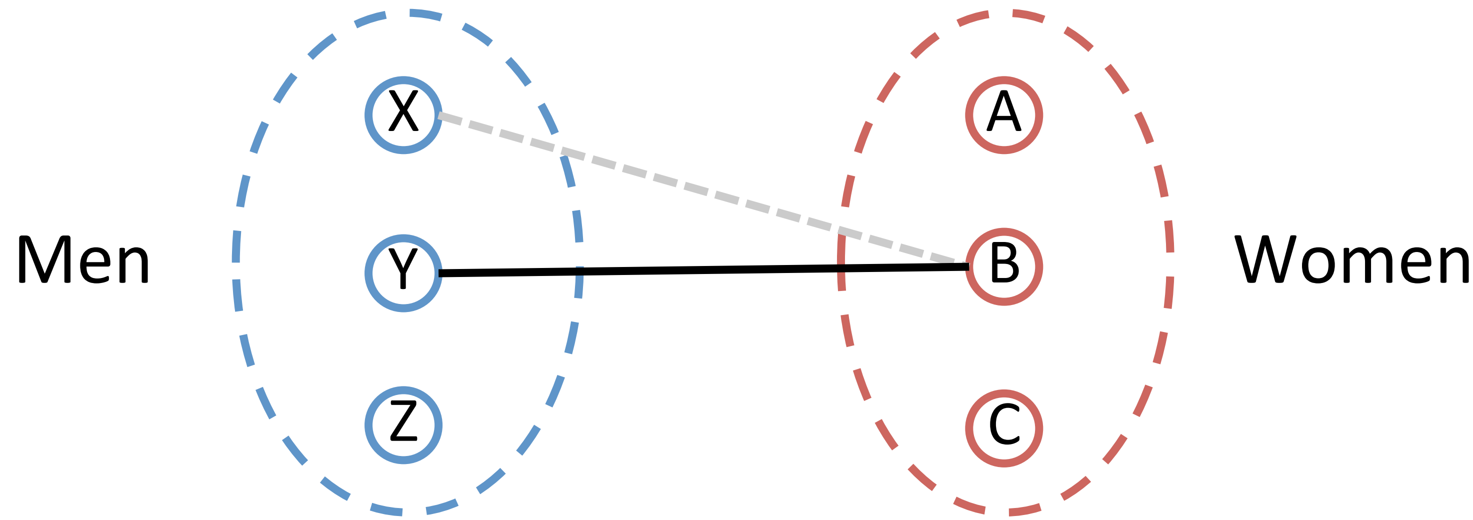

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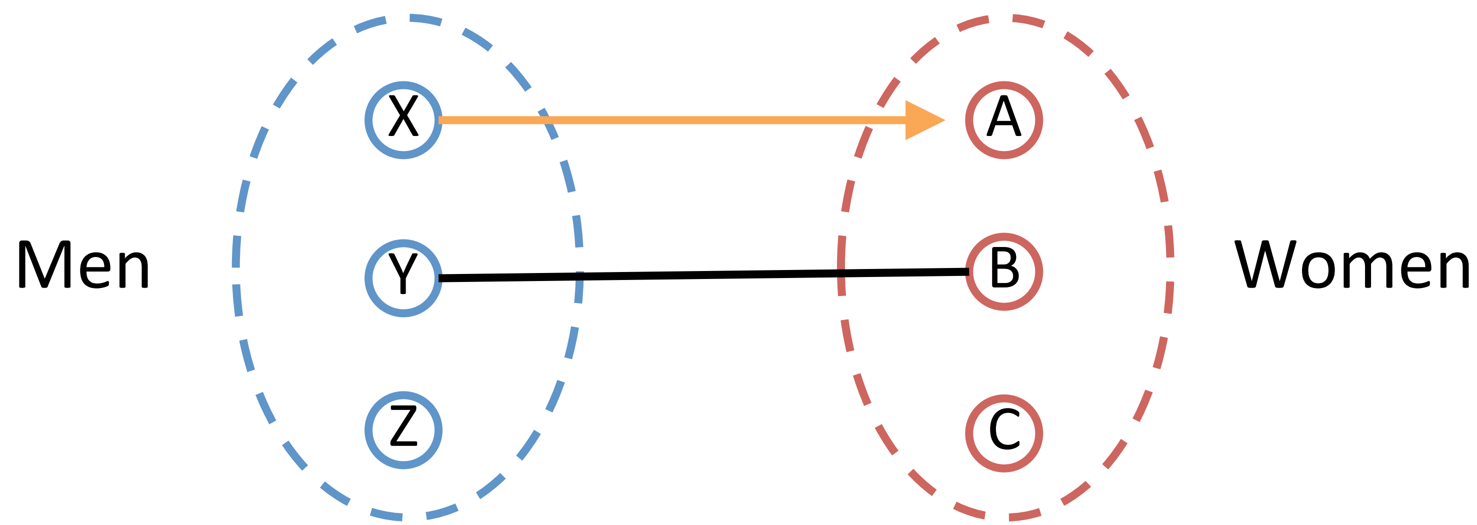

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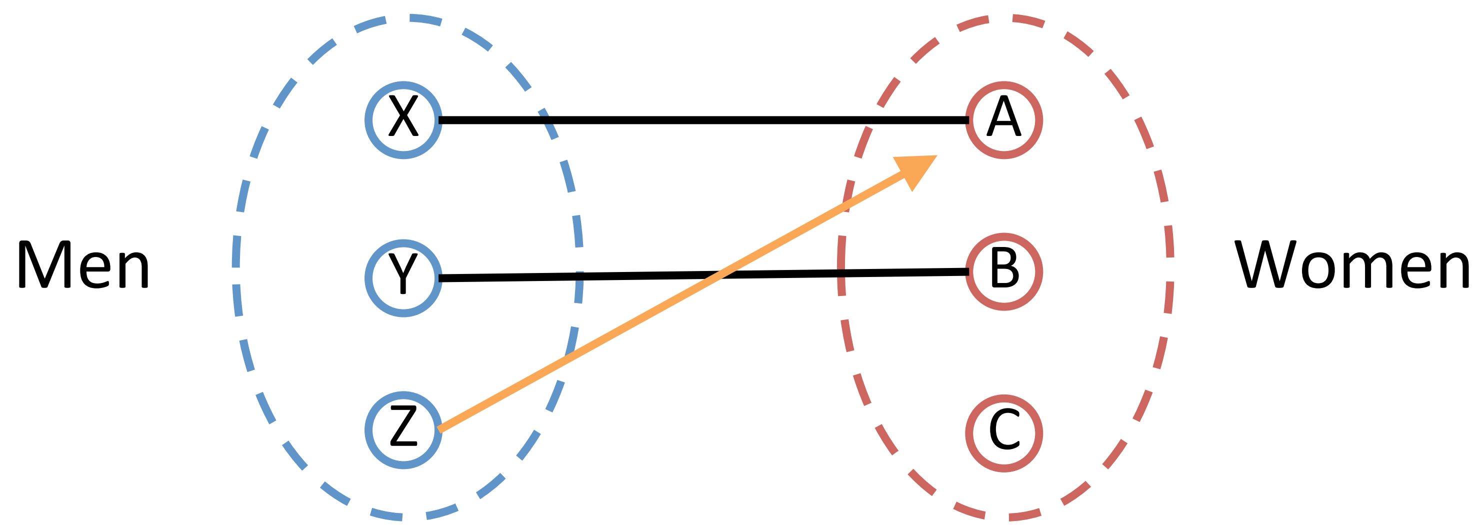

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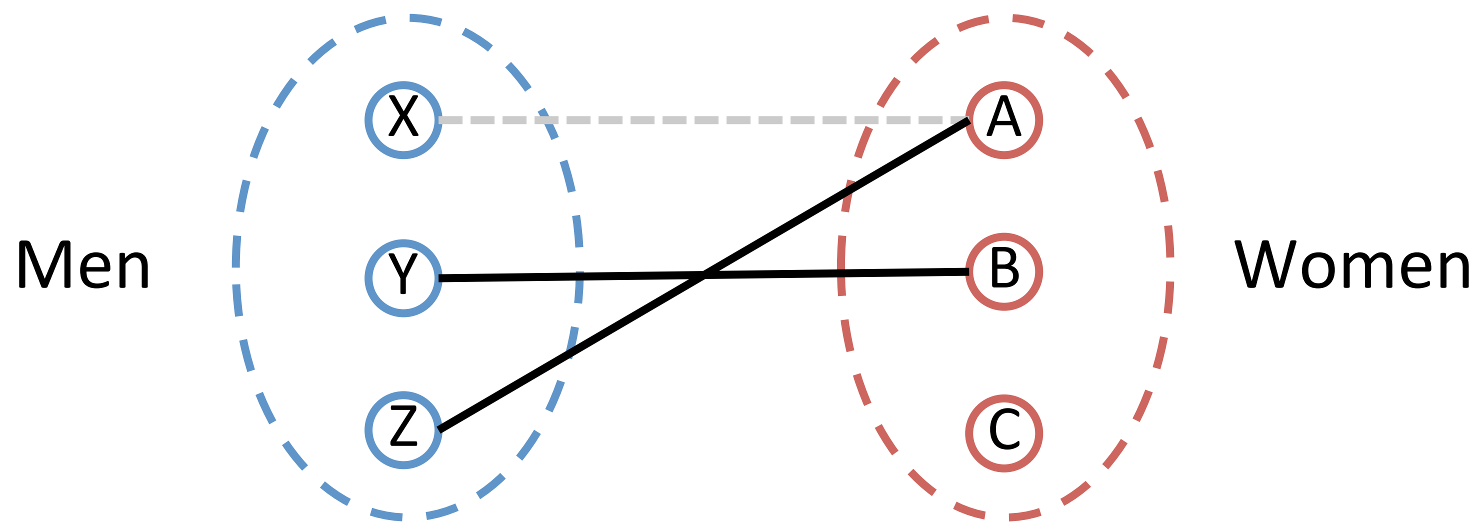

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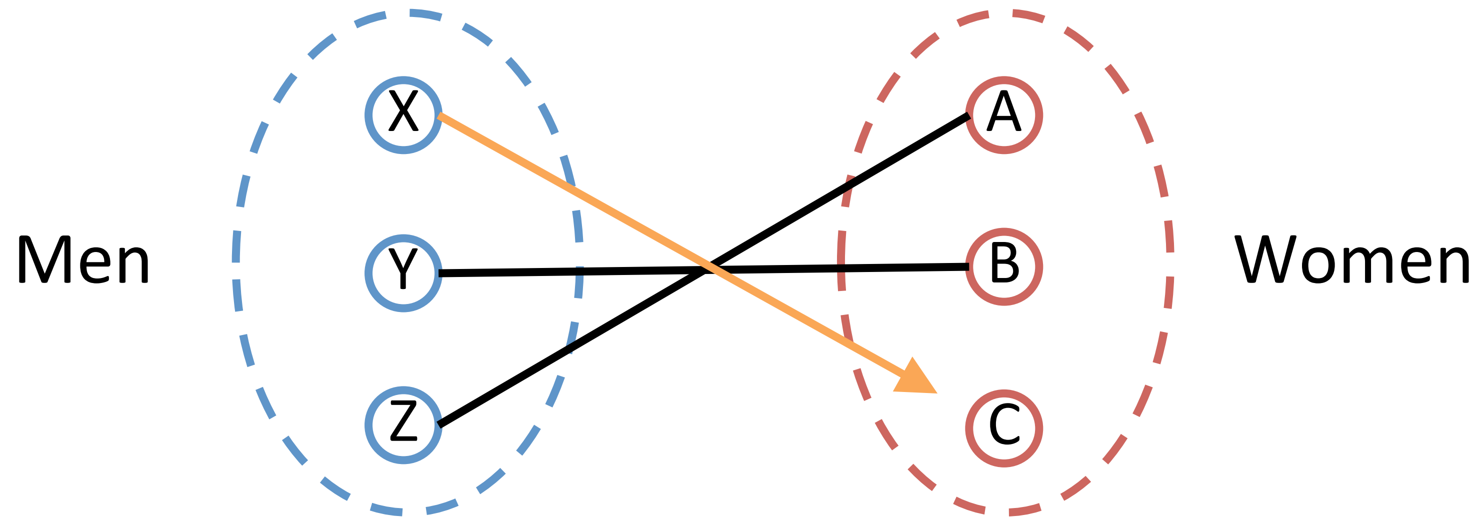

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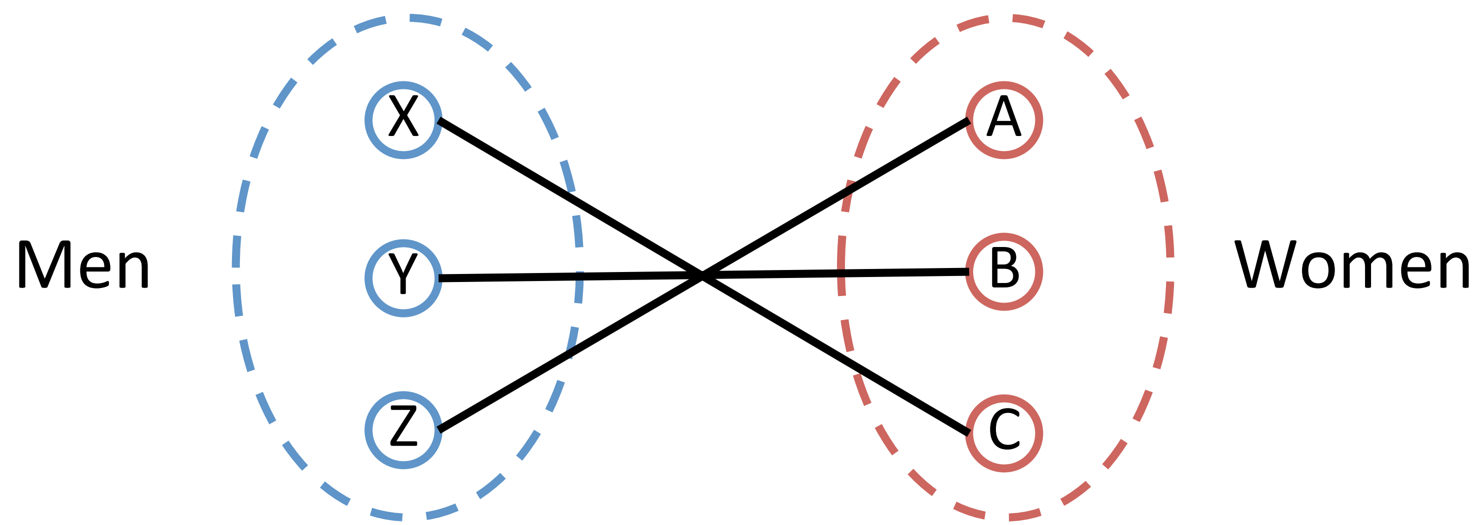

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Correctness (termination)

Observations:

1. Men propose to women in decreasing order of preference.
2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim: Algorithm terminates after at most n^2 iterations of while loop (i.e. $O(n^2)$ running time).

Proof: Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ■

Correctness (perfection)

Claim: All men and women get matched.

Proof: (by contradiction)

- Suppose, for sake of contradiction, that Zoran is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2 (only trading up, never becoming unmatched), Amy was never proposed to.
- But, Zoran proposes to everyone. Contradiction. ■

Correctness (stability)

Claim: No unstable pairs.

Proof: (by contradiction)

- Suppose **A**-**Z** is an unstable pair: each prefers each other to partner in Gale-Shapley `matching`.
- Case 1: **Z** never proposed to **A**.
  - ⇒ **Z** prefers his GS partner to **A**.
  - ⇒ **A**-**Z** is stable.
- Case 2: **Z** proposed to **A**.
  - ⇒ **A** rejected **Z** (right away or later)
  - ⇒ **A** prefers her GS partner to **Z**.
  - ⇒ **A**-**Z** is stable.
- In either case **A**-**Z** is stable. Contradiction. ■

Optimality

Definition: Man **m** is a valid partner of woman **w** if there exists some stable matching in which they are matched.

Man-optimal assignment: Each man receives **best** valid partner (according to his preferences).

Claim: All executions of GS yield a **man-optimal** assignment, which is a stable matching!

Man-Optimality

Claim: GS matching **S*** is man-optimal.

Proof: (by contradiction)

- Suppose some man is paired with someone other than his best partner. Men propose in decreasing order of preference ⇒ some man is rejected by a valid partner.
- Let **Y** be first such man, and let **A** be the first valid woman that rejects him.
- Let **S** be a stable matching where **A** and **Y** are matched.
- In building `matching`, when **Y** is rejected, **A** forms (or reaffirms) engagement with a man, say **Z**, whom she prefers to **Y**.
- Let **B** be **Z**'s partner in **S**.
- In building `matching`, **Z** is not rejected by any valid partner at the point when **Y** is rejected by **A**.
- Thus, **Z** prefers **A** to **B**.
- But **A** prefers **Z** to **Y**.
- Thus **A**-**Z** is unstable in **S**.