COMP251: Topological Sort & Strongly Connected Components

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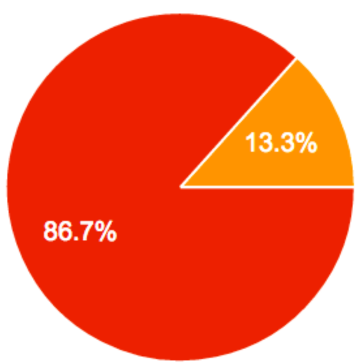
Based on (Cormen *et al.*, 2002)

Based on slides from D. Plaisted (UNC)

We prefer to use an adjacency matrix vs a adjacency list to represent a graph when:

- The graph is sparse ✗
- The graph is dense ✓
- The graph is a weighted graph ✗
- The graph is directed ✗

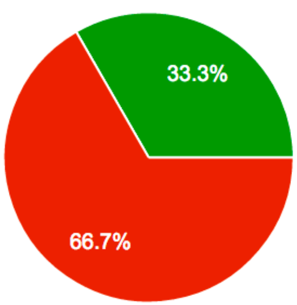
Representation of graphs



The graph is sparse	**0**	0%
The graph is dense	**13**	86.7%
The graph is a weighted graph	**2**	13.3%
The graph is directed	**0**	0%

Let G be a directed graph. We explore G using the BFS algorithm. Which of the following assertions are true?

- The best case running time of BFS is $\Omega(V+E)$ ✓ (if connected)
- All vertices at distance d from the source s are visited before vertices at distance d+1 ✓
- All vertices of G are visited even if G has disconnected components ✗
- The source s can be any vertex of G ✓

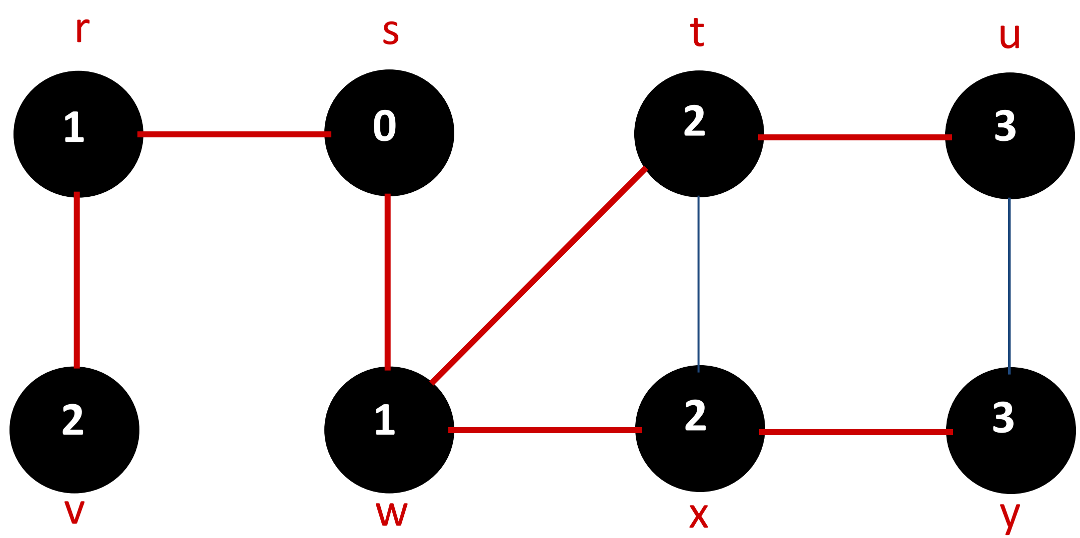


The best case running time of BFS is $\Omega(V+E)$	**0**	0%
All vertices at distance d from the source s are visited before vertices at distance d+1	**6**	42.9%
All vertices of G are visited even if G has disconnected components	**0**	0%
The source s can be any vertex of G	**3**	21.4%

Recap: Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \leadsto v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Recap: BFS Example

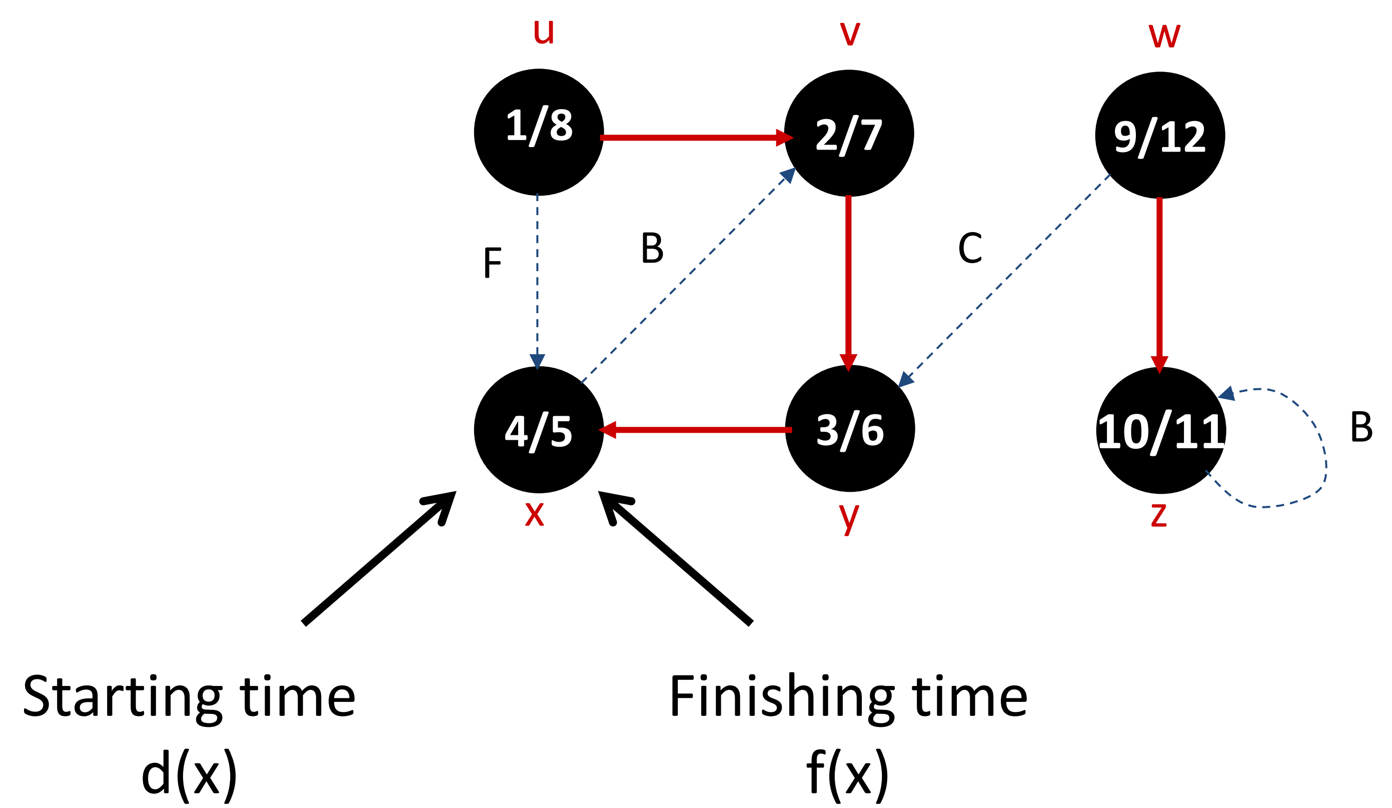


Q: $\emptyset$

Recap: Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given.
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f\,[v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u' s adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Recap: DFS Example



Parenthesis Theorem

Theorem 1:

For all u, v, exactly one of the following holds:

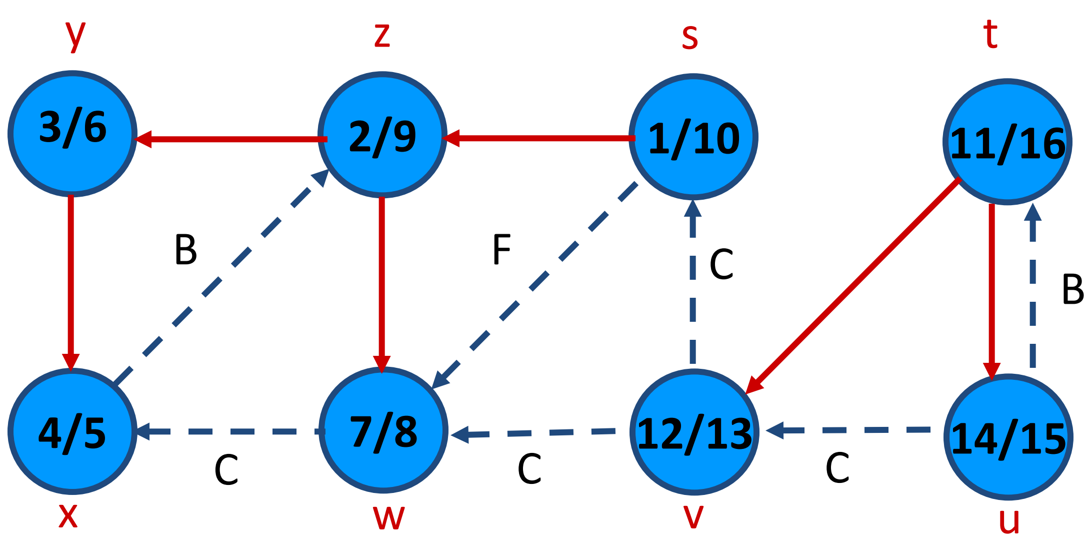
1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)



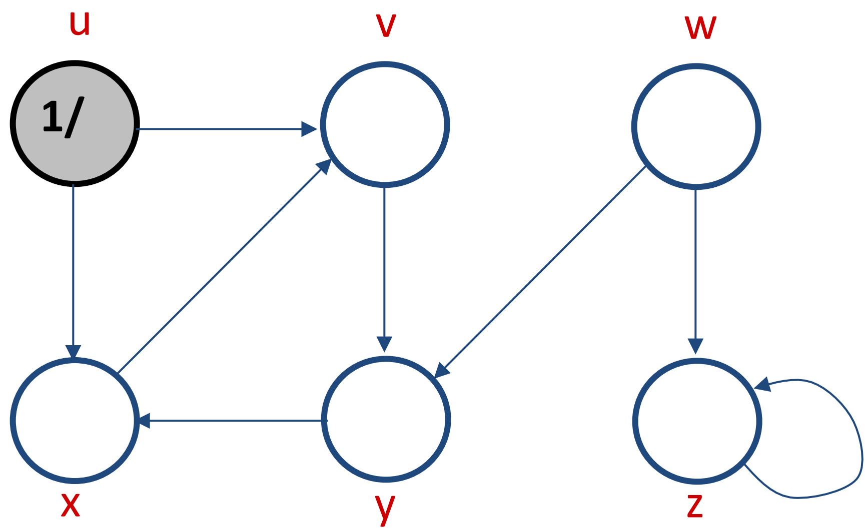
(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

White-path Theorem

Theorem ***2***

v is a descendant of *u* if and only if at time *d*[*u*], there is a path *u* $\rightsquigarrow$ *v* consisting of only white vertices. (Except for *u*, which was *just* colored gray.)

Example (DFS)



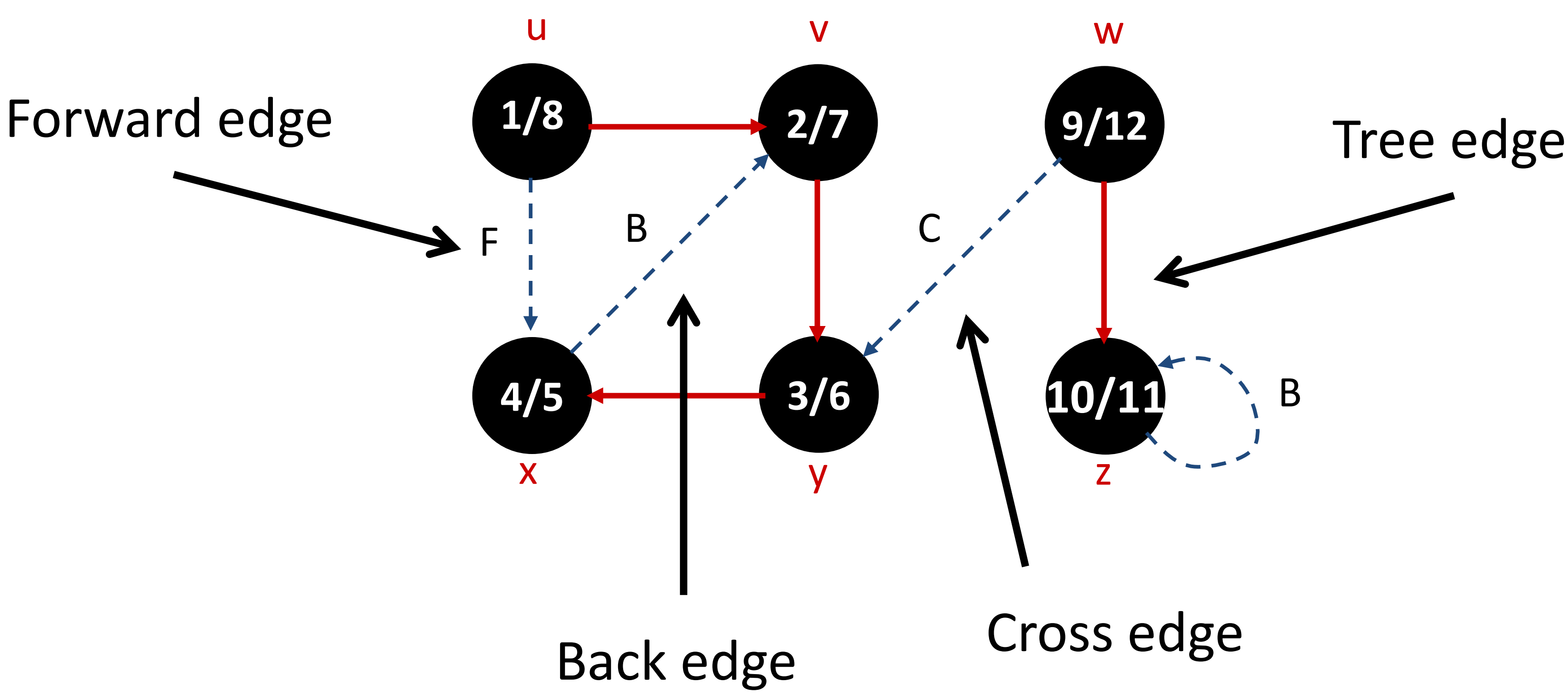
v, y, and x are descendants of u.

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Theorem 3
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Example (DFS)



u
v
w
1/8
2/7
9/12
Forward edge
Tree edge
F
B
C
4/5
3/6
10/11
B
x
y
z
Back edge
Cross edge

Identification of Edges

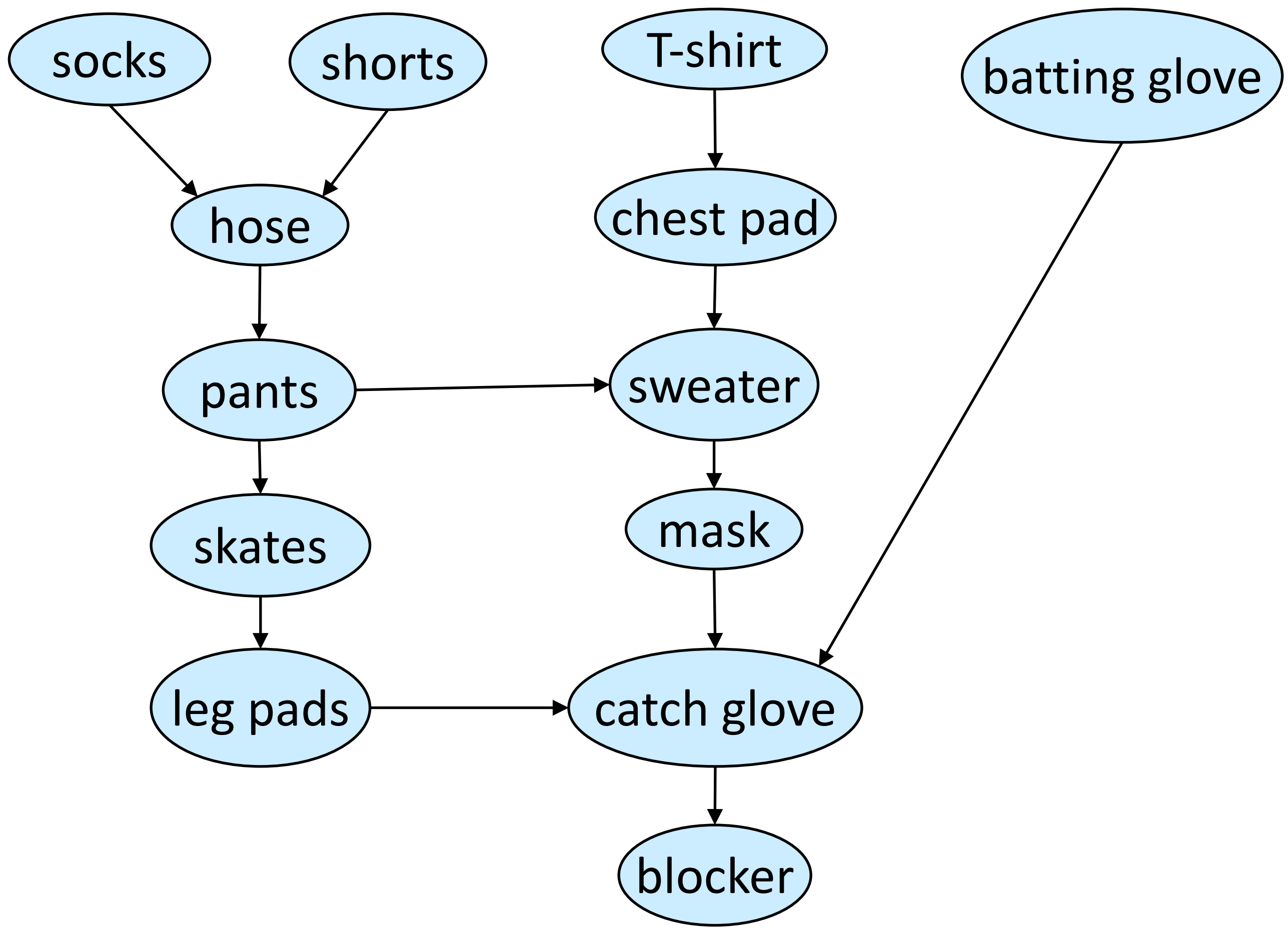
- Edge type for edge (u, v) can be identified when it is first explored by DFS.
- Identification is based on the color of v.
  - White – tree edge.
  - Gray – back edge.
  - Black – forward or cross edge.

Directed Acyclic Graph

- DAG – Directed graph with no cycles.
- Good for modeling processes and structures that have a **partial order:**
  - $a > b$ and $b > c \Rightarrow a > c$.
  - But may have a and b such that neither $a > b$ nor $b > a$.
- Can always make a **total order** (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order.

Example

DAG of dependencies for putting on goalie equipment.

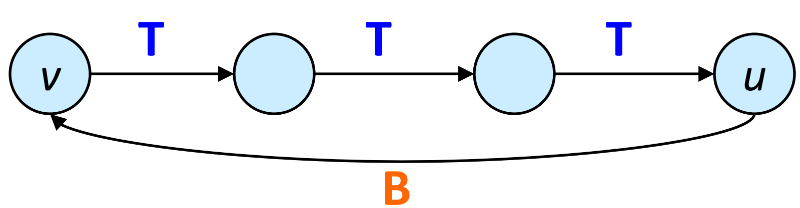


Characterizing a DAG

> **Lemma 1**
> A directed graph *G* is acyclic iff a DFS of G yields no back edges.

Proof:

- $\Rightarrow$: Show that back edge $\Rightarrow$ cycle.
  - Suppose there is a back edge (u, v). Then v is ancestor of *u* in depth-first forest.
  - Therefore, there is a path $v \rightsquigarrow u$, so $v \rightsquigarrow u \rightsquigarrow v$ is a cycle.



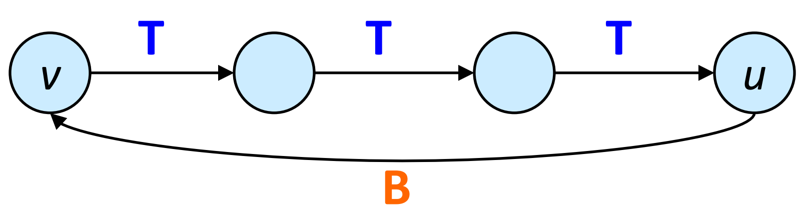
Characterizing a DAG

Lemma 1

A directed graph *G* is acyclic iff a DFS of G yields no back edges.

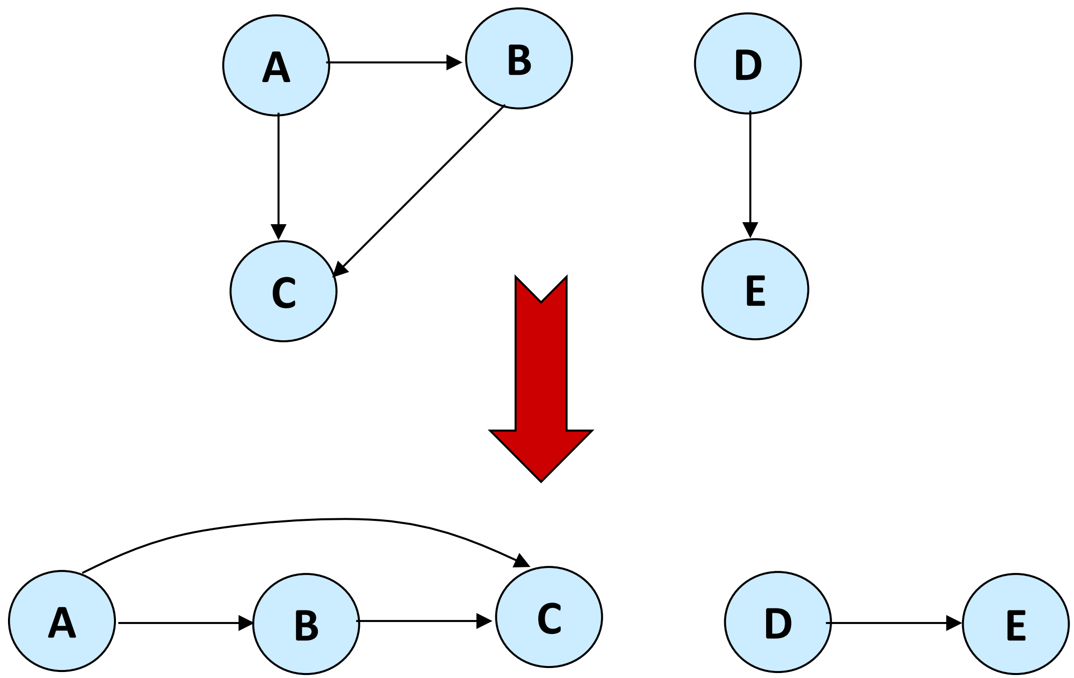
Proof (Contd.):

- $\Leftarrow$: Show that a cycle implies a back edge.
  - *c* : cycle in *G*, *v* : first vertex discovered in *c*, (u, v) : preceding edge in c.
  - At time *d*[*v*], vertices of *c* form a white path $v \rightsquigarrow u$.
  - By **white-path theorem**, *u* is a descendent of *v* in depth-first forest.
  - Therefore, (u, v) is a back edge.



Topological Sort

Want to "sort" a directed acyclic graph (DAG).



Think of original DAG as a **partial order**.

Want a **total order** that extends this partial order.

Topological Sort

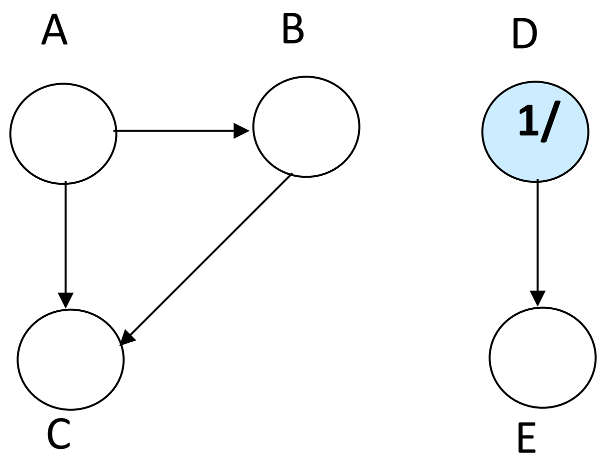
- Performed on a DAG.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Topological-Sort (G)

1. call DFS(G) to compute finishing times $f[v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices

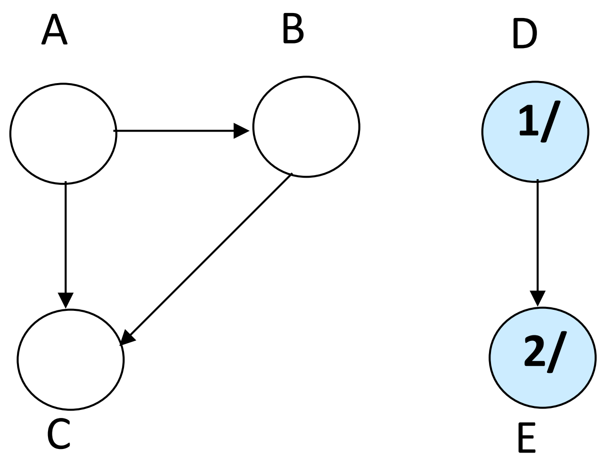
Time: $\Theta(V + E)$.

Example 1



Linked List:

Example 1



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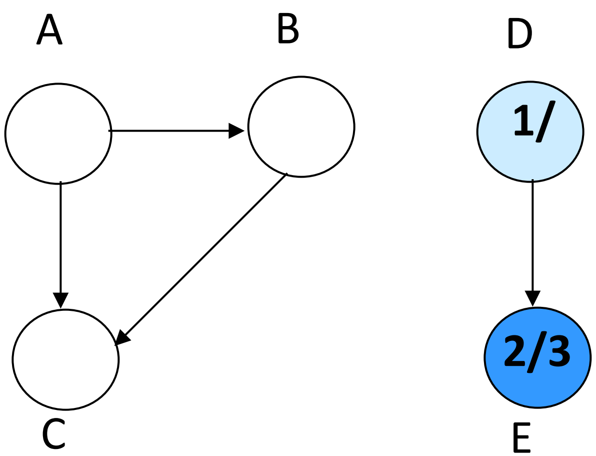

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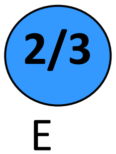

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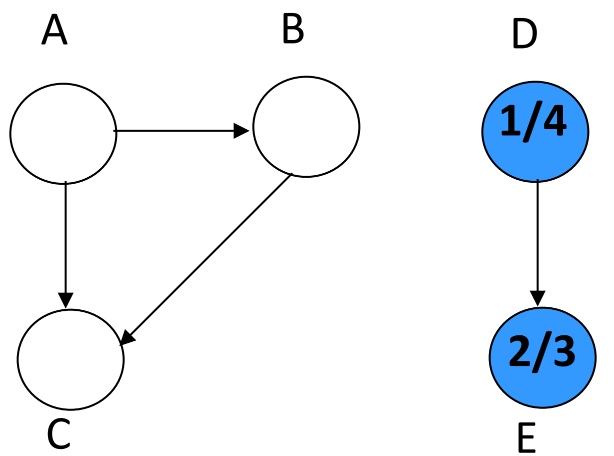

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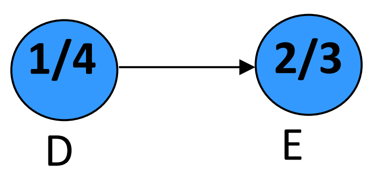

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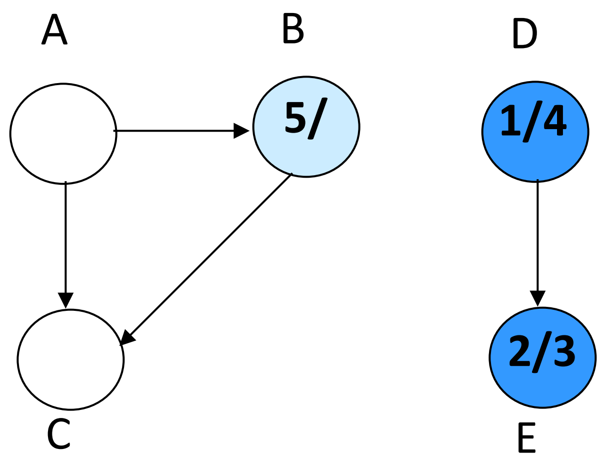

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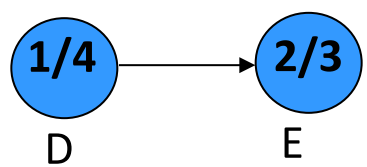

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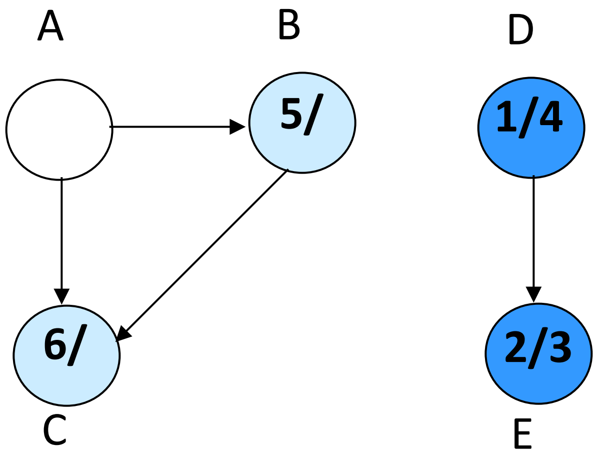

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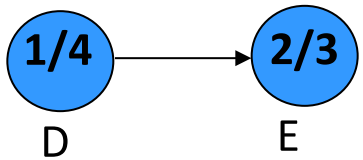

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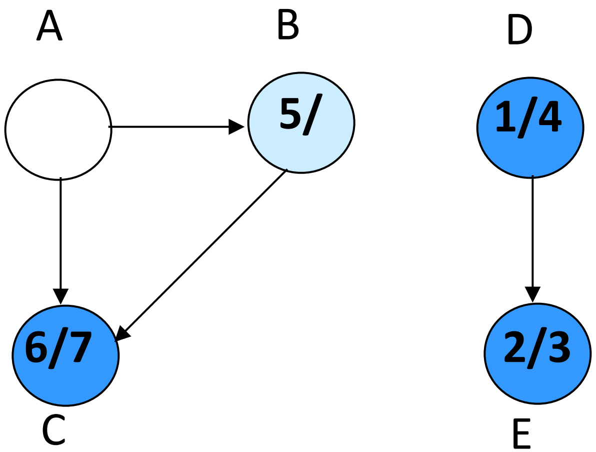

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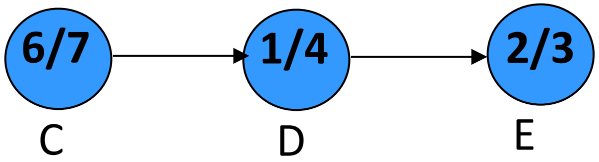

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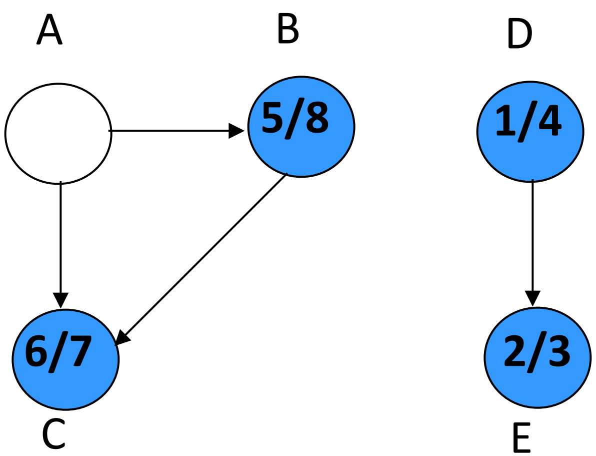

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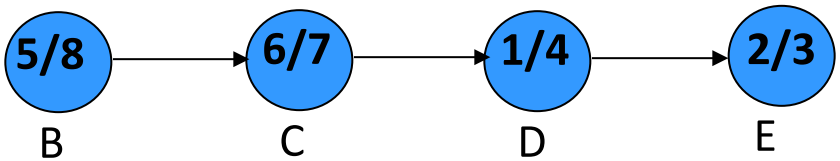

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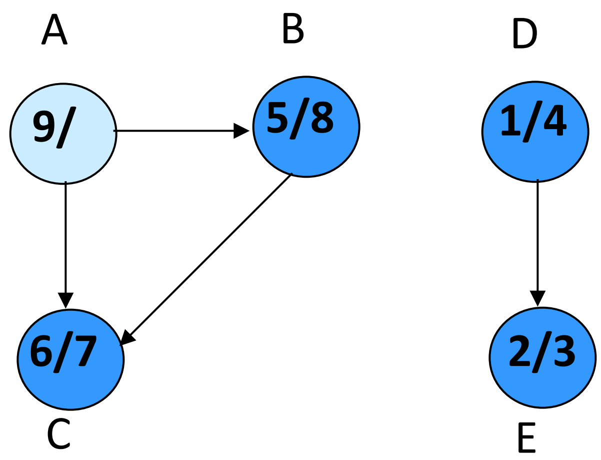

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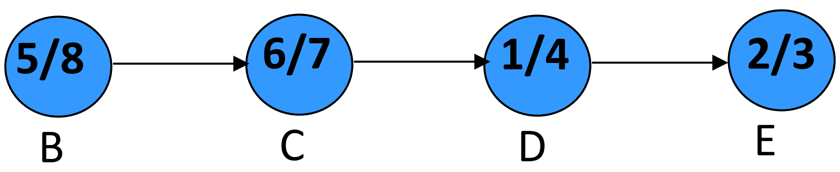
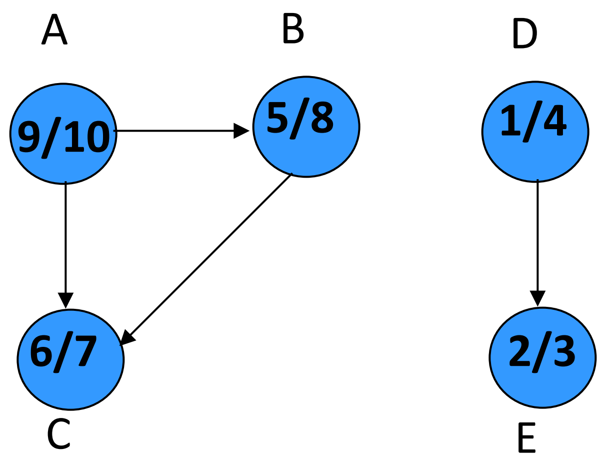


Example 1



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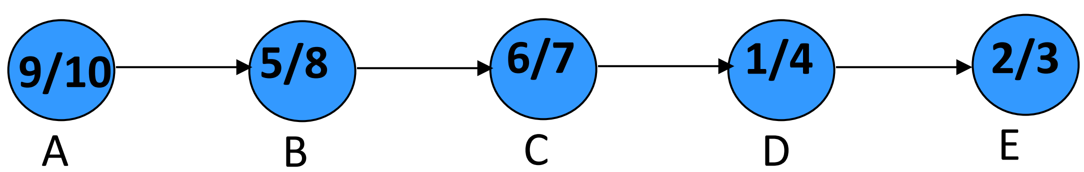
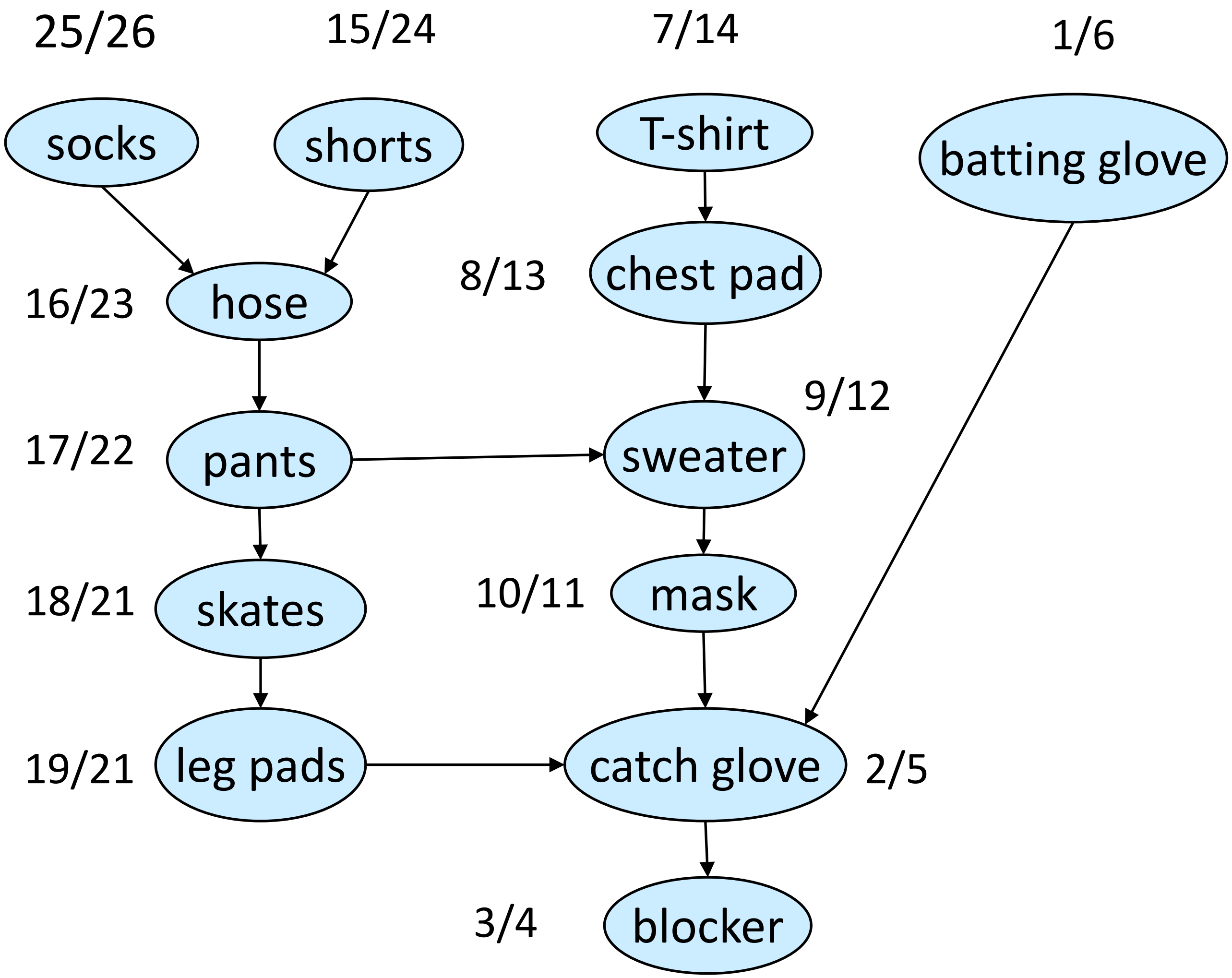


Example 2



26 socks
24 shorts
23 hose
22 pants
21 skates
20 leg pads
14 t-shirt
13 chest pad
12 sweater
11 mask
6 batting glove
5 catch glove
4 blocker

Correctness Proof

- Just need to show if $(u, v) \in E$, then $f[v] < f[u]$.
- When we explore (u, v), what are the colors of u and v?
  - u is gray.
  - Is v gray, too?

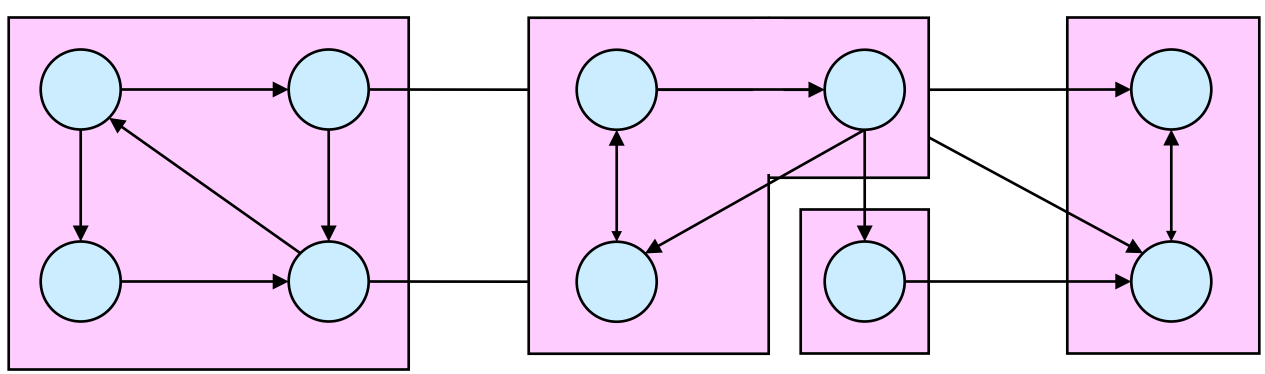
    No, because then v would be ancestor of u.

    $\Rightarrow (u, v)$ is a back edge.

    $\Rightarrow$ contradiction of **Lemma 1** (DAG has no back edges).
  - Is v white?
    - Then becomes descendant of u.
    - By **parenthesis theorem**, $d[u] < d[v] < \boldsymbol{f[v] < f[u]}$.
  - Is v black?
    - Then v is already finished.
    - Since we're exploring (u, v), we have not yet finished u.
    - Therefore, $f[v] < f[u]$.

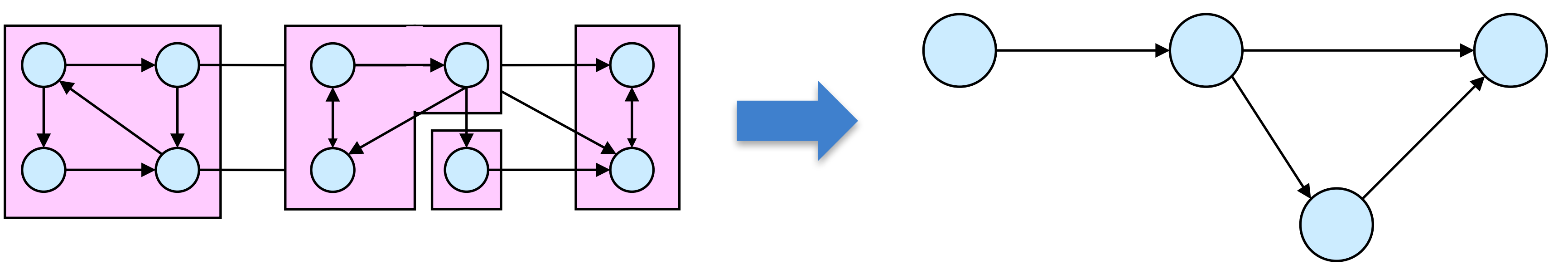
Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.
- A **strongly connected component (*SCC*)** of G is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$ exist.



Component Graph

- $G^{\text{SCC}} = (V^{\text{SCC}}, E^{\text{SCC}})$.
- V^{SCC} has one vertex for each SCC in G.
- E^{SCC} has an edge if there is an edge between the corresponding SCC's in G.
- G^{SCC} for the example considered:



G^{SCC} is a DAG

Lemma 2
Let C and C' be distinct SCC's in G, let $u, v \in C$, $u', v' \in C'$, and suppose there is a path $u \leadsto u'$ in G. Then there cannot also be a path $v' \leadsto v$ in G.

Proof:

- Suppose there is a path $v' \leadsto v$ in G.
- Then there are paths $u \leadsto u' \leadsto v'$ and $v' \leadsto v \leadsto u$ in G.
- Therefore, u and v' are reachable from each other, so they are not in separate SCC's.

Transpose of a Directed Graph

- G^{T} = **transpose** of directed G.
  - $G^{\mathrm{T}} = (V, E^{\mathrm{T}})$, $E^{\mathrm{T}} = \{(u, v) : (v, u) \in E\}$.
  - G^{T} is G with all edges reversed.
- Can create G^{T} in $\Theta(V + E)$ time if using adjacency lists.
- G and G^{T} have the *same* SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^{T}.)

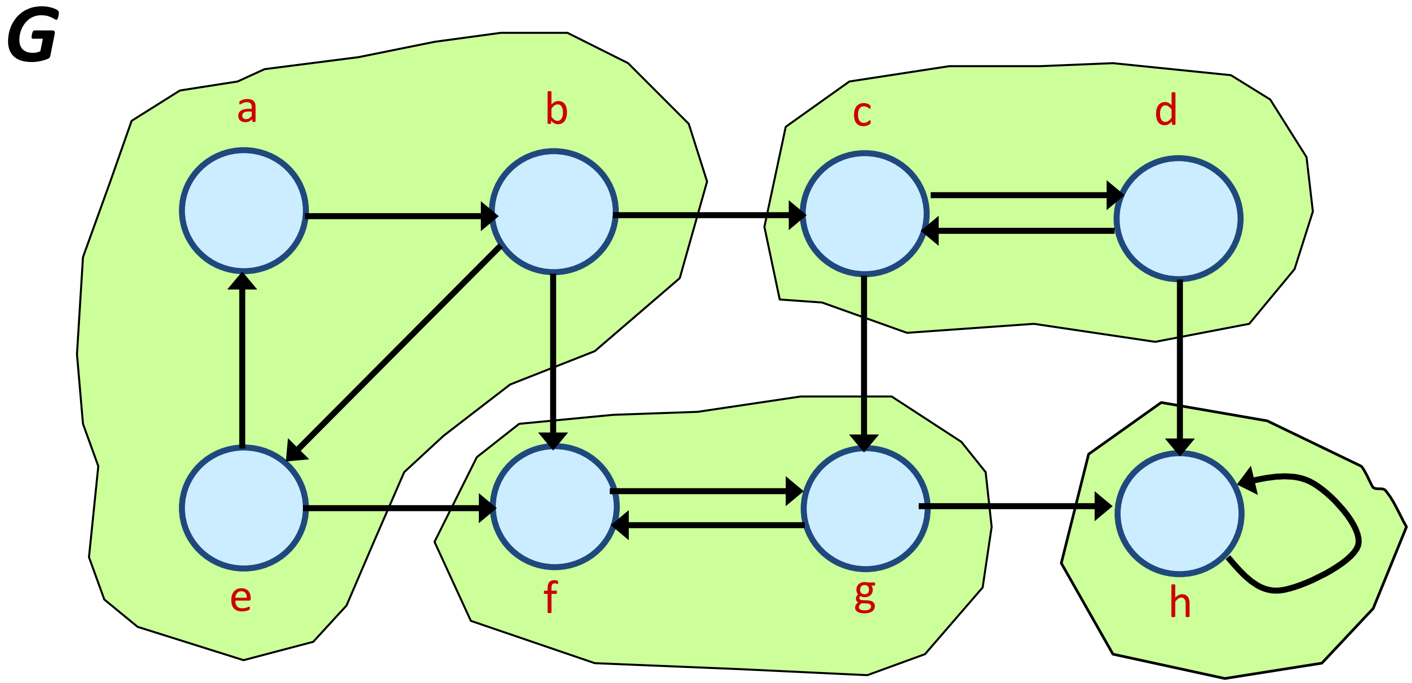
Algorithm to determine SCCs

SCC(*G*)

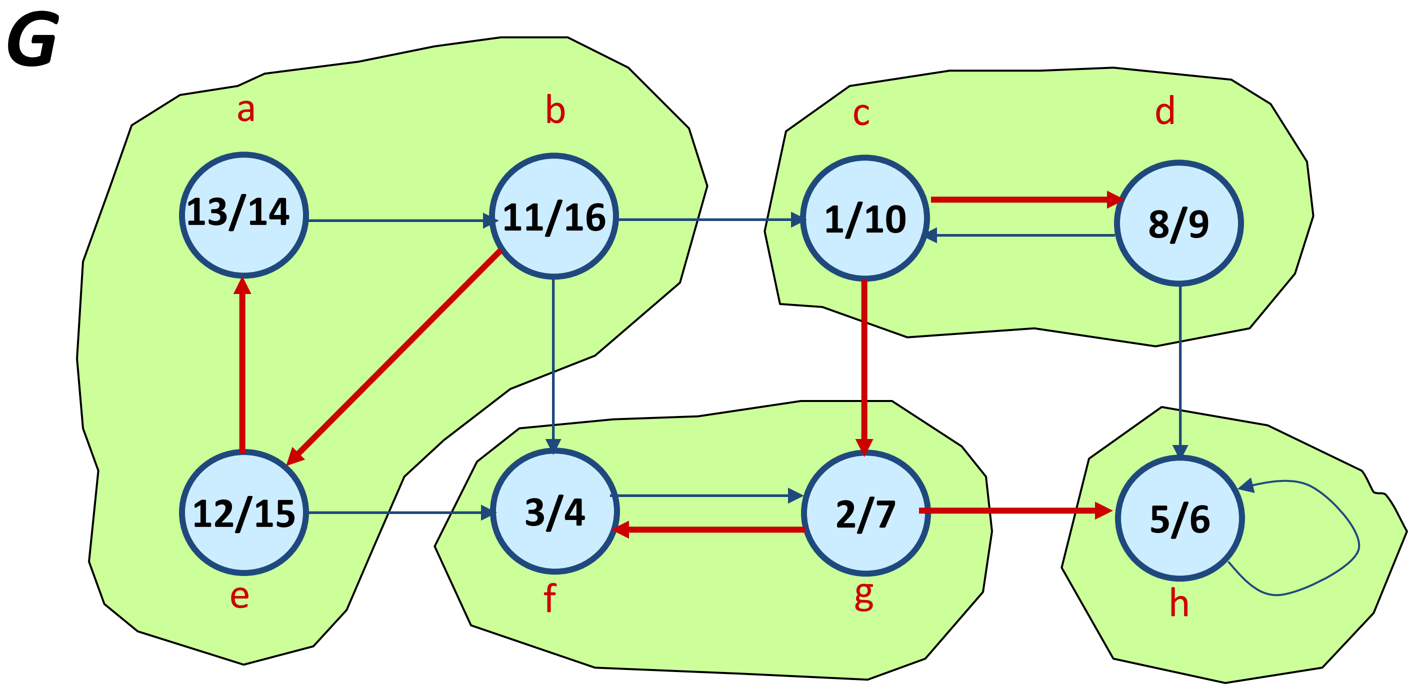
1. call DFS(G) to compute finishing times $f[u]$ for all u
2. compute G^{T}
3. call DFS(G^{T}), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

Time: $\Theta(V + E)$.

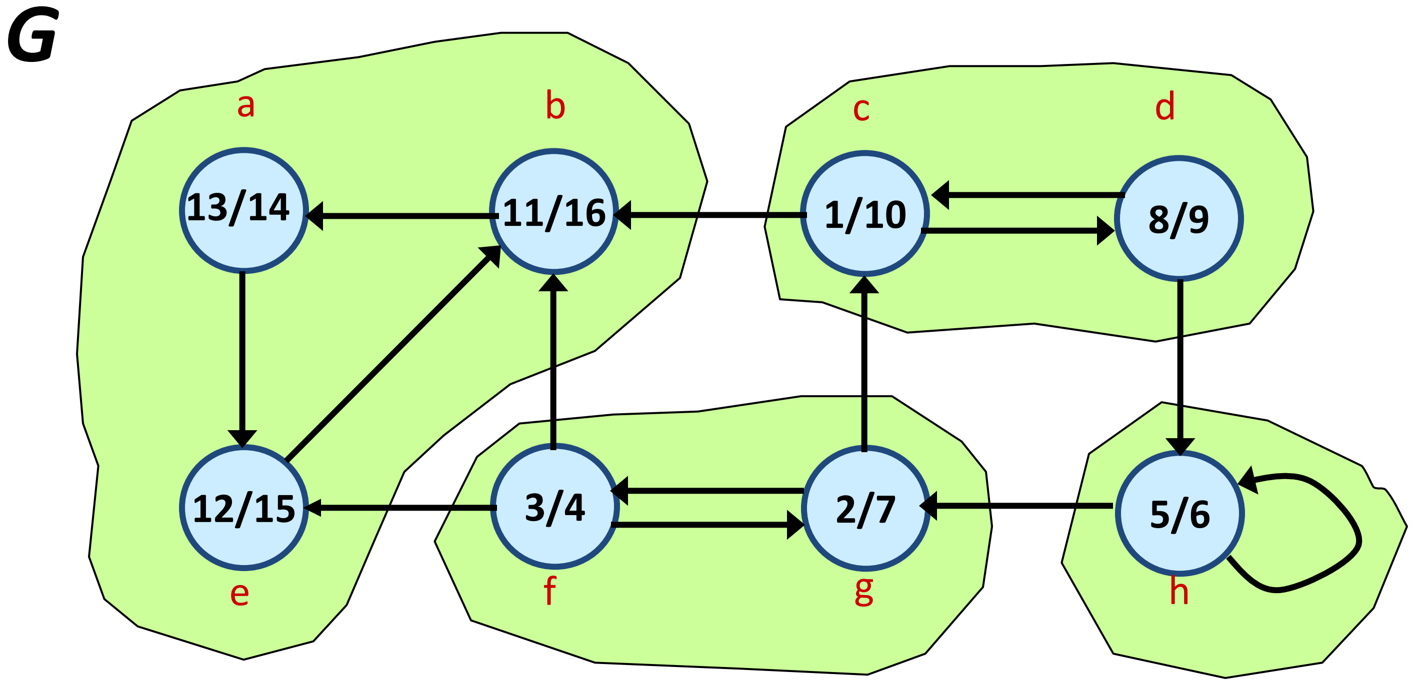
Example



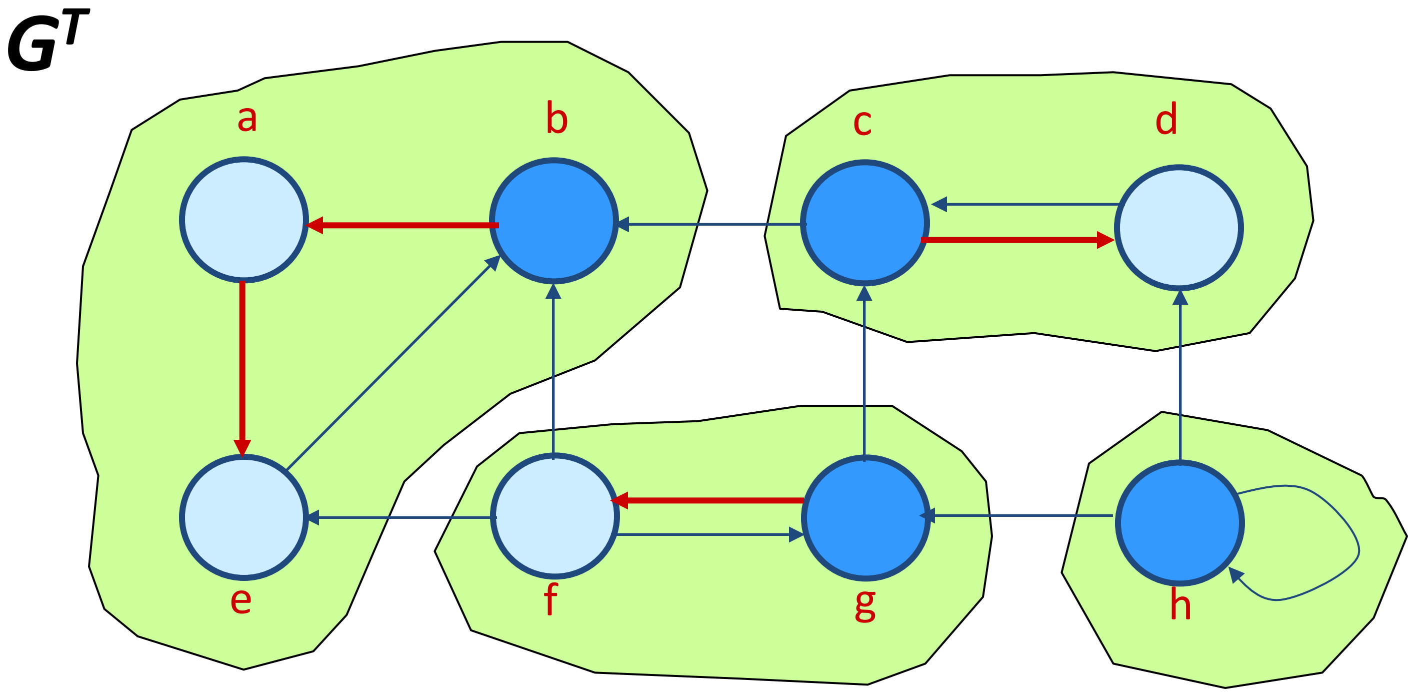
Example



Example

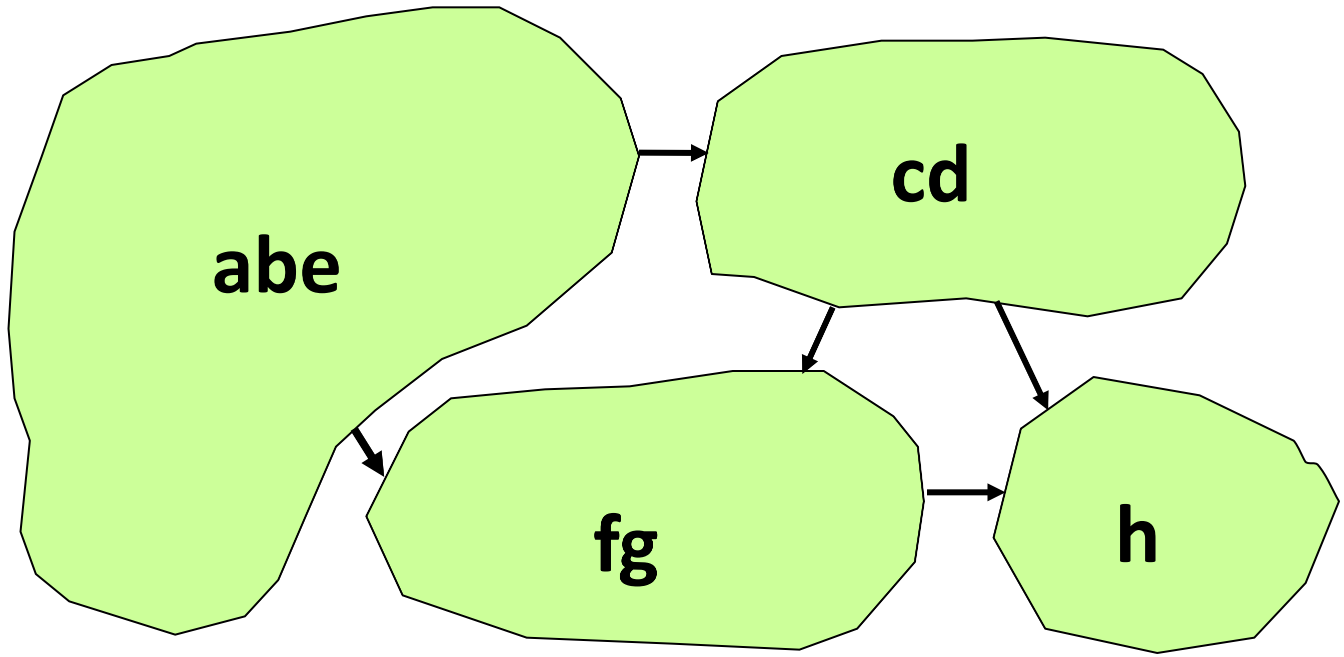


Example



(b (a (e e) a) b) (c (d d) c) (g (f f) g) (h)

Example



How does it work?

- **Idea:**
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - Because we are running DFS on G^T, we will not be visiting any v from a u, where v and u are in different components.
- **Notation:**
  - $d[u]$ and $f[u]$ always refer to *first* DFS.
  - Extend notation for d and f to sets of vertices $U \subseteq V$:
  - $d(U) = \min_{u \in U}\{d[u]\}$ (earliest discovery time)
  - $f(U) = \max_{u \in U}\{f[u]\}$ (latest finishing time)

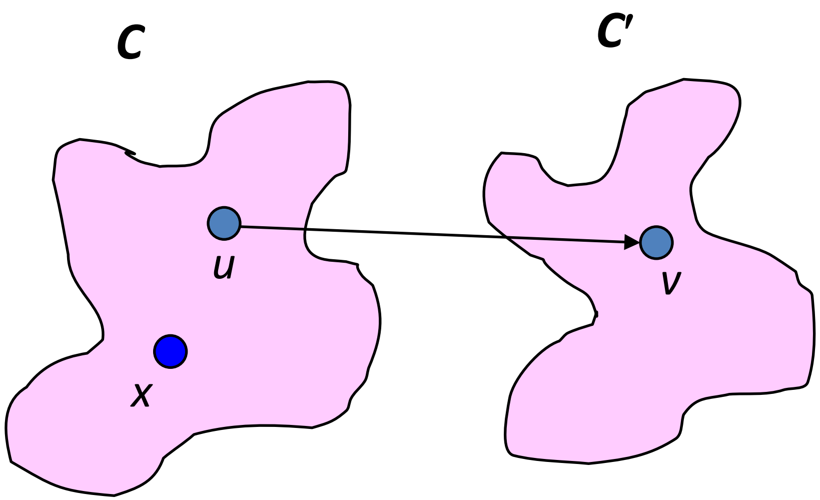
SCCs and DFS finishing times

Lemma 3

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 1: $d(C) < d(C')$
  - Let x be the first vertex discovered in C.
  - At time $d[x]$, all vertices in C and C' are white. Thus, there exist paths of white vertices from x to all vertices in C and C'.
  - By the white-path theorem, all vertices in C and C' are descendants of x in depth-first tree.
  - By the parenthesis theorem, $f[x] = f(C) > f(C')$.

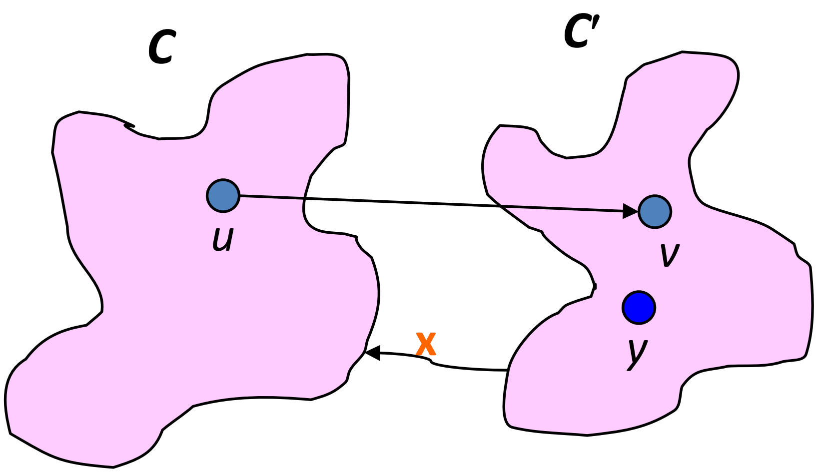


SCCs and DFS finishing times

Lemma 4
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 2: $d(C) > d(C')$
  - Let y be the first vertex discovered in C'.
  - At $d[y]$, all vertices in C' are white and there is a white path from y to each vertex in C' $\Rightarrow$ all vertices in C' become descendants of y. Again, $f[y] = f(C')$.
  - At $d[y]$, all vertices in C are also white.
  - By lemma 2, since there is an edge (u, v), we cannot have a path from C' to C.
  - So no vertex in C is reachable from y.
  - Therefore, at time $f[y]$, all vertices in C are still white.
  - Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.



SCCs and DFS finishing times

Corollary 1
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof:

- $(u, v) \in E^T \Rightarrow (v, u) \in E$.
- Since SCC's of G and G^T are the same, $f(C') > f(C)$, by Lemma.

Correctness of SCC

- When we do the second DFS, on G^T, start with SCC C such that $f(C)$ is maximum.
  - The second DFS starts from some $x \in C$, and it visits all vertices in C.
  - Corollary 1 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
  - Therefore, DFS will visit *only* vertices in C.
  - Which means that the depth-first tree rooted at x contains *exactly* the vertices of C.

Correctness of SCC

- The next root chosen in the second DFS is in SCC C' such that $f(C')$ is maximum over all SCC's other than C.
  - DFS visits all vertices in C', but the only edges out of C' go to C, *which we've already visited*.
  - Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only
  - vertices in its SCC—get tree edges to these,
  - vertices in SCC's *already visited* in second DFS—get *no* tree edges to these.