COMP251: Greedy algorithms

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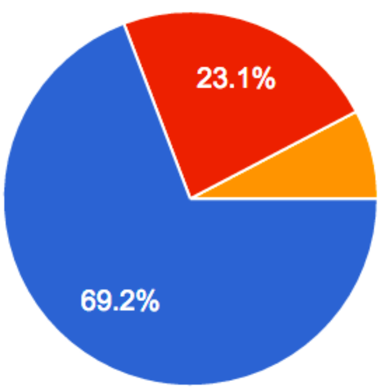
Based on (Cormen *et al.*, 2002)

Based on slides from D. Plaisted (UNC) & (goodrich & Tamassia, 2009)

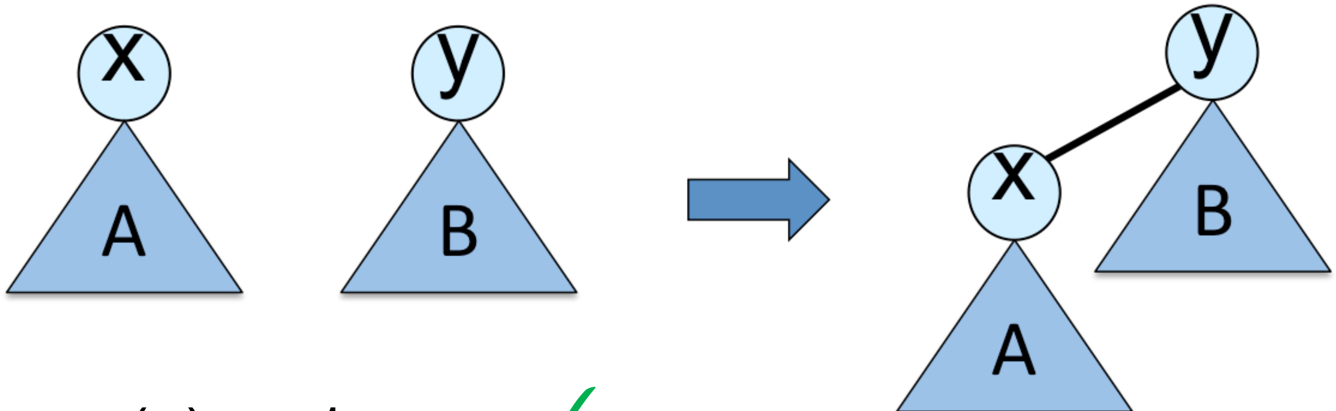
Disjoint sets are represented with an array rep[], that stores the representative rep[i] of each element i. The running time of the function find(i) that returns the representative of the set containing i is:

- $\boldsymbol{\Omega}(1)$ ✓ (More interestingly $\boldsymbol{\Theta}(1)$)
- O(log n)
- $\boldsymbol{\Theta}$(log n)

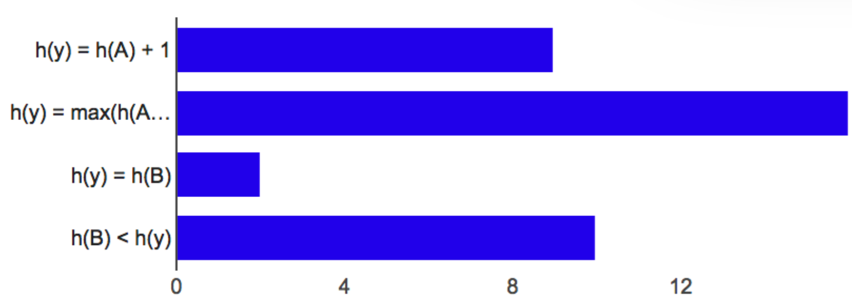


Omega(1)	**18**	69.2%
O(log n)	**6**	23.1%
Theta(log n)	**2**	7.7%

Let h(A) (resp. h(B)) be the height of the tree A (resp. B) rooted at x (resp. y). We assume that h(B) <= h(A) + 1. After union(x,y), which assertion are true?



- h(y) = h(A) + 1 ✓
- h(y) = max(h(A)+1, h(B)) ✓
- h(y) = h(B) ✗
- h(B) < h(y) ✗



h(y) = h(A) + 1	**9**	34.6%
h(y) = max(h(A)+1, h(B))	**16**	61.5%
h(y) = h(B)	**2**	7.7%
h(B) < h(y)	**10**	38.5%

Overview

- Algorithm design technique to solve optimization problems.
- Problems exhibit optimal substructure.
- Idea (the greedy choice):
  - When we have a choice to make, make the one that looks best right now.
  - Make a locally optimal choice in hope of getting a globally optimal solution.

Greedy Strategy

The choice that seems best at the moment is the one we go with.

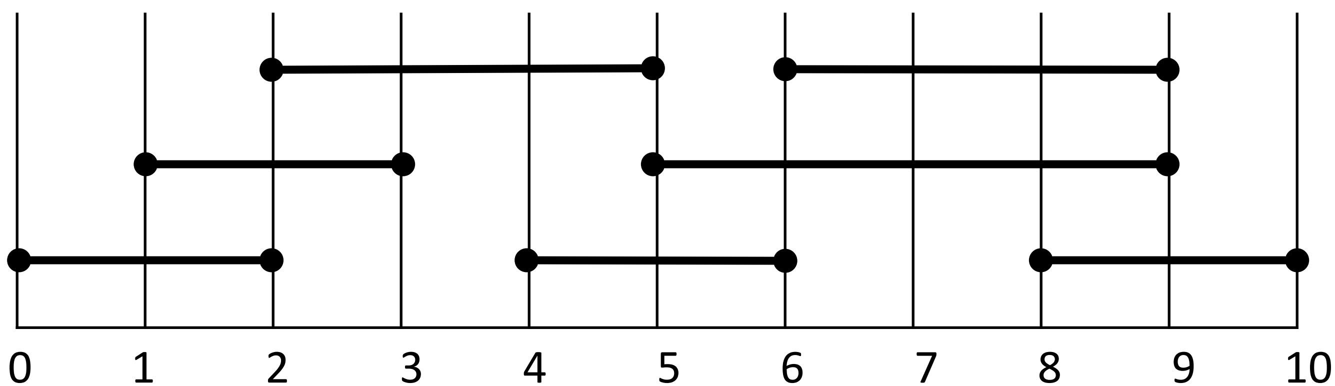
- Prove that when there is a choice to make, one of the optimal choices is the greedy choice. Therefore, it is always safe to make the greedy choice.
- Show that all but one of the sub-problems resulting from the greedy choice are empty.

Activity-selection Problem

- Input: Set S of n activities, $a_1, a_2, \ldots, a_n$.
  - s_i = start time of activity i.
  - f_i = finish time of activity i.
- Output: Subset A of maximum **number** of compatible activities.
  - 2 activities are compatible, if their intervals do not overlap.

Example:

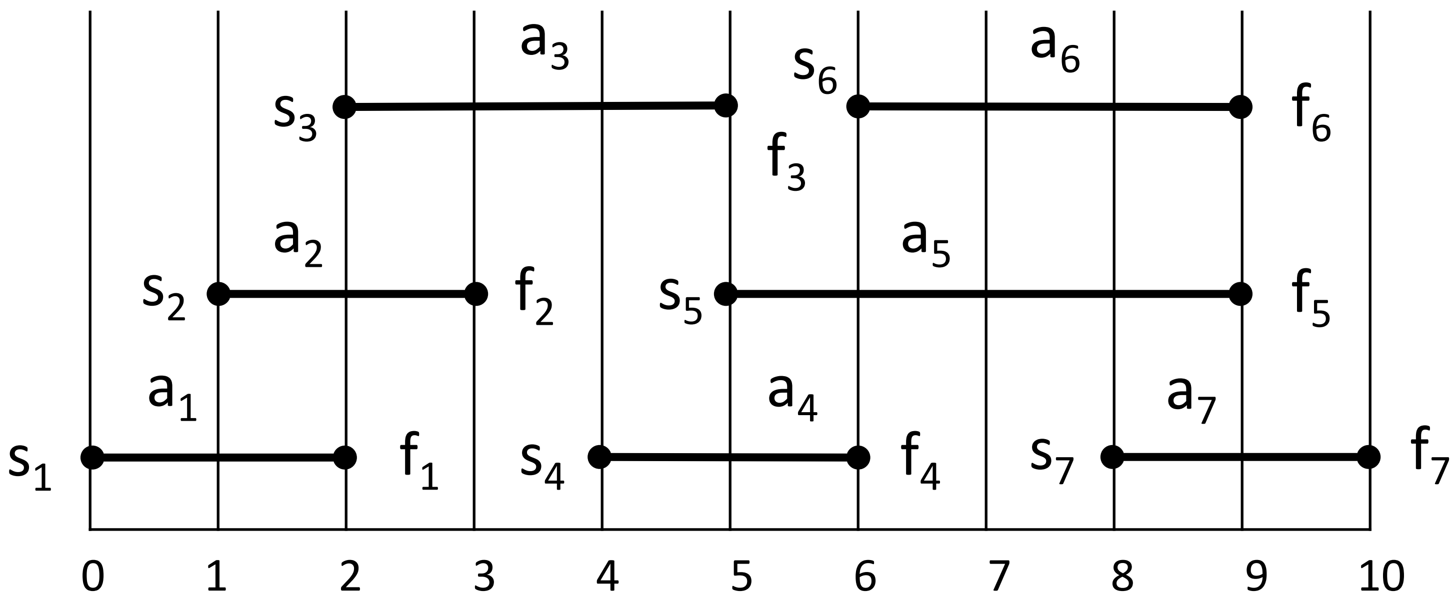
Activities in each line are compatible.



Activity-selection Problem

i	1	2	3	4	5	6	7
s_i	0	1	2	4	5	6	8
f_i	2	3	5	6	9	9	10

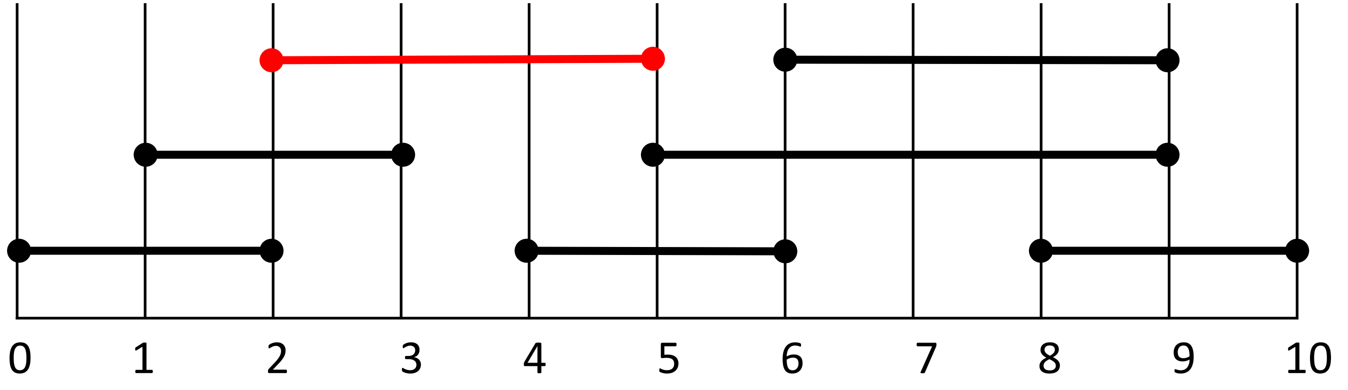
Activities sorted by finishing time.



Optimal compatible set: $\{ a_1 , a_3 , a_5 \}$

Optimal Substructure

- Assume activities are sorted by finishing times.
- Suppose an optimal solution includes activity a_k. This solution is obtained from:
  - An optimal selection of $a_1, \ldots, a_{k-1}$ activities compatible with one another, and that finish **before** a_k starts.
  - An optimal solution of $a_{k+1}, \ldots, a_n$ activities compatible with one another, and that start **after** a_k finishes.



Optimal Substructure

- Let S_{ij} = subset of activities in S that start after a_i finishes and finish before a_j starts.

$$S_{ij} = \left\{ a_k \in S : \forall i, j \quad f_i \leq s_k < f_k \leq s_j \right\}$$

- A_{ij} = optimal solution to S_{ij}

- $A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}$

Recursive Solution

- Subproblems: Selecting maximum number of mutually compatible activities from S_{ij}.
- Let $c[i, j]$ = size of maximum-size subset of mutually compatible activities in S_{ij}.

Recursive solution:

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset \\ \max\limits_{i<k<j \text{ and } a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \emptyset \end{cases}$$

Note: Here, we do not know which k to use for the optimal solution.

Greedy choice

Theorem:

Let $S_{ij} \neq \varnothing$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\{ f_k : a_k \in S_{ij}\}$. Then:

1. a_m is used in some maximum-size subset of mutually compatible activities of S_{ij}.

2. $S_{im} = \varnothing$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem.

Greedy choice

Proof:

(1) a_m is used in some maximum-size subset of mutually compatible activities of S_{ij}.

- Let A_{ij} be a maximum-size subset of mutually compatible activities in S_{ij} (i.e. an optimal solution of S_{ij}).
- Order activities in A_{ij} in monotonically increasing order of finish time, and let a_k be the first activity in A_{ij}.
- If $a_k = a_m \Rightarrow$ done.
- Otherwise, let $A'_{ij} = A_{ij} - \{ a_k \} \cup \{ a_m \}$
- A'_{ij} is valid because a_m finishes before a_k
- Since $|A_{ij}|=|A'_{ij}|$ and A_{ij} maximal $\Rightarrow$ A'_{ij} maximal too.

Greedy choice

Proof:

(2) $S_{im} = \varnothing$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem.

If there is $a_k \in S_{im}$ *then* $f_i \leq s_k < f_k \leq s_m < \boldsymbol{f_m} \Rightarrow f_k < f_m$ which contradicts the hypothesis that a_m has the earlier finish.

Greedy choice

	Before theorem	After theorem
# subproblems in optimal solution	2	1
# choices to consider	j-i-1	1

$$\underbrace{\hphantom{xxxxxxxxxx}}_{A_{ij} = A_{ik} \cup \{ a_k \} \cup A_{kj}} \qquad \underbrace{\hphantom{xxxxxxxxxx}}_{A_{ij} = \{ a_m \} \cup A_{mj}}$$

We can now solve the problem S_{ij} top-down:

- Choose $a_m \in S_{ij}$ with the earliest finish time (greedy choice).
- Solve S_{mj}.

Activity-selection Problem

i	1	2	3	4	5	6	7
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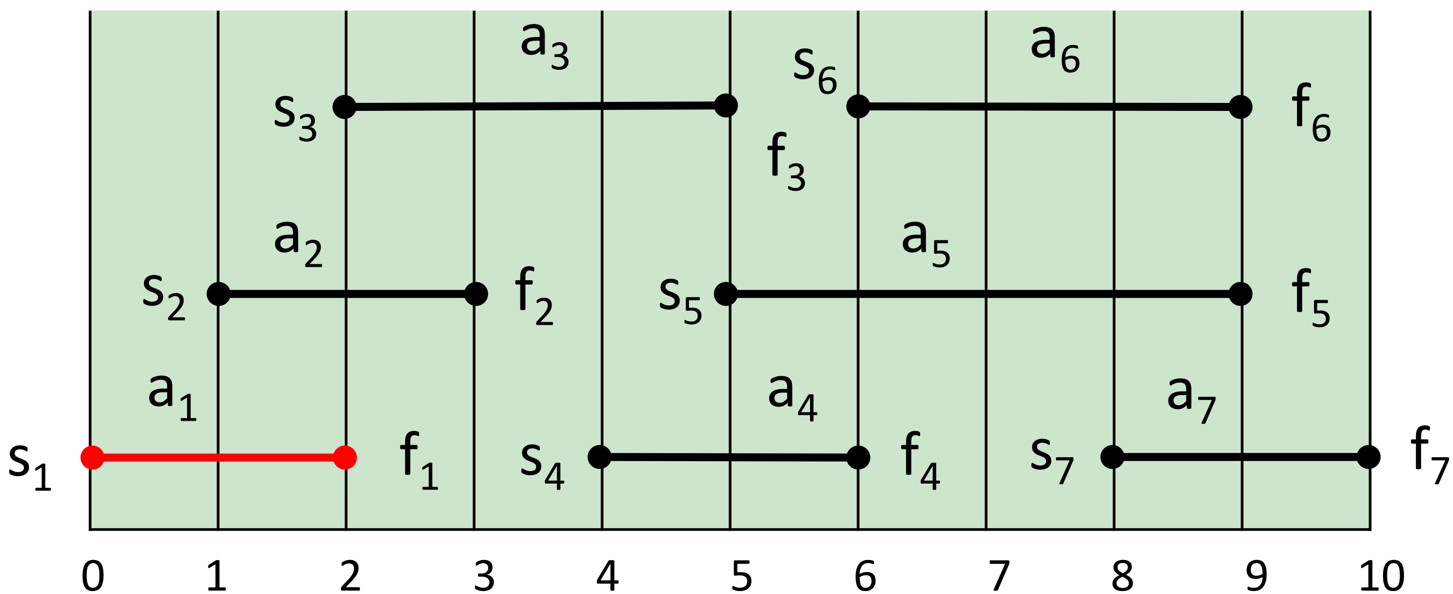
Activities sorted by finishing time.



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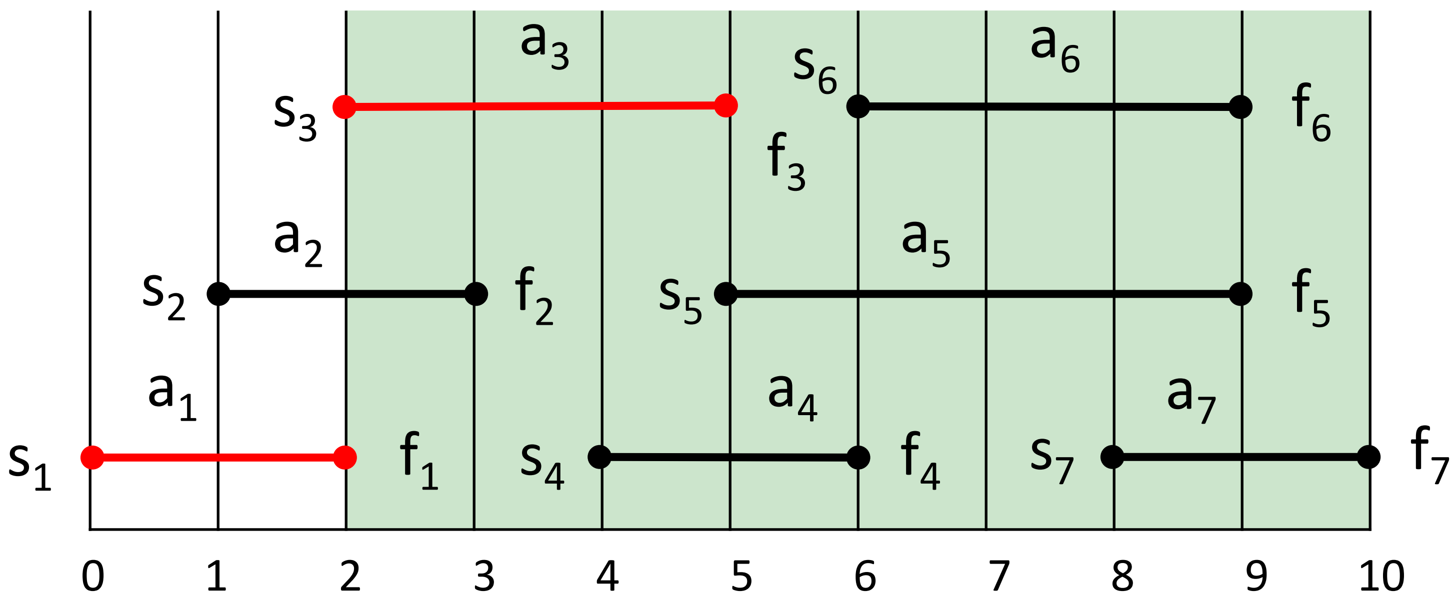

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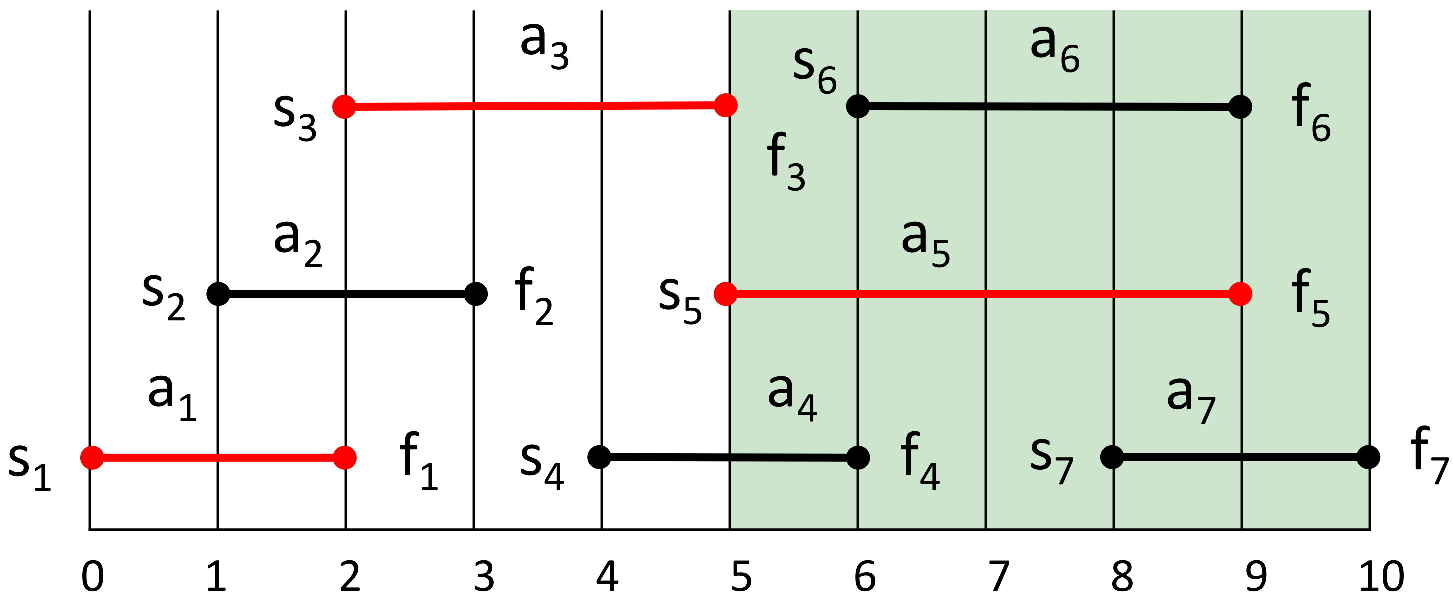

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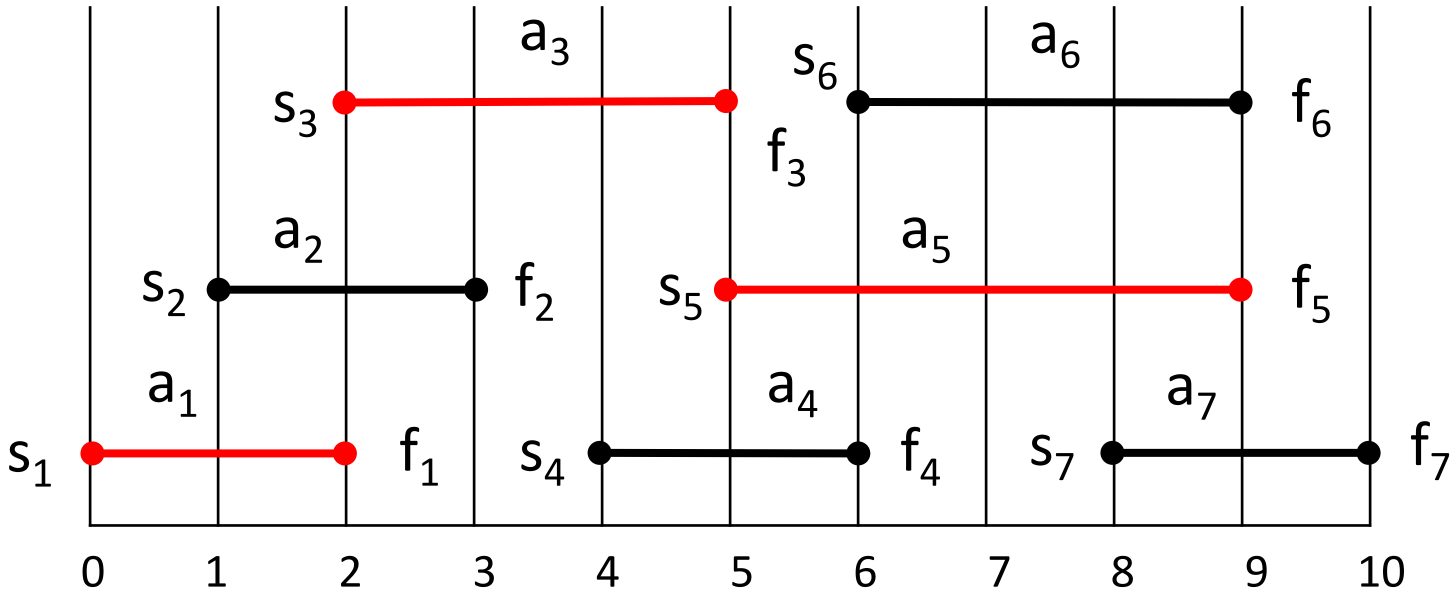


Recursive Algorithm

Recursive-Activity-Selector (*s, f, i, n*)

1. $m \leftarrow i+1$
2. **while** $m \leq n$ and $s_m < f_i$ // Find first activity in $S_{i,n+1}$
3. **do** $m \leftarrow m+1$
4. **if** $m \leq n$
5. **then return** $\{a_m\} \cup$ Recursive-Activity-Selector(s, f, m, n)
6. else return $\varnothing$

Initial Call: Recursive-Activity-Selector (s, f, 0, n+1)
Complexity: Θ(n)

Note 1: We assume activities are already ordered by finishing time.
Note 2: Straightforward to convert the algorithm to an iterative one.

Typical Steps

- Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.
- Prove that there is always an optimal solution that makes the greedy choice (greedy choice is safe).
- Show that greedy choice and optimal solution to subproblem $\Rightarrow$ optimal solution to the problem.
- Make the greedy choice and **solve top-down**.
- You may have to preprocess input to put it into greedy order (e.g. sorting activities by finish time).

Elements of Greedy Algorithms

No general way to tell if a greedy algorithm is optimal, but two key ingredients are:

- Greedy-choice Property.
  - A globally optimal solution can be arrived at by making a locally optimal (greedy) choice.
- Optimal Substructure.

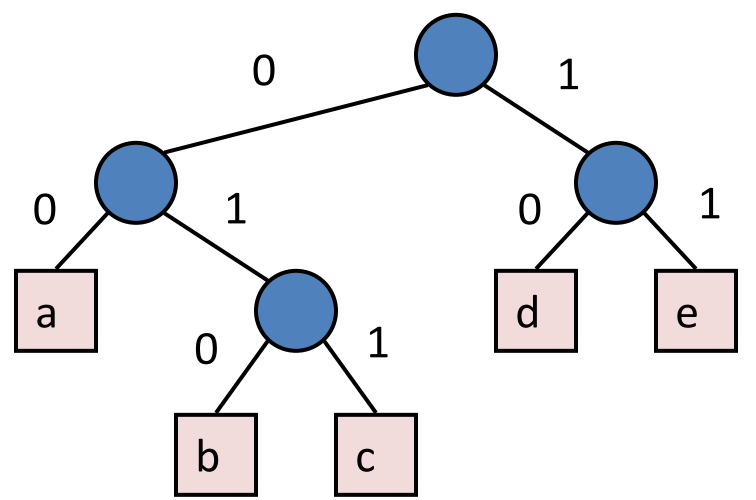
Text Compression

- Given a string X, efficiently encode X into a smaller string Y
  - Saves memory and/or bandwidth
- A good approach: **Huffman encoding**
  - Compute frequency f(c) for each character c.
  - Encode high-frequency characters with short code words
  - No code word is a prefix for another code
  - Use an optimal encoding tree to determine the code words

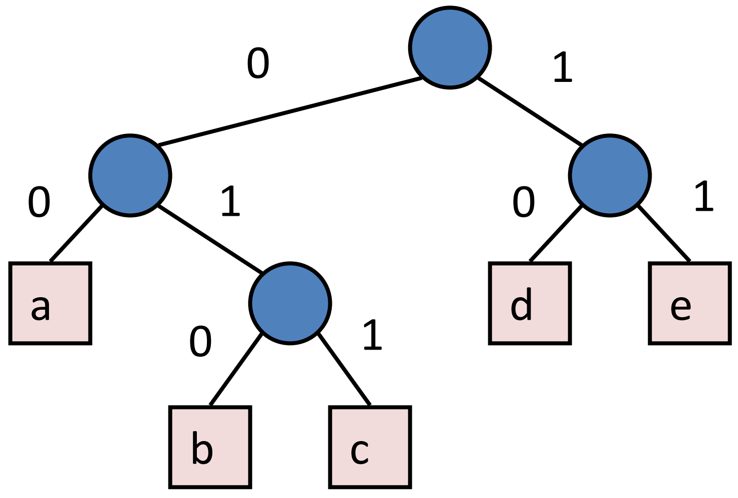
Encoding Tree Example

- A **code** is a mapping of each character of an alphabet to a binary code-word
- A **prefix code** is a binary code such that no code-word is the prefix of another code-word
- An **encoding tree** represents a prefix code
  - Each external node (leaf) stores a character
  - The code word of a character is given by the path from the root to the external node storing the character (0 for a left child and 1 for a right child)

00	010	011	10	11
a	b	c	d	e



Encoding Example

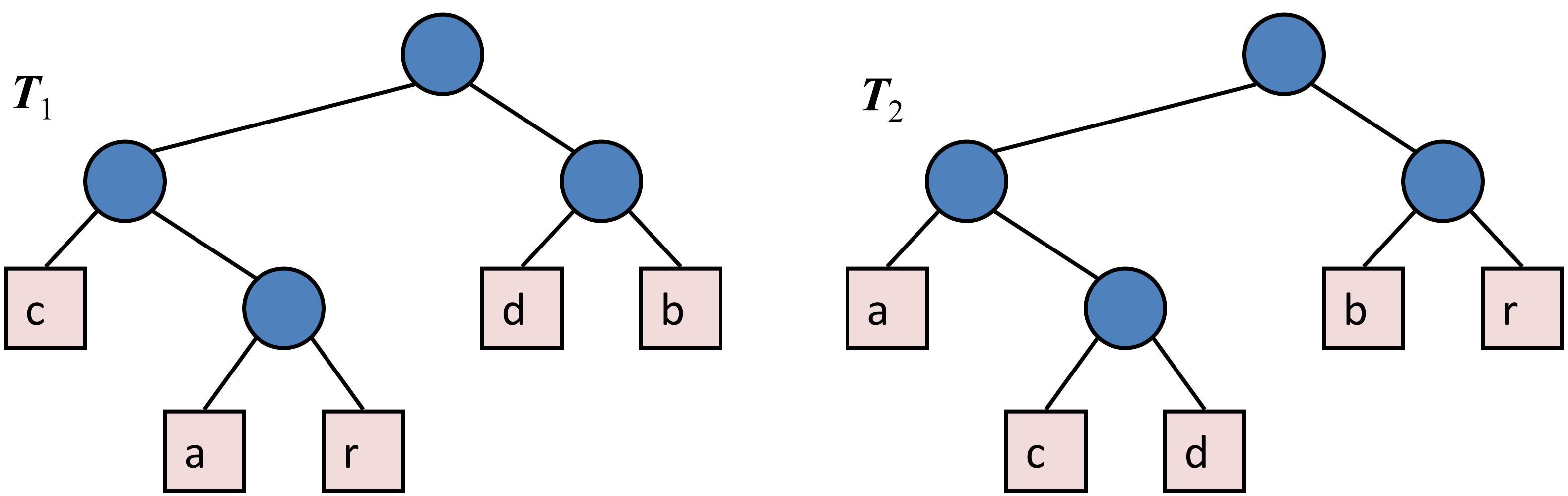


Initial string: X = acda

Encoded string: X = 00 011 10 00

Encoding Tree Optimization

- Given a text string X, we want to find a prefix code for the characters of X that yields a small encoding for X
  - Frequent characters should have long code-words
  - Rare characters should have short code-words
- Example
  - X = abracadabra
  - T_1 encodes X into 29 bits
  - T_2 encodes X into 24 bits

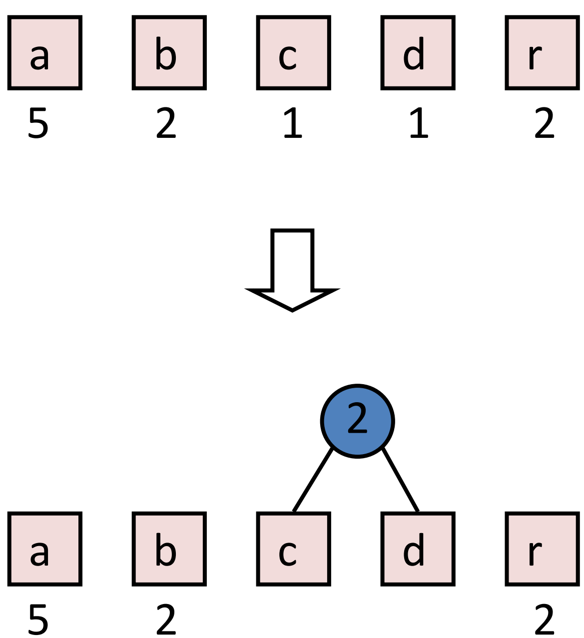


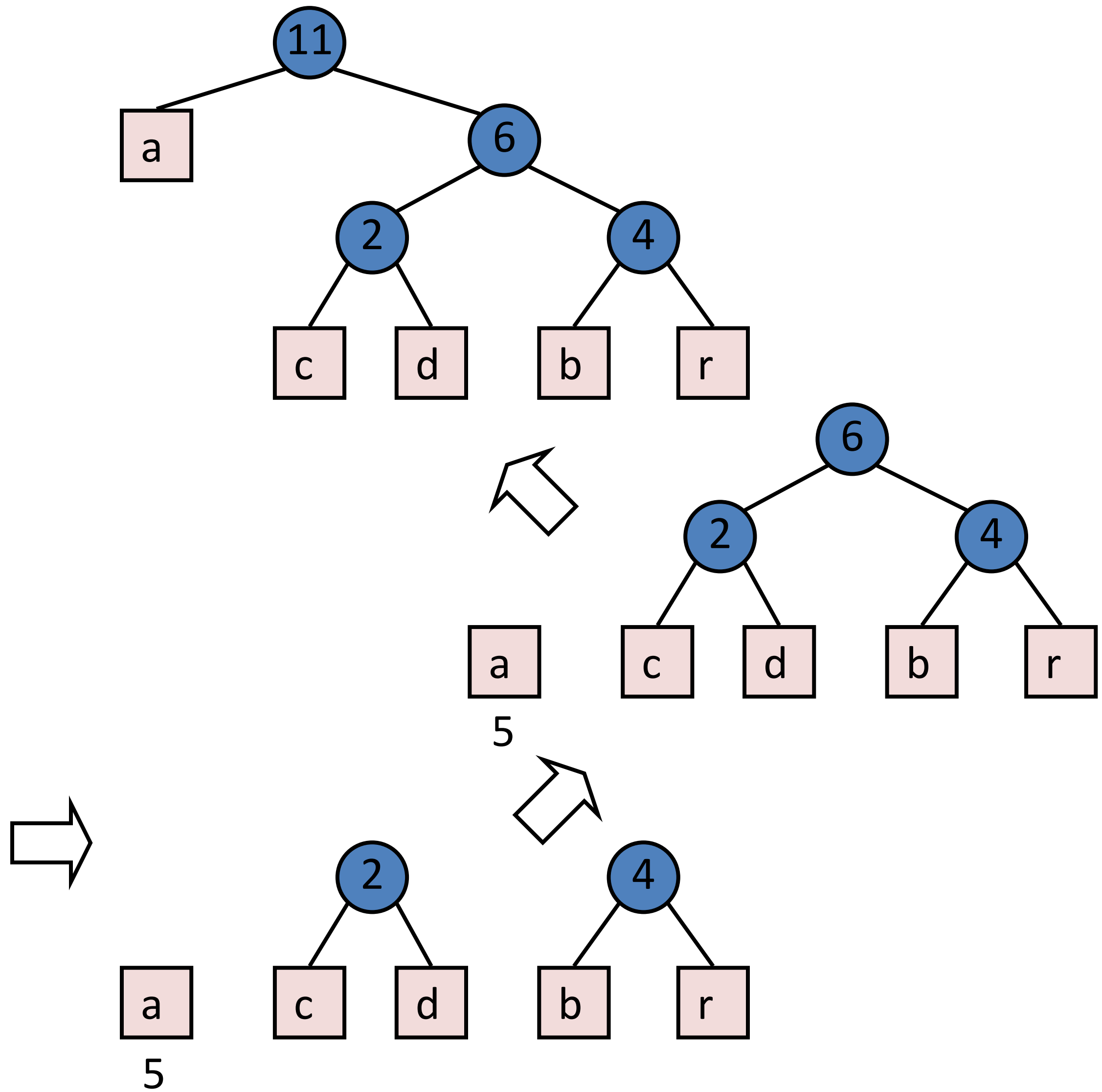
Example

$X =$ abracadabra

Frequencies

a	b	c	d	r
5	2	1	1	2

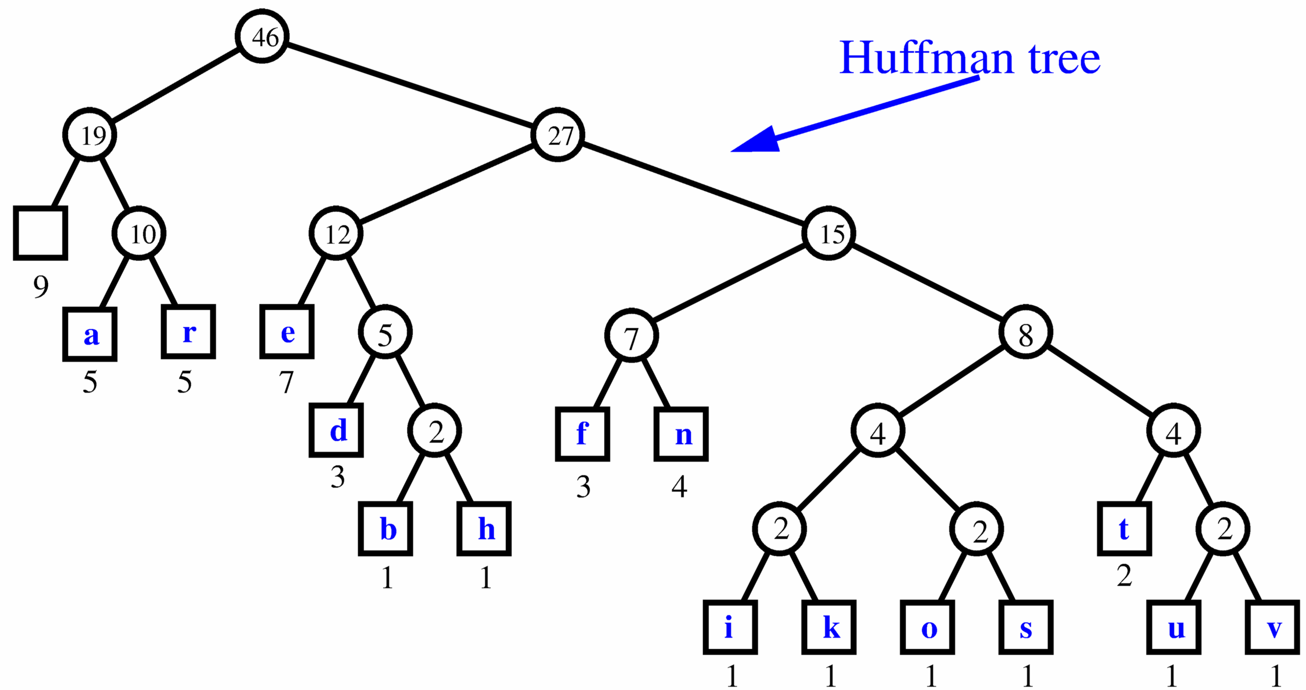




Extended Huffman Tree Example

String: **a fast runner need never be afraid of the dark**

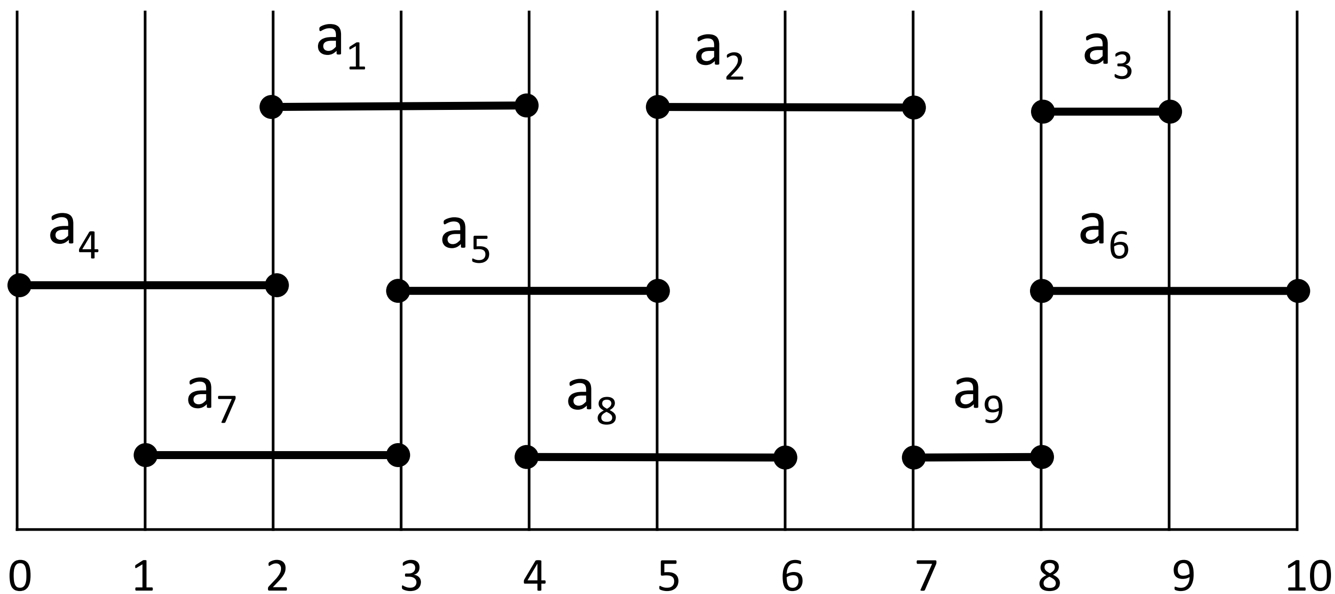
Character		**a**	**b**	**d**	**e**	**f**	**h**	**i**	**k**	**n**	**o**	**r**	**s**	**t**	**u**	**v**
Frequency	9	5	1	3	7	3	1	1	1	4	1	5	1	2	1	1



Huffman's Algorithm

- Given a string X, Huffman's algorithm construct a prefix code the minimizes the size of the encoding of X
- It runs in time $O(n + d \log d)$, where n is the size of X and d is the number of distinct characters of X
- A heap-based priority queue is used as an auxiliary structure

Algorithm *HuffmanEncoding*(X)
  Input string X of size n
  Output optimal encoding trie for X
  $C \leftarrow$ *distinctCharacters*(X)
  computeFrequencies(C, X)
  $Q \leftarrow$ new empty heap
  for all $c \in C$
    $T \leftarrow$ new single-node tree storing c
    Q.insert(*getFrequency*(c), T)
  while *Q.size*() > 1
    $f_1 \leftarrow$ *Q.minKey*()
    $T_1 \leftarrow$ *Q.removeMin*()
    $f_2 \leftarrow$ *Q.minKey*()
    $T_2 \leftarrow$ *Q.removeMin*()
    $T \leftarrow$ *join*(T_1, T_2)
    Q.insert($f_1 + f_2, T$)
  return *Q.removeMin*()



a1
a2
a3
a4
a5
a6
a7
a8
a9
0
1
2
3
4
5
6
7
8
9
10