COMP251: Red-black trees

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Based on (Cormen *et al.*, 2002)

Based on slides from D. Plaisted (UNC)

The running time of insertions in BST trees with n nodes is:

- $\Omega(\log(n))$
- $\Theta(\log(n))$
- $O(\log(n))$
- $O(n)$
- $\Omega(n)$

Which assertion(s) are true?

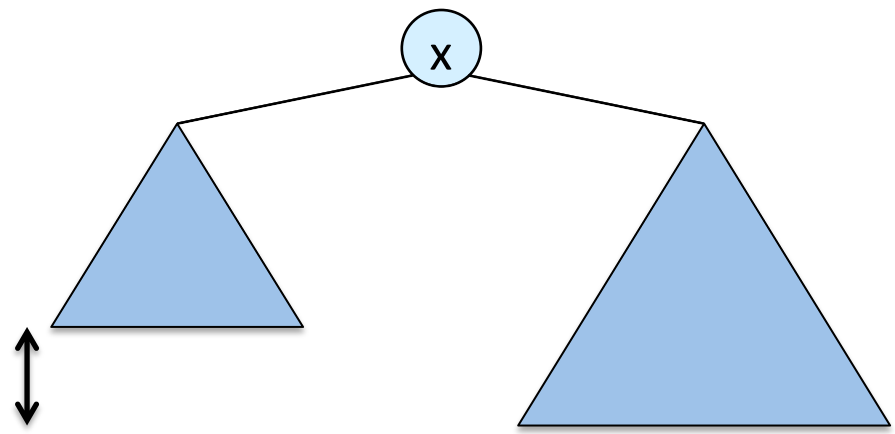
- Rotations preserve BST properties ✔
- Rotations preserve AVL tree properties ✗
- AVL properties can be restored using rotations ✔
- In the worst case, a rotation has a O(log n) running time ✗

How should we modify BST sort to sort numbers in decreasing order?

- Use post-order traversal
- Reverse the order of recursive calls in in-order traversal
- Use an AVL tree instead of a BST

Recap lecture 3

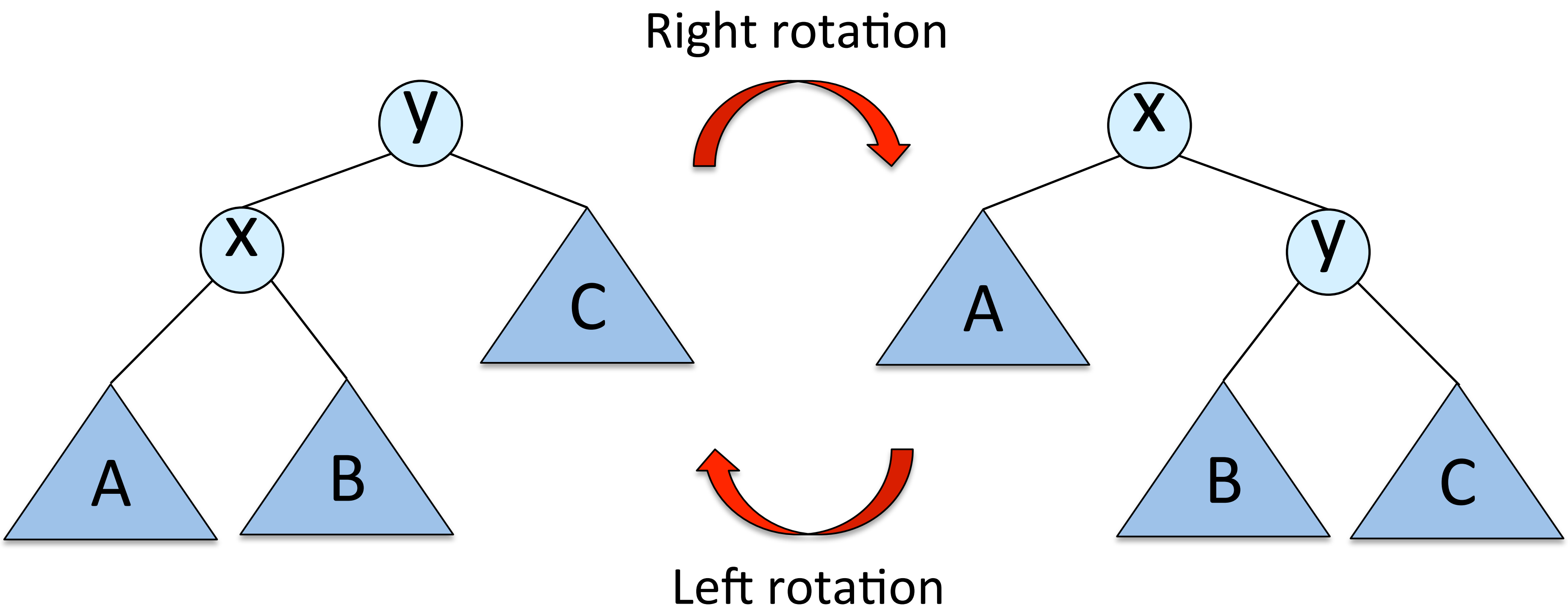
Definition: An AVL tree is a BST such that the heights of the two child subtrees of any node differ by at most one.



$$|h_{left} - h_{right}| \leq 1$$

- AVL trees are self-balanced binary search trees.
- Insert, Delete & Search take O(log n) in average and worst cases.

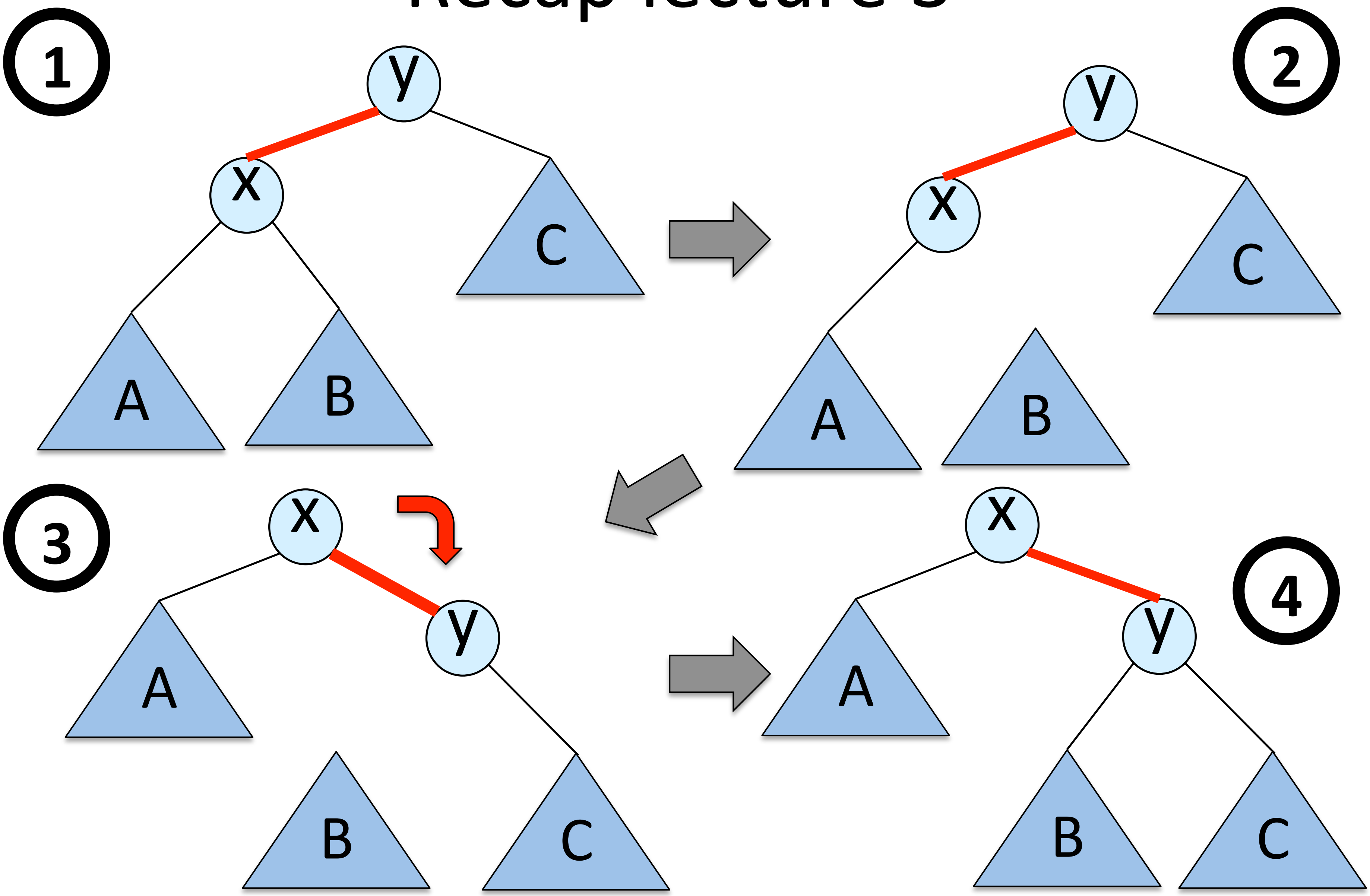
Recap lecture 3



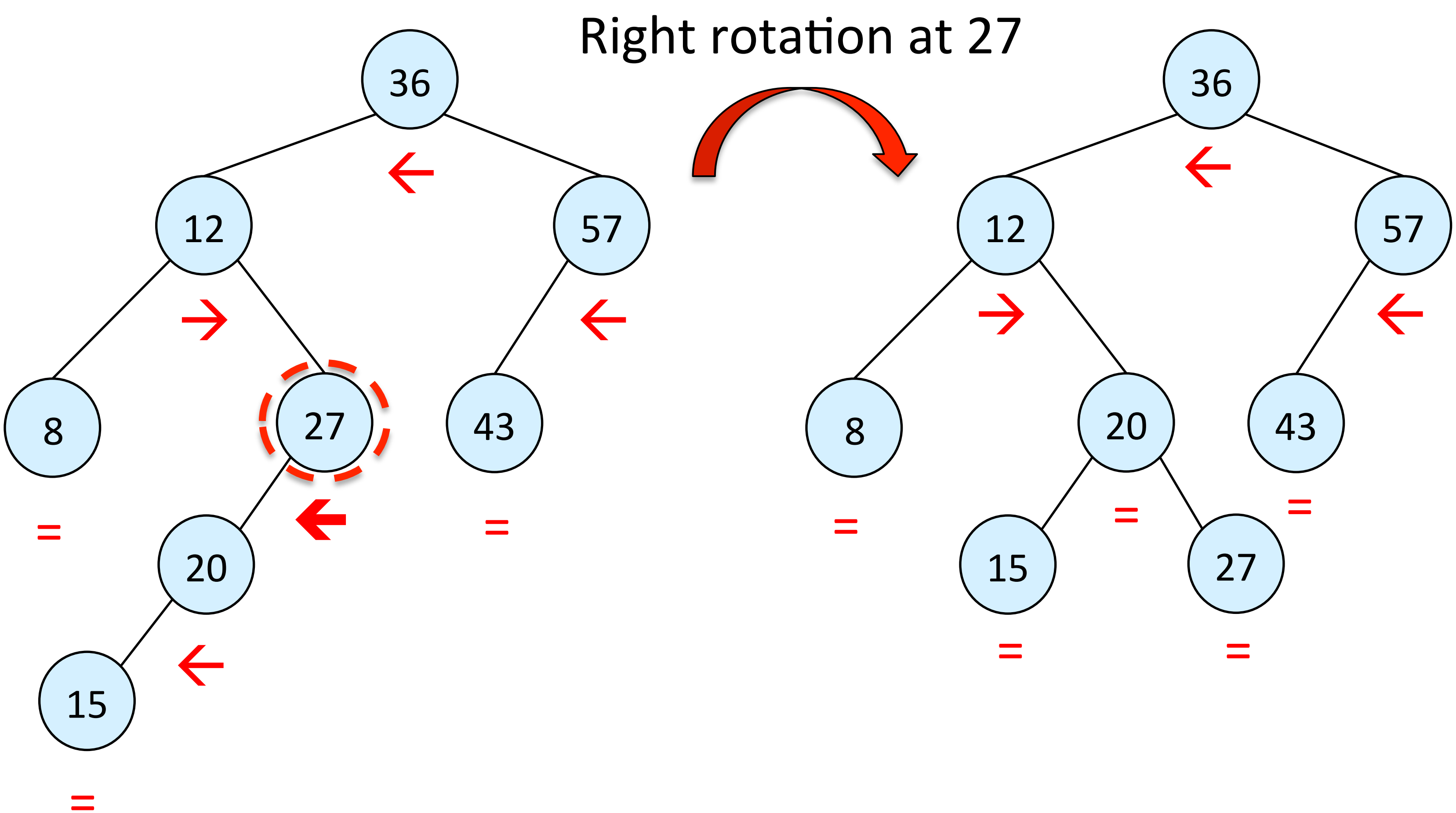
Rotations preserve the BST property.

Proof: elements in B are ≥ x and ≤ y…

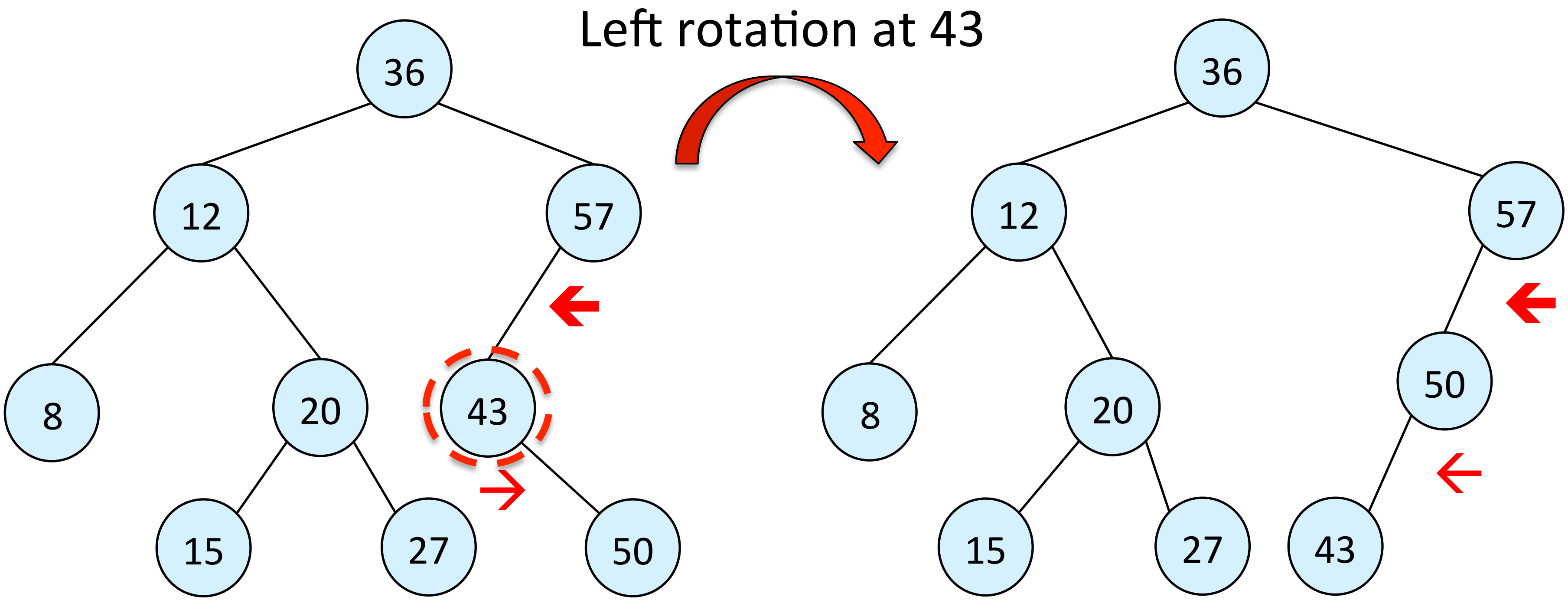
Recap lecture 3



Insert in AVL trees

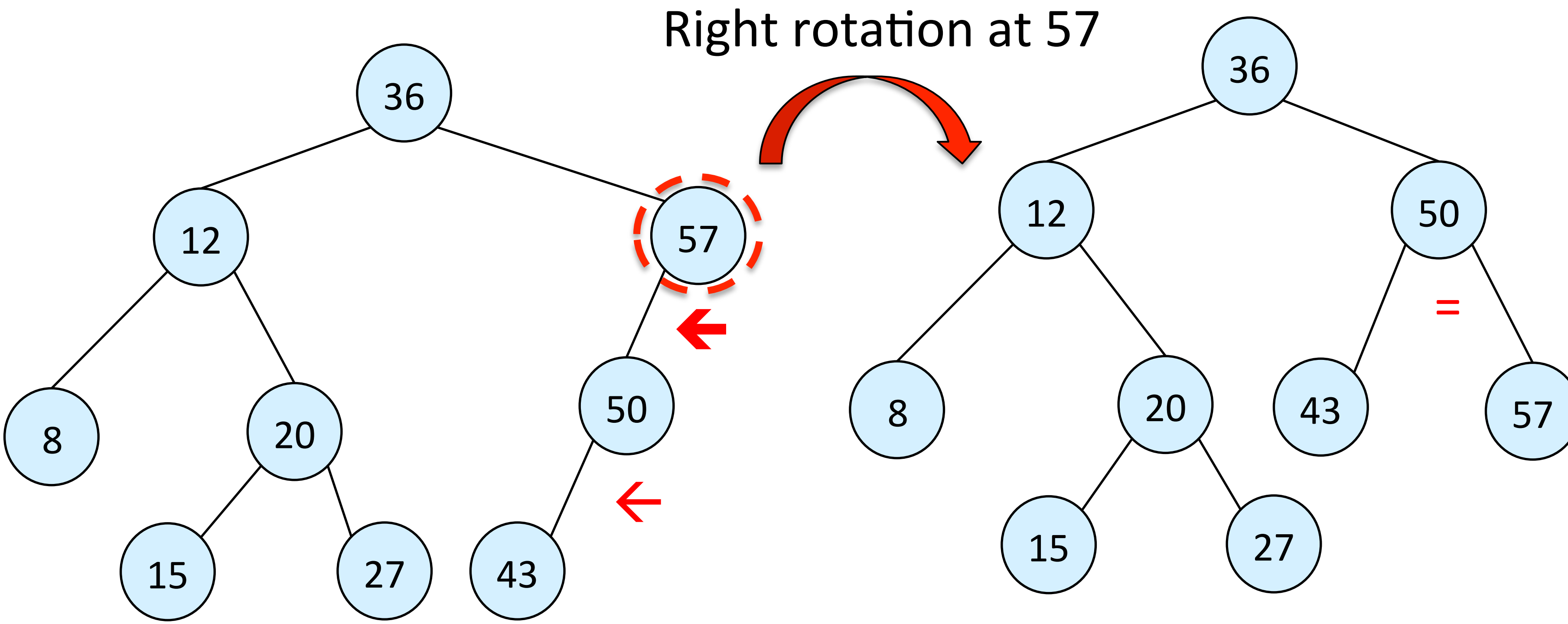


Insert in AVL trees



RotateLeft(T,43)

Insert in AVL trees



RotateRight(T,57)

Red-black trees: Overview

- Red-black trees are a variation of binary search trees to ensure that the tree is **balanced**.
  - Height is $O(\lg n)$, where n is the number of nodes.
- Operations take $O(\lg n)$ time in the worst case.
- Invented by R. Bayer (1972).
- Modern definition by L.J. Guibas & R. Sedgewick (1978).

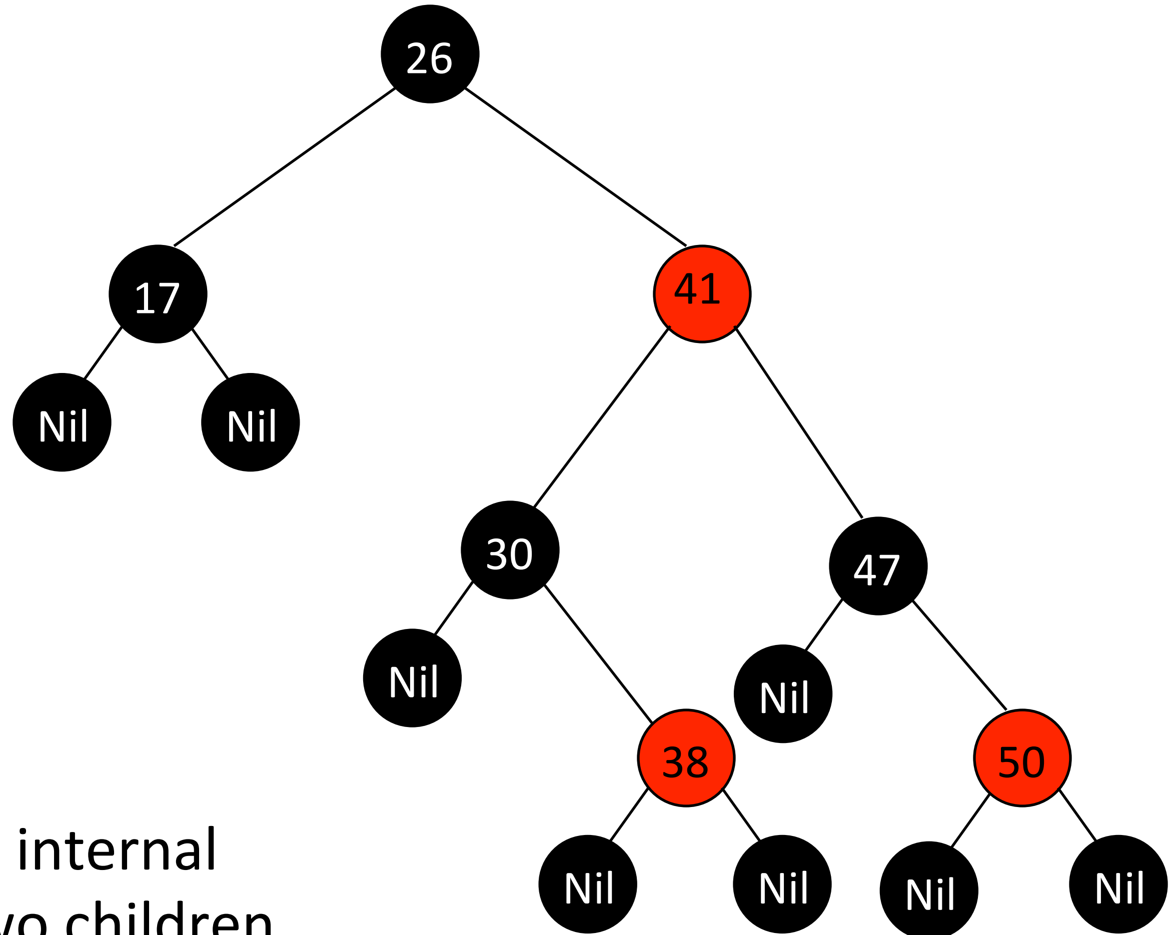
Red-black Tree

- Binary search tree + 1 bit per node: the attribute color, which is either **red** or **black**.
- All other attributes of BSTs are inherited:
  - *key*, *left*, *right*, and *parent*.
- All empty trees (leaves) are colored black.
  - Note: We can use a single sentinel, *nil*, for all the leaves of red-black tree *T*, with *color*[*nil*] = black. The root's parent is also *nil*[*T*].

Red-black Properties

1. Every node is either red or black.
2. The root is black.
3. Every leaf (*nil*) is black.
4. If a node is red, then its children are black (i.e. no 2 consecutive red nodes).
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes (i.e. same black height).

Red-black Tree – Example

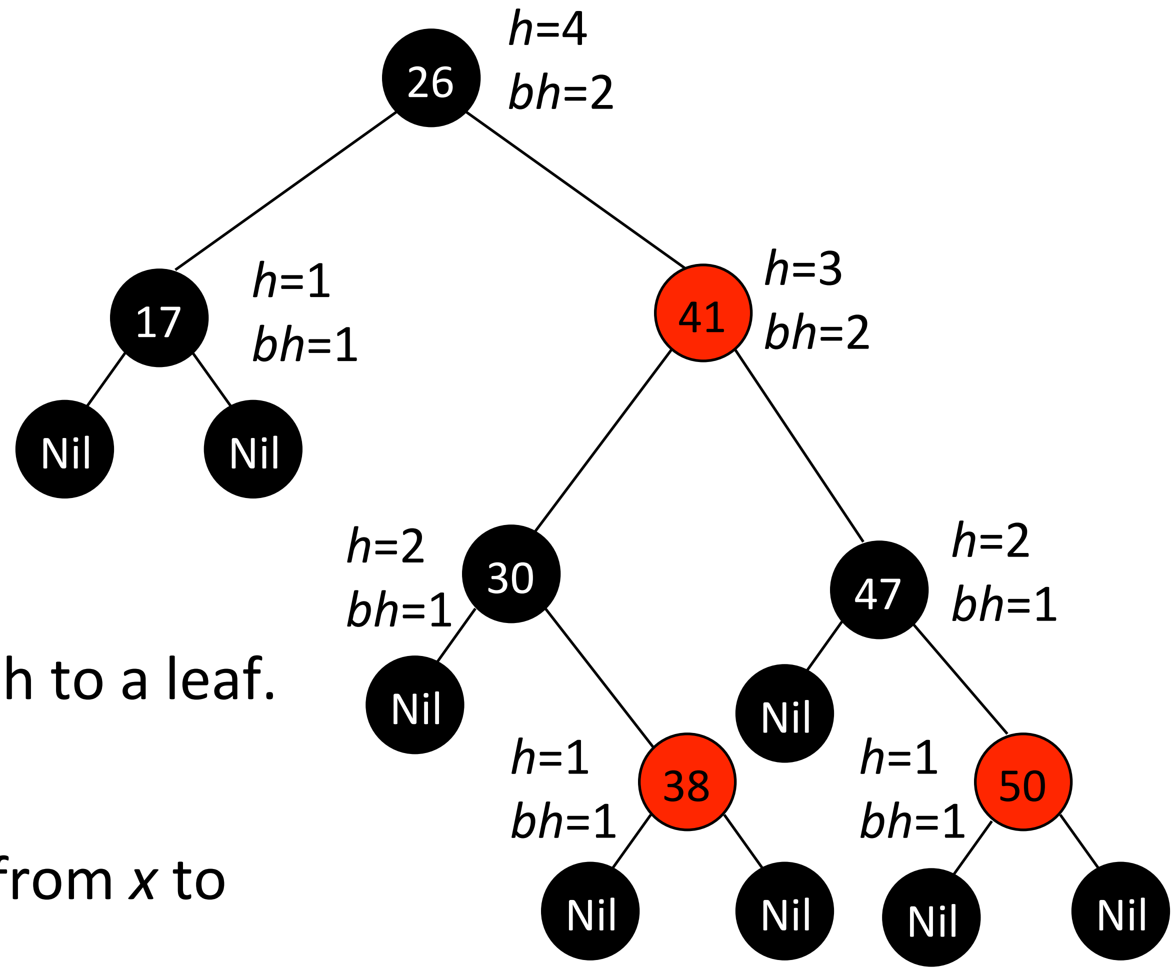


Note: every internal node has two children, even though nil leaves are not usually shown.

Height of a Red-black Tree

- Height of a node:
  - $h(x)$ = number of edges in the longest path to a leaf.
- Black-height of a node x, $bh(x)$:
  - $bh(x)$ = number of black nodes (including $nil[T]$) on the path from x to leaf, not counting x.
- Black-height of a red-black tree is the black-height of its root.
  - By Property 5, black height is well defined.

Height of a Red-black Tree



- Height h(x):

  #edges in a longest path to a leaf.

- Black-height bh(x):

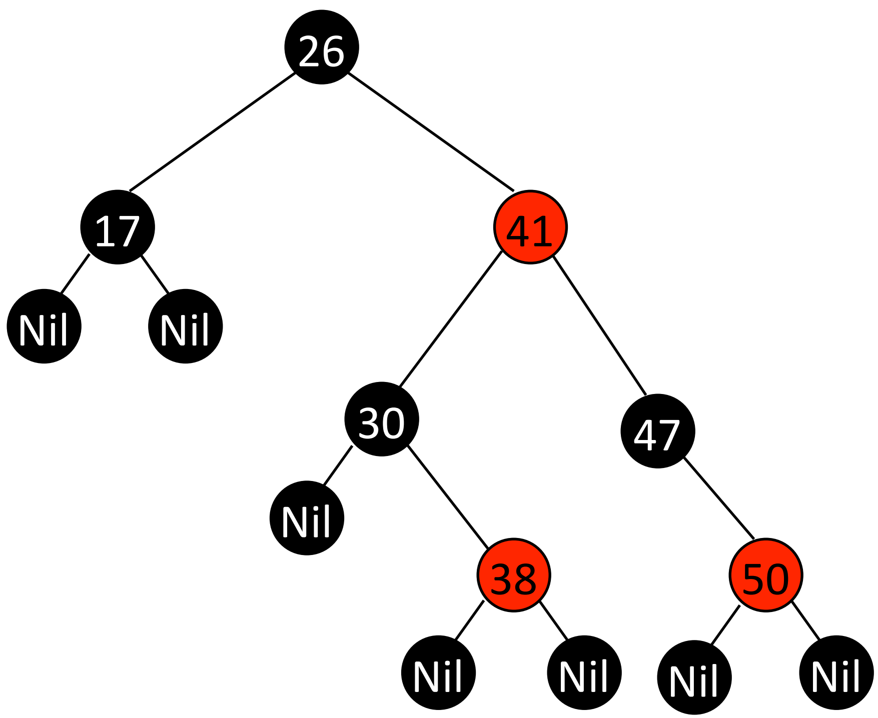
  # black nodes on path from *x* to leaf, *not counting x*.

- Property: bh(x) ≤ h(x) ≤ 2 bh(x*)*

Bound on RB Tree Height

Lemma 1: Any node x with height h(x) has a black-height bh(x) ≥ h(x)/2.

Proof: By property 4, ≤ h / 2 nodes on the path from the node to a leaf are red. Hence ≥ h/2 are black. ■



Bound on RB Tree Height

Lemma 2: The subtree rooted at any node *x* contains $\geq 2^{bh(x)}–1$ internal nodes.

Proof: By induction on height of *x*.

- **Base Case:** Height $h(x) = 0 \Rightarrow x$ is a leaf $\Rightarrow \text{bh}(x) = 0$.
  Subtree has $\geq 2^0–1 = 0$ nodes.
- **Induction Step:**
  – Each child of *x* has height $h(x) - 1$ and black-height either $b(x)$ (child is red) or $b(x) - 1$ (child is black).
  – By ind. hyp., each child has $\geq 2^{\text{bh}(x)-1}–1$ internal nodes.
  – Subtree rooted at *x* has $\geq 2 \cdot (2^{\text{bh}(x)-1} - 1) + 1$
  $= 2^{\text{bh}(x)} - 1$ internal nodes. (The +1 is for *x* itself) ■

Bound on RB Tree Height

Lemma 1: Any node x with height h(x) has a black-height bh(x) ≥ h(x)/2.

Lemma 2: The subtree rooted at any node x has ≥ $2^{bh(x)}–1$ internal nodes.

Lemma 3: A red-black tree with n internal nodes has height at most 2 lg(n+1).

Proof:

- By lemma 2, $n \geq 2^{bh} - 1$,
- By lemma 1, $bh \geq h/2$, thus $n \geq 2^{h/2} - 1$.
- $\Rightarrow$ $h \leq 2 \lg(n + 1)$.

Insertion in RB Trees

- Insertion must preserve all red-black properties.
- Should an inserted node be colored Red? Black?
- Basic steps:
  - Use BST Tree-Insert to insert a node *x* into *T.*
    - Procedure **RB-Insert(*x*)**.
  - Color the node *x* red.
  - Fix the new tree by (1) re-coloring nodes, and (2) performing rotation to preserve RB tree property.
    - Procedure **RB-Insert-Fixup**.

Insertion

RB-Insert(*T*, *z*)

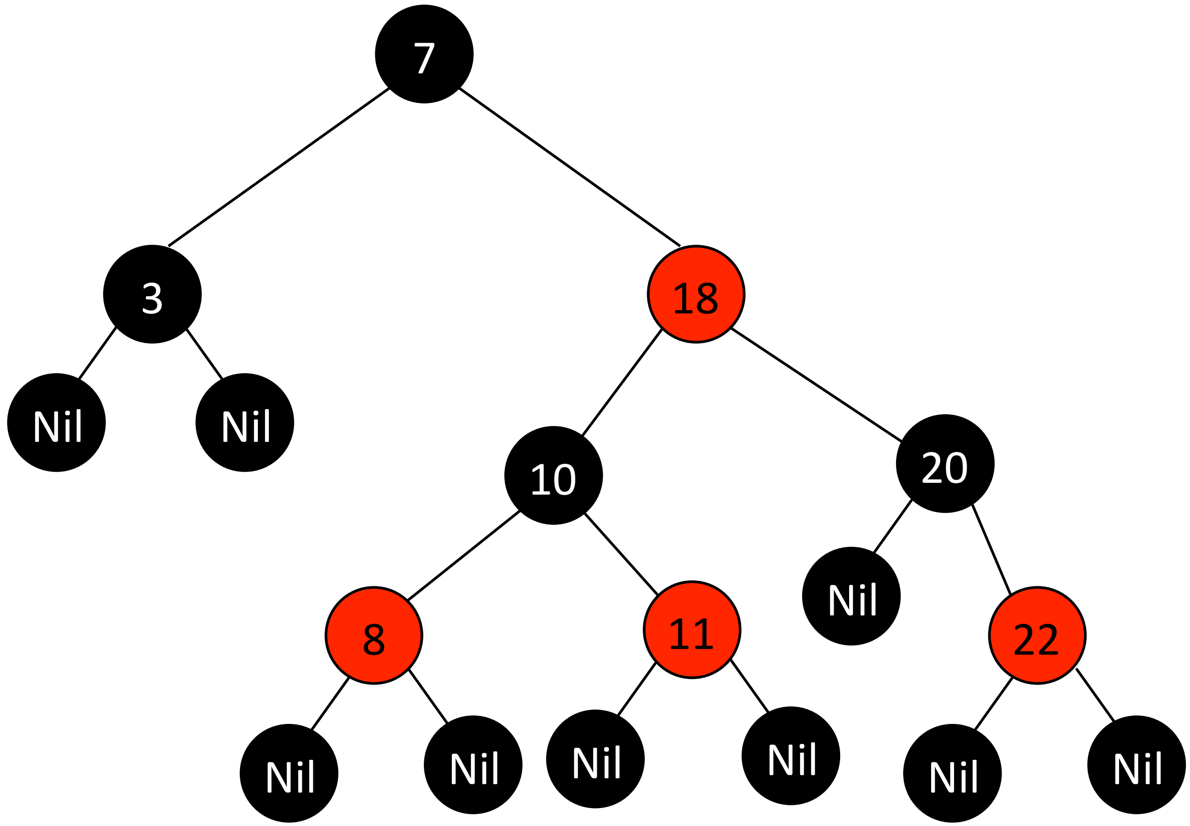
```
y ← nil[T]
x ← root[T]
while x ≠ nil[T]
    do y ← x
        if key[z] < key[x]
            then x ← left[x]
            else x ← right[x]
p[z] ← y
if y = nil[T]
    then root[T] ← z
    else if key[z] < key[y]
        then left[y] ← z
        else right[y] ← z
```

RB-Insert(*T*, *z*) Contd.

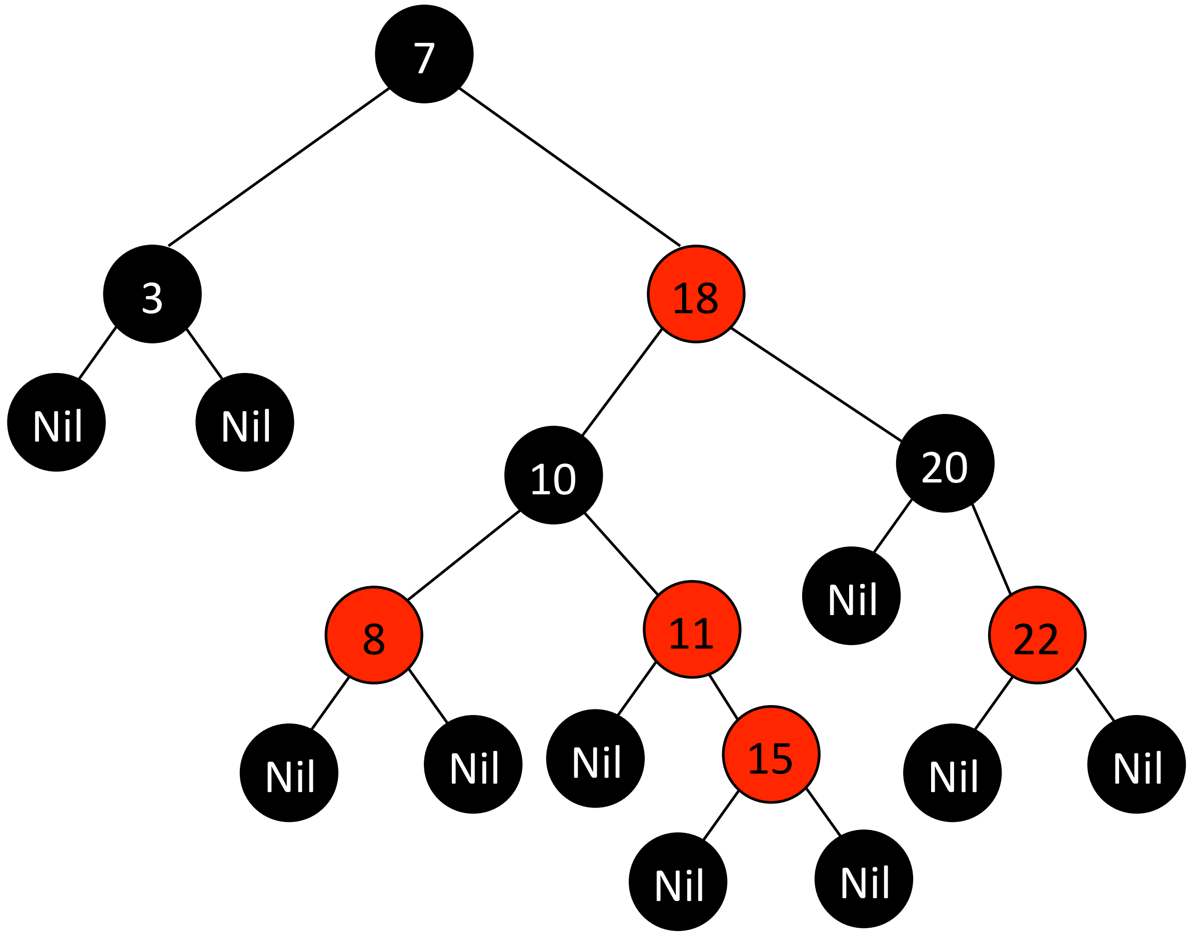
```
left[z] ← nil[T]
right[z] ← nil[T]
color[z] ← RED
RB-Insert-Fixup (T, z)
```

Regular BST insert + color assignment + fixup.

Insert RB Tree – Example

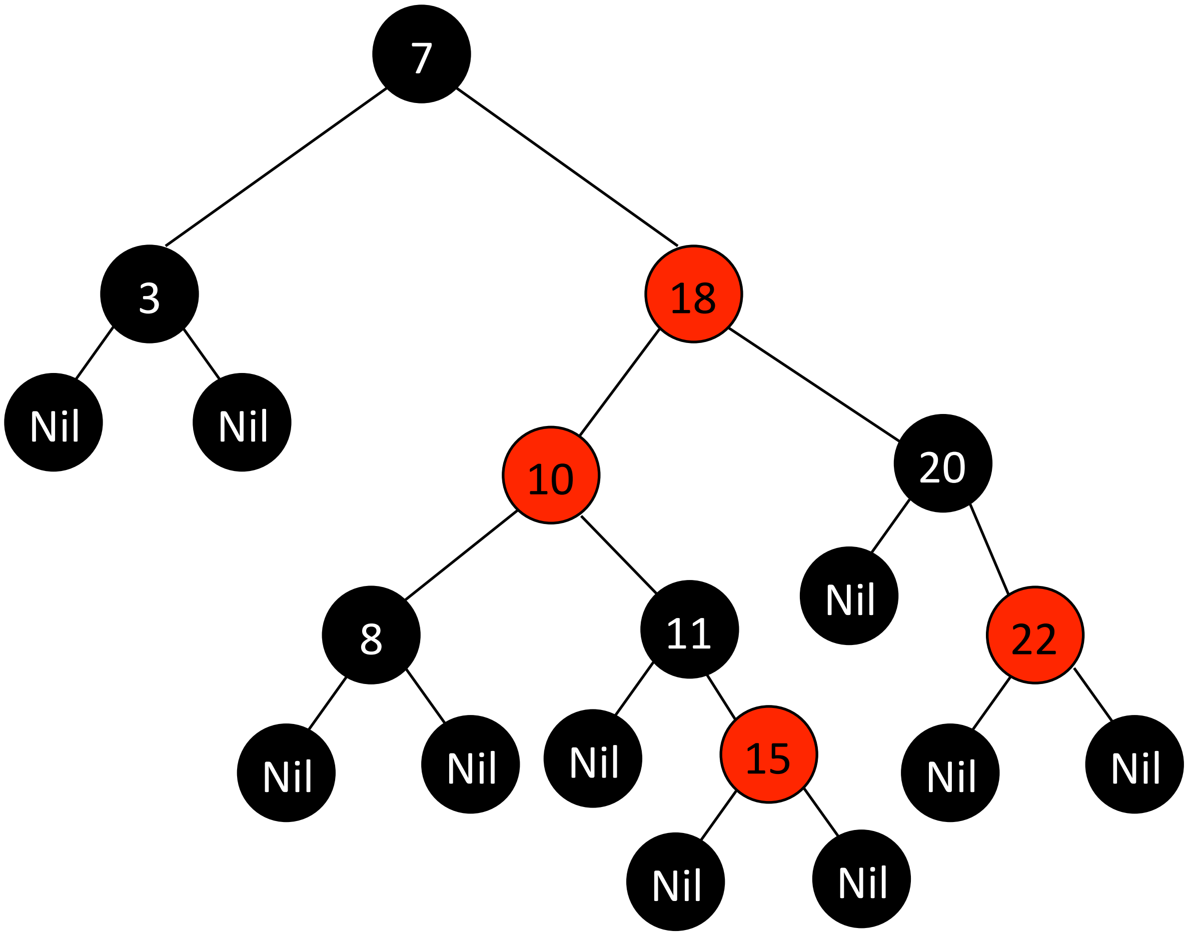


Insert RB Tree – Example



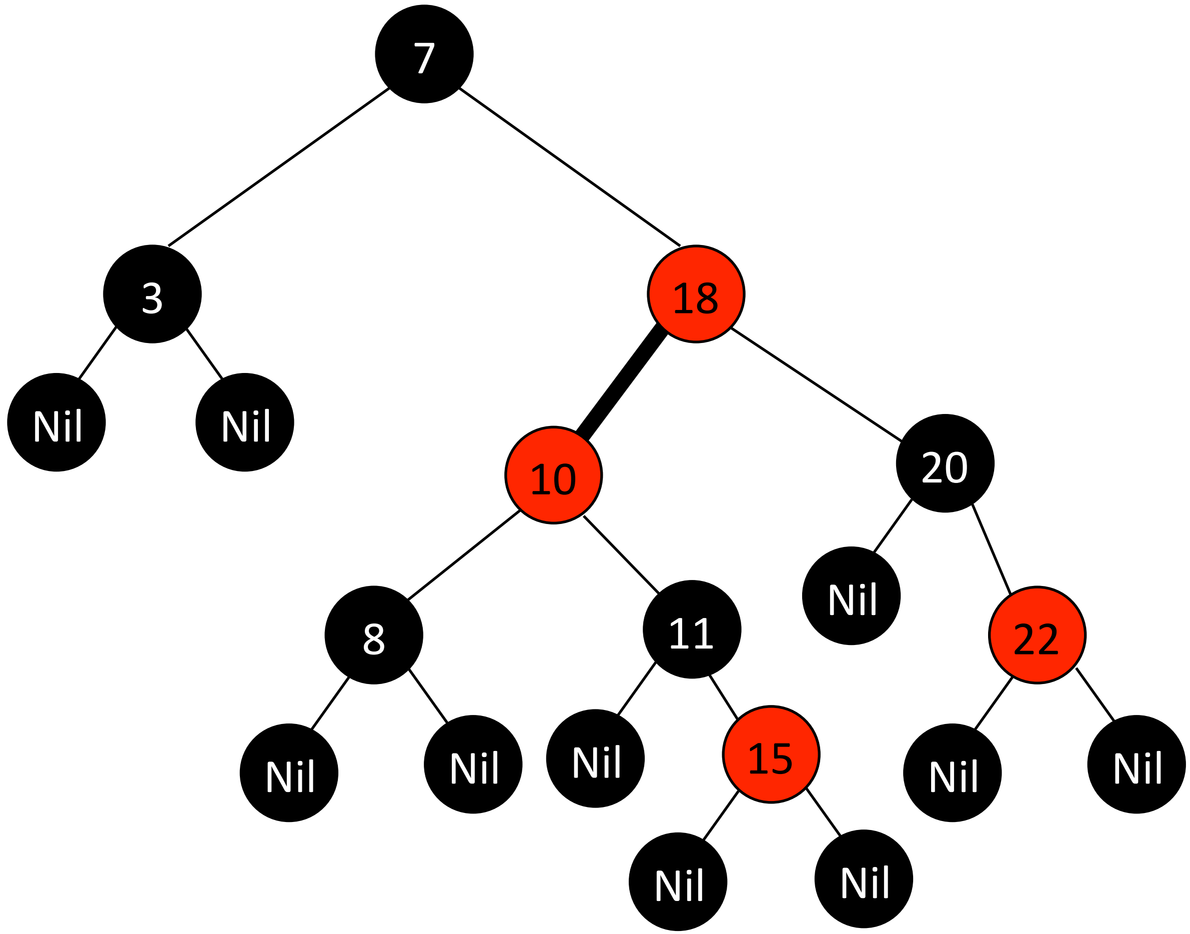
Insert(T,15)

Insert RB Tree – Example



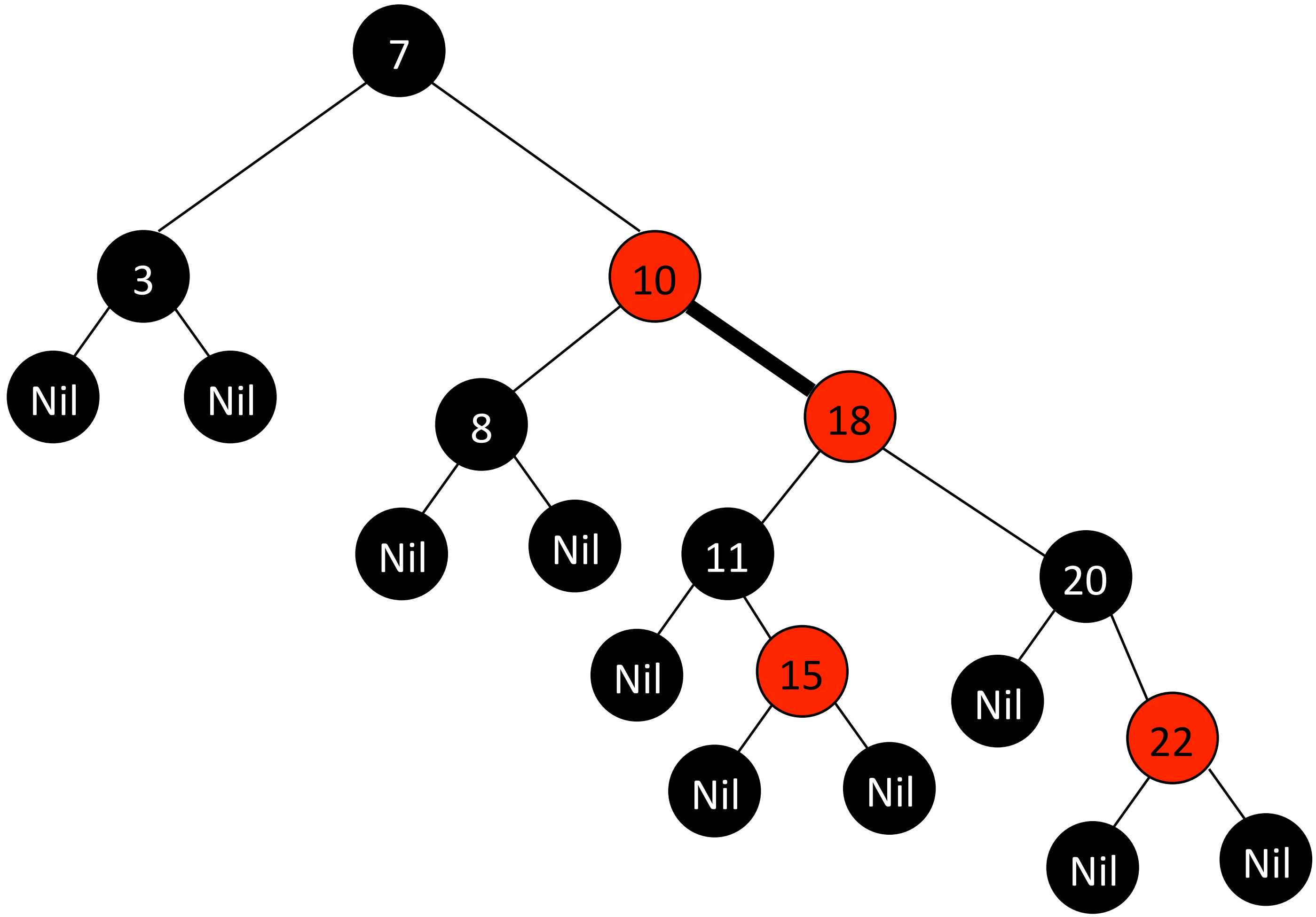
Recolor 10, 8 &11

Insert RB Tree – Example



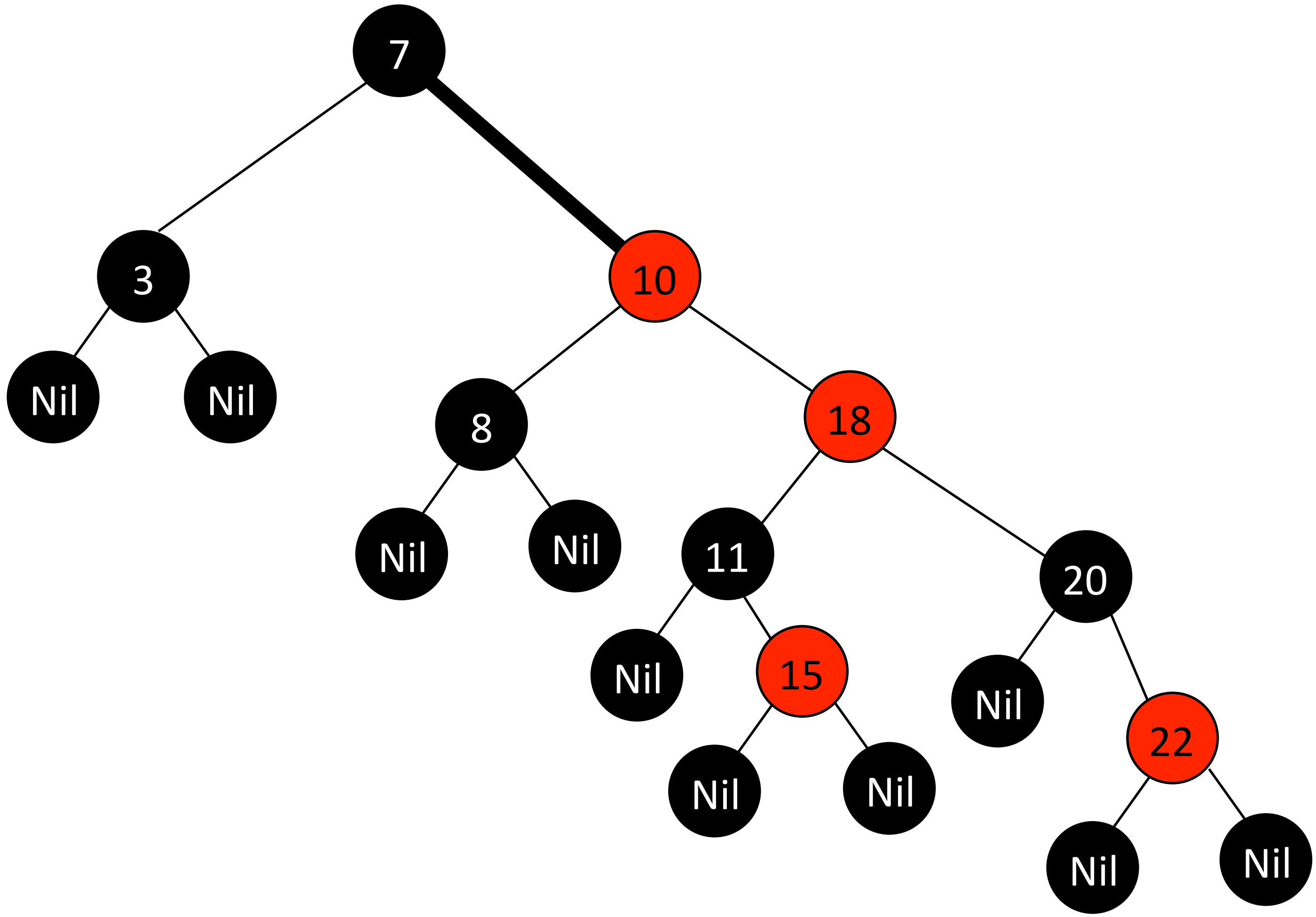
Right rotate at 18

Insert RB Tree – Example



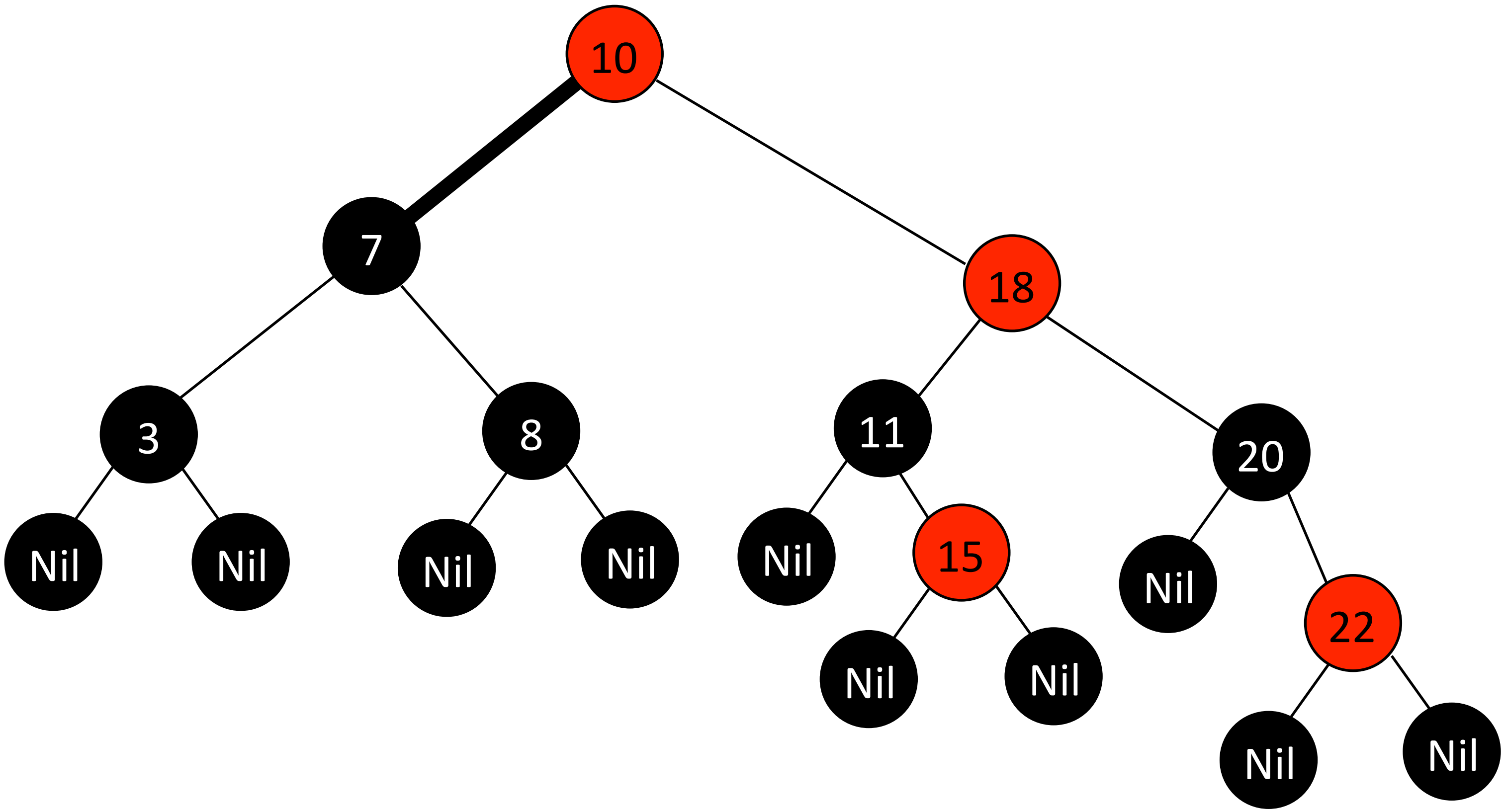
Right rotate at 18 (parent & child with conflict are aligned)

Insert RB Tree – Example



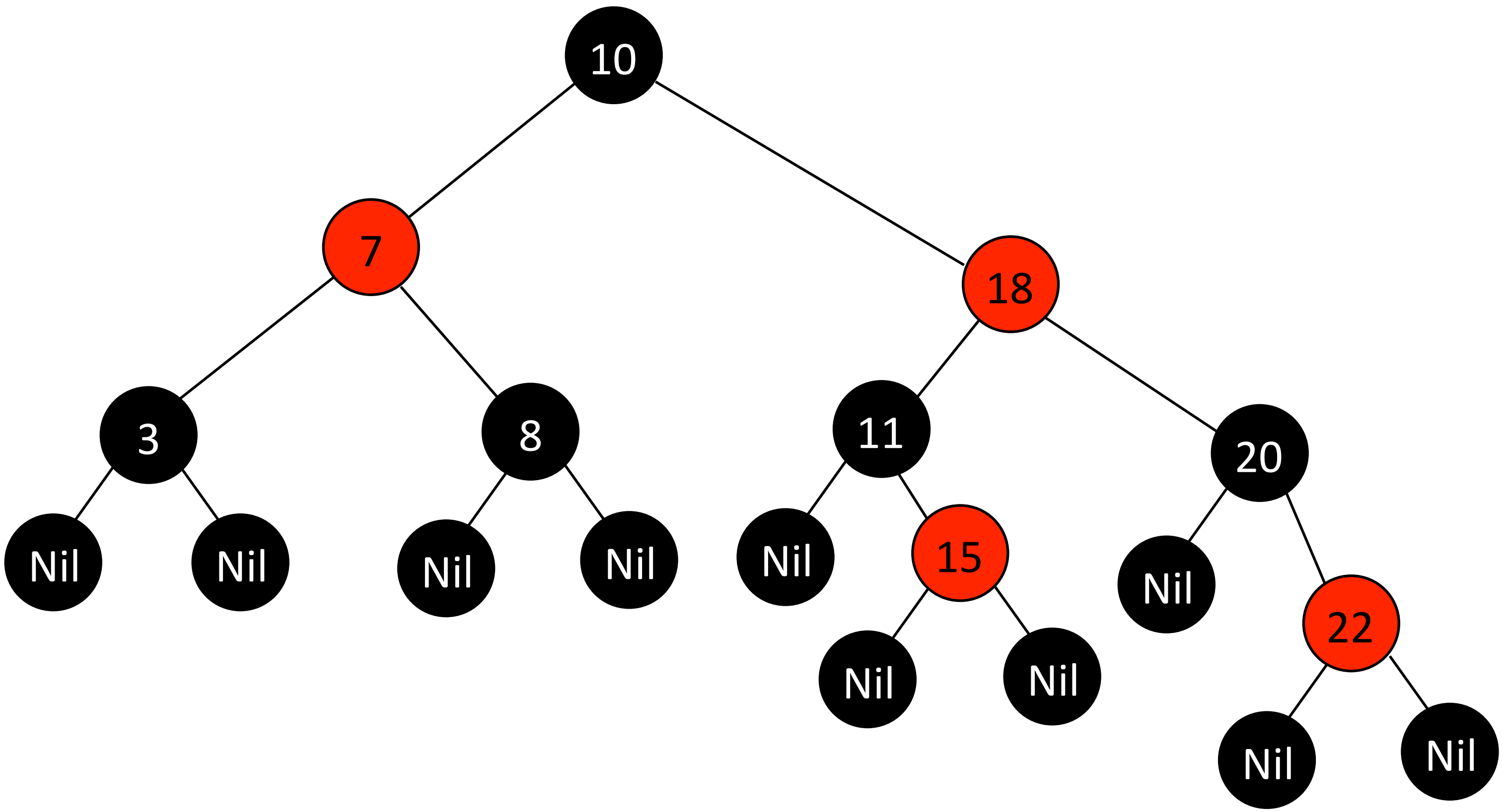
Left rotate at 7

Insert RB Tree – Example



Left rotate at 7

Insert RB Tree – Example



Recolor 10 & 7 (root must be black!)

Insertion – Fixup

RB-Insert-Fixup (*T, z*)

```
while color[p[z]] = RED
    do if p[z] = left[p[p[z]]]
        then y ← right[p[p[z]]]
            if color[y] = RED
                then color[p[z]] ← BLACK  // Case 1
                    color[y] ← BLACK       // Case 1
                    color[p[p[z]]] ← RED   // Case 1
                    z ← p[p[z]]            // Case 1
```

Insertion – Fixup

RB-Insert-Fixup(*T*, *z*) (Contd.)

9. **else if** $z = right[p[z]]$ // color[y] $\neq$ RED

10. **then** $z \leftarrow p[z]$ // Case 2

11. LEFT-ROTATE(T, z) // Case 2

12. $color[p[z]] \leftarrow$ BLACK // Case 3

13. $color[p[p[z]]] \leftarrow$ RED // Case 3

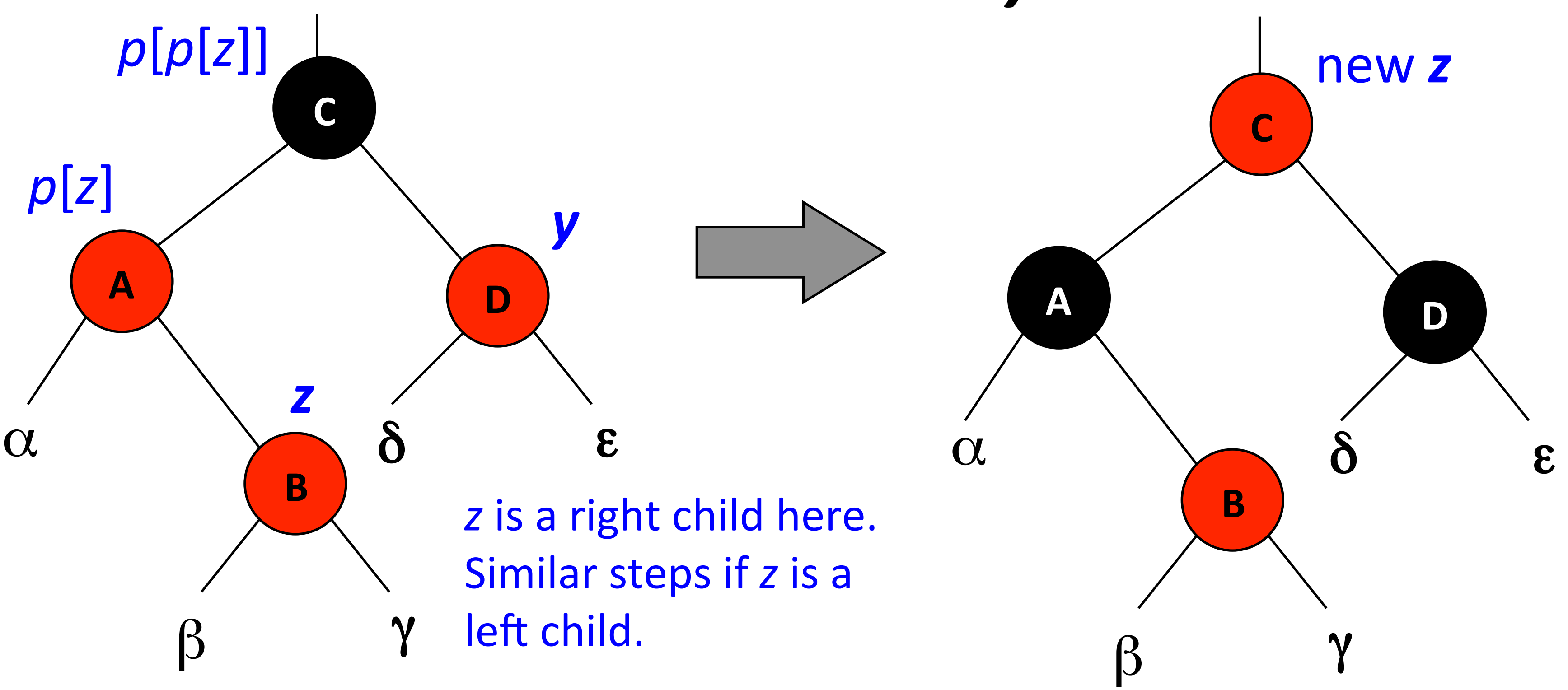
14. RIGHT-ROTATE(T, $p[p[z]]$) // Case 3

15. **else** (if $p[z] = right[p[p[z]]]$)(same as **10-14**

16. with "right" and "left" exchanged)

17. $color[root[T\,]] \leftarrow$ BLACK

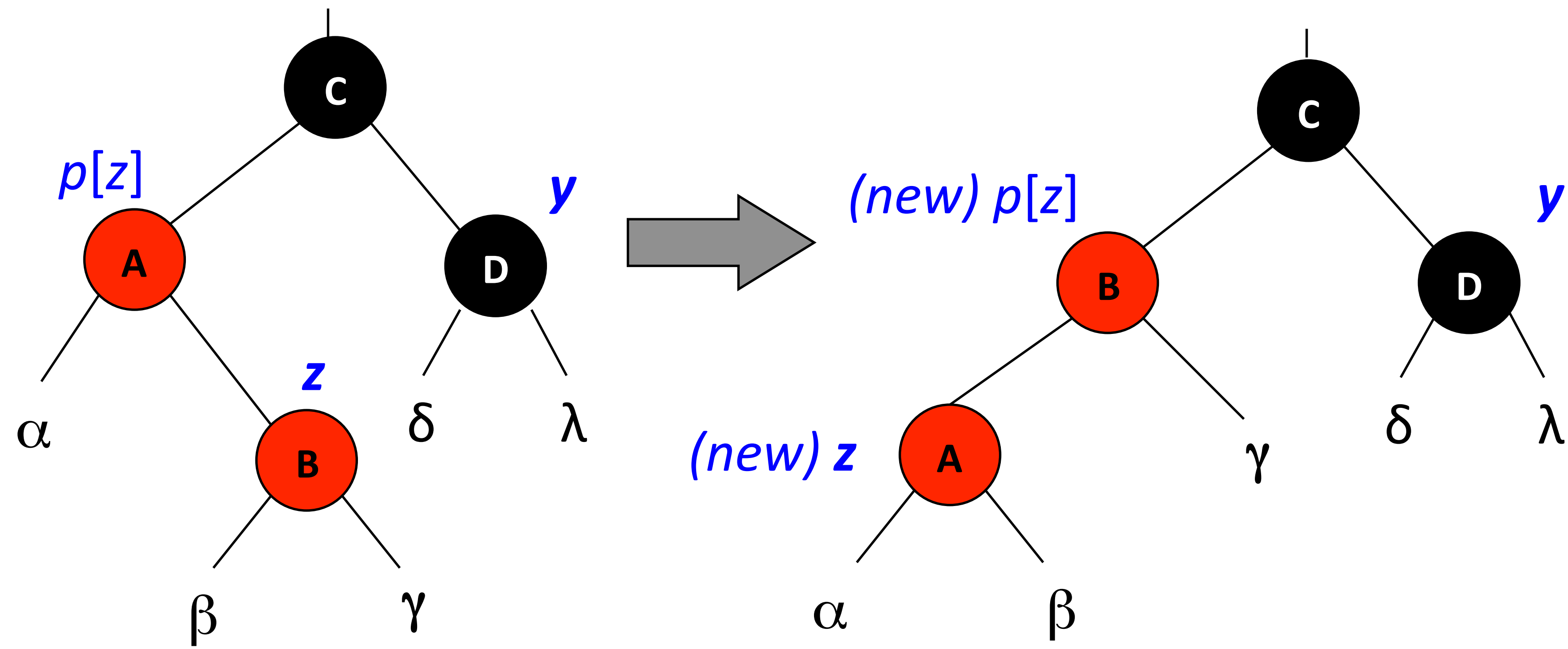
Case 1 – uncle y is red



z is a right child here. Similar steps if *z* is a left child.

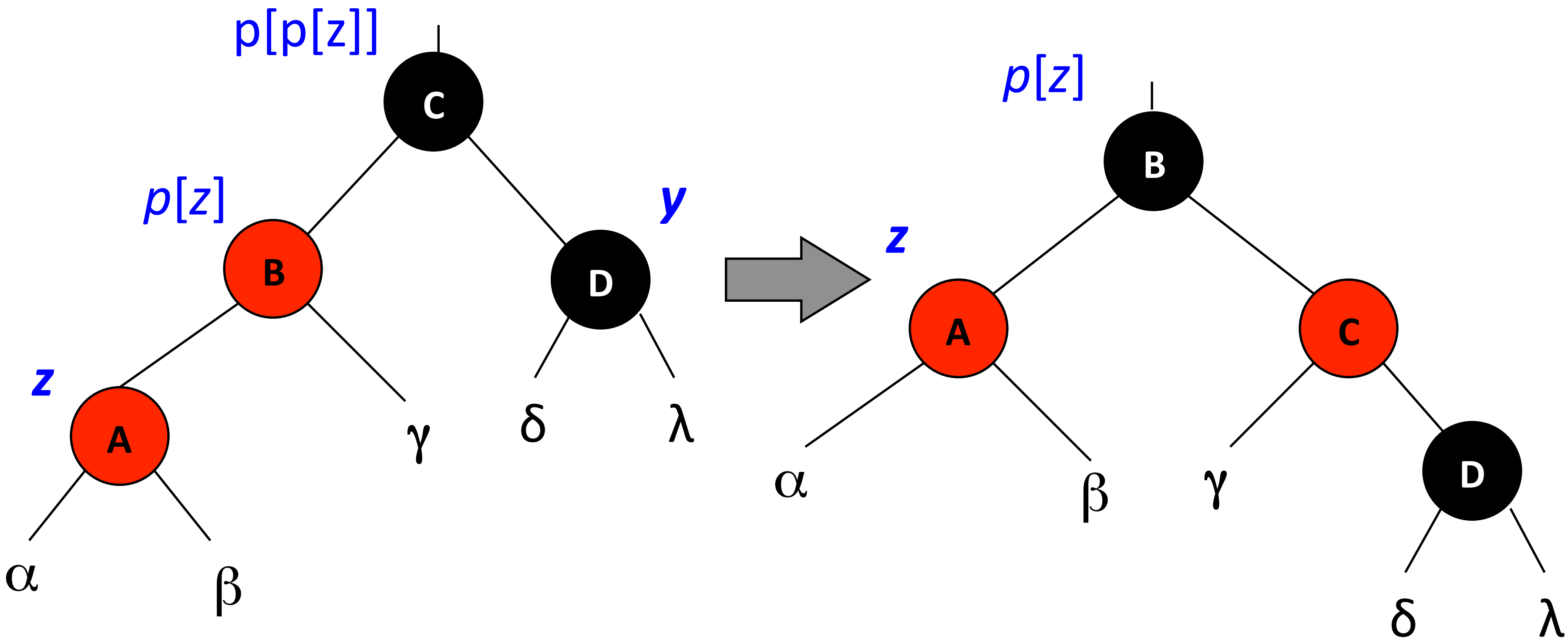
- $p[p[z]]$ (z's grandparent) must be black, since z and $p[z]$ are both red and there are no other violations of property 4.
- Make $p[z]$ and y black $\Rightarrow$ now z and $p[z]$ are not both red. But property 5 might now be violated.
- Make $p[p[z]]$ red $\Rightarrow$ restores property 5.
- The next iteration has $p[p[z]]$ as the new z (i.e., z moves up 2 levels).

Case 2 – *y* is black, *z* is a right child



- Left rotate around *p*[*z*], *p*[*z*] and *z* switch roles ⇒ now *z* is a left child, and both *z* and *p*[*z*] are red.
- Takes us immediately to case 3.

Case 3 – *y* is black, *z* is a left child



- Make *p*[*z*] black and *p*[*p*[*z*]] red.
- Then right rotate right on *p*[*p*[*z*]] (in order to maintain property 4).
- No longer have 2 reds in a row.
- *p*[*z*] is now black $\Rightarrow$ no more iterations.

Algorithm Analysis

- $O(\lg n)$ time to get through RB-Insert up to the call of RB-Insert-Fixup.
- Within RB-Insert-Fixup:
  - Each iteration takes $O(1)$ time.
  - Each iteration but the last moves z up 2 levels.
  - $O(\lg n)$ levels $\Rightarrow O(\lg n)$ time.
  - Thus, insertion in a red-black tree takes $O(\lg n)$ time.
  - Note: there are at most 2 rotations overall.

Correctness

Loop invariant:

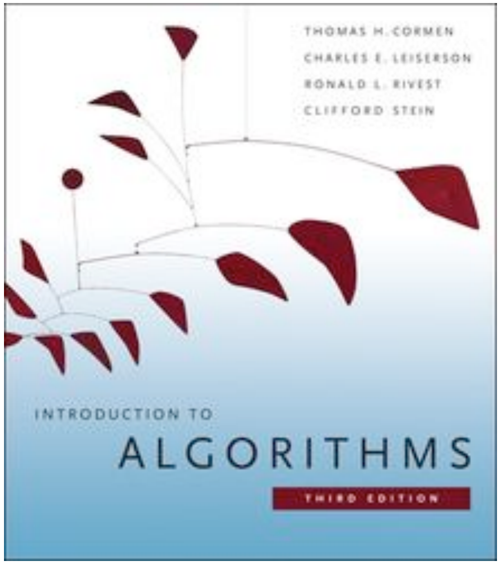
- At the start of each iteration of the **while** loop,
  - *z* is red.
  - There is at most one red-black violation:
    - Property 2: *z* is a red root, or
    - Property 4: *z* and *p*[*z*] are both red.

Correctness – Contd.

- **Initialization:** ✓
- **Termination:** The loop terminates only if *p*[*z*] is black. Hence, property 4 is OK.
  The last line ensures property 2 always holds.
- **Maintenance:** We drop out when *z* is the root (since then *p*[*z*] is sentinel *nil*[*T*], which is black). When we start the loop body, the only violation is of property 4.
  - There are 6 cases, 3 of which are symmetric to the other 3. We consider cases in which *p*[*z*] is a left child.
  - See cases 1, 2, and 3 described above.

Further Readings

[CLRS2009] Cormen, Leiserson, Rivest, & Stein, *Introduction to Algorithms*. (available as E-book)



See Chapter 13 for the complete proofs & deletion