COMP251: Hashing

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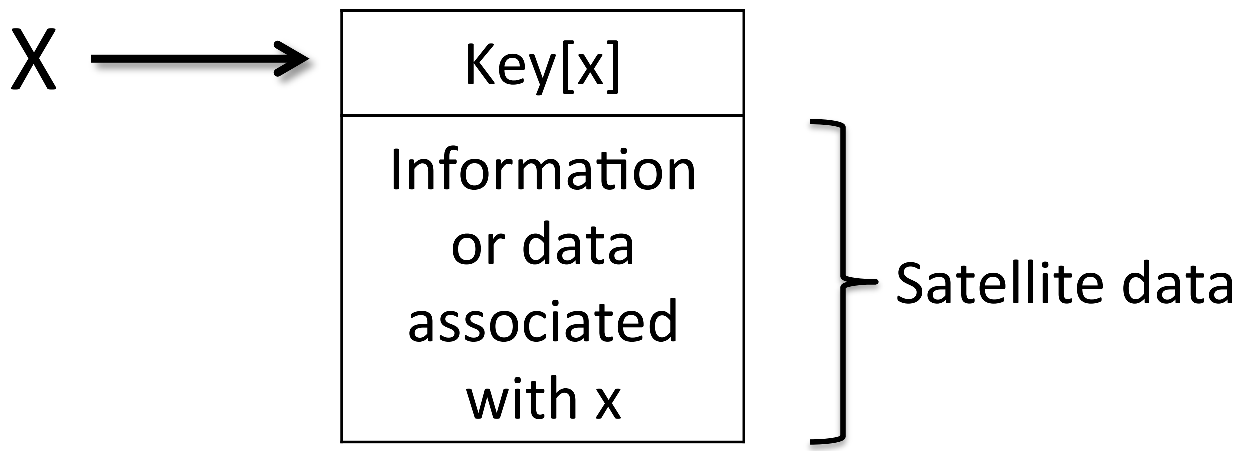
Based on (Cormen *et al.*, 2002)

Problem Definition

Table S with *n* records *x*:



We want a data structure to store and retrieve these data.

Operations:

- $insert(S,x): S \leftarrow S \cup \{x\}$
- $delete(S,x): S \leftarrow S \setminus \{x\}$
- $search(S,k)$

Dynamic set

Direct Address Table

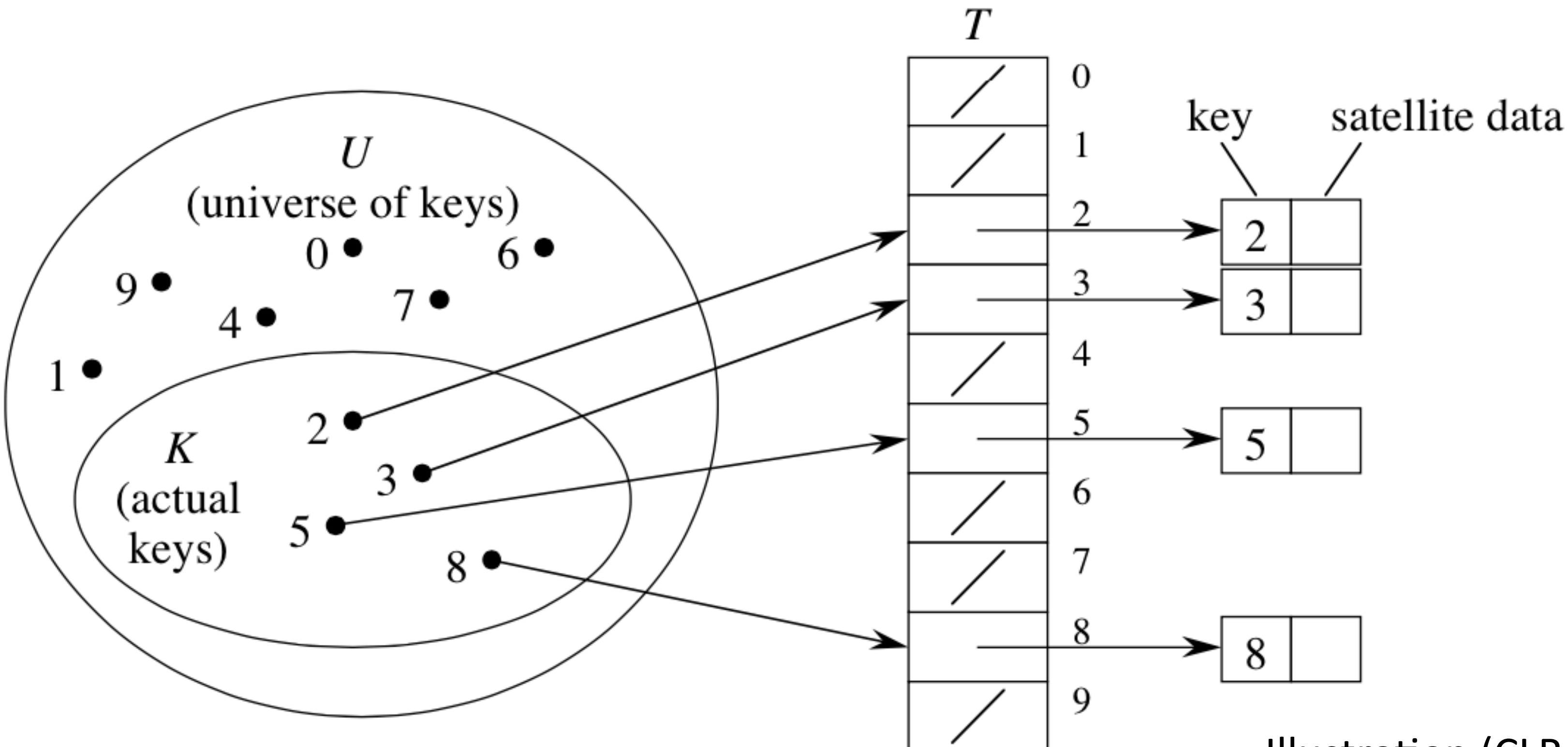


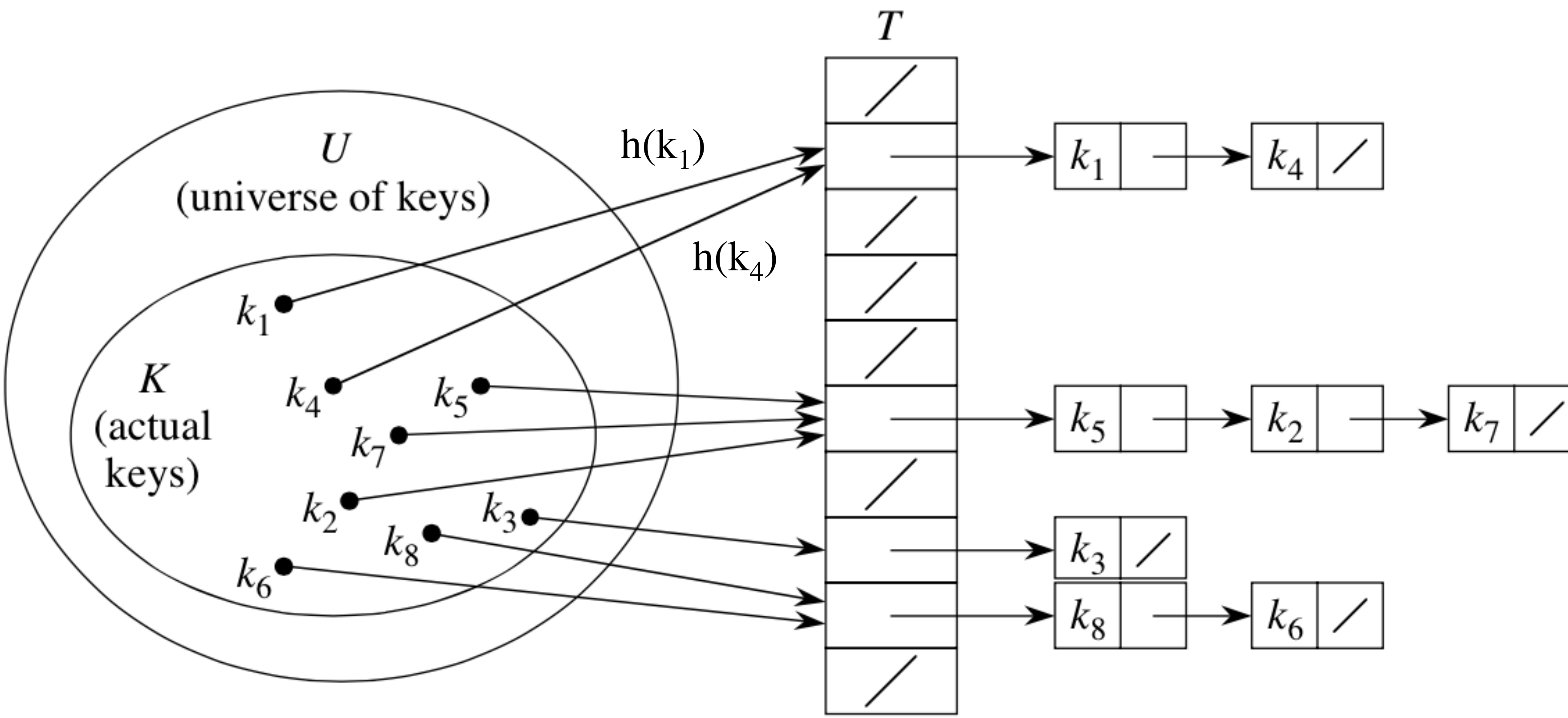
Illustration (CLR, 2005)

- Each slot, or position, corresponds to a key in U .
- If there is an element x with key k, then T[k] contains a pointer to x.
- If T [k] is empty, represented by NIL.

All operations in O(1), but if n (#keys) < m (#slots), lot of wasted space.

Hash Tables

- Reduce storage to O(n) keys.
- Resolve conflicts by chaining.
- Search time in $O(1)$ time in average, but not the *worst case*.



Hash function: $h : U \rightarrow \{0,1,\ldots,m-1\}$

Analysis of Hashing with Chaining

Insertion: O(1) time (Insert at the beginning of the list).

Deletion: Search time + O(1) if we use a double linked list.

Search:

- Worst case: Worst search time to is O(n).

  Search time = time to compute hash function +

  time to search the list.

  Assuming the time to compute hash function is O(1).

  Worst time happens when all keys go the same slot (list of size n), and we need to scan the full list => O(n).

- Average case: It depends how keys are distributed among slots.

Average case Analysis

Assume a **simple uniform hashing**: *n* keys are distributed uniformly among *m* slots.

Let *n* be the number of keys, and *m* the number of slots.

Average number of element per linked list?

Load factor: $\alpha = \dfrac{n}{m}$

Theorem:

The expected time of a search is $\Theta(1 + \alpha)$.

Note: $O(1)$ if $\alpha < 1$, but $O(n)$ if α is $O(n)$.

Average case Analysis

Theorem:

The expected time of a search is $\Theta(1 + \alpha)$.

Proof?

Distinguish two cases:

- search is unsuccessful
- search is successful

Unsuccessful search

- Assume that we can compute the hash function in $O(1)$ time.
- An unsuccessful search requires to scan all the keys in the list.

Search time = O(1 + average length of lists)

Let n_i be the length of the list attached to slot i.

Average value of n_i? $E(n_i) = \alpha = \dfrac{n}{m}$ (Load factor)

$$\Rightarrow O(\ 1\ ) + O(\ \alpha\ ) = O(\ 1 + \alpha\ )$$

Successful search

- Assume the position of the searched key x is equally likely to be any of the elements stored in the list.
- Keys scanned (in the list) *after* finding x have been inserted in the hash table before x (i.e. we insert at the head).

$$X_{ij} = I\left\{h(k_i) = h(k_j)\right\};\ E(X_{ij}) = \frac{1}{m} \quad \text{(probability of a collision)}$$

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E\left[X_{ij}\right]\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)$$

$$= 1+\frac{\alpha}{2}+\frac{\alpha}{2n}$$

Search time:

$$\Theta(1+1+\frac{\alpha}{2}+\frac{\alpha}{2n}) = \Theta(1+\alpha)$$

Choosing a hash function

Properties:

1. Uniform distribution of keys into slots
2. Regularity in key disturb should not affect uniformity.

List of functions:

- Division method
- Multiplication methods
- Open addressing:
  - Linear probing
  - Quadratic probing
  - Double hashing

Binary Numbers (reminder)

Each integer x accepts an unique decomposition $x = \sum_i a_i \cdot 2^i$ where $0 \leq a_i < 2$

Example: $x = 11 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3$

The binary number representation of an integer x is its (reversed) sequence of a's.

Example: $x = 11 \rightarrow$

2^3	2^2	2^1	2^0
1	0	1	1

$\rightarrow 1011$

Binary number operations:

101101 >> 1 = 10110 (right shift) : quotient of division by 2^k

101101 << 2 = 10110100 (left shift) : multiplication by 2^k

101101 mod 2^2 = 01 (modulo 2^k) : remainder of division by 2^k

Division Method

$$h(k) = k \bmod d$$

d must be chosen carefully.

Example 1: *d* = 2 and all keys are even?

Odd slots are never used!

Example 2: *d* = 2^r

k = 100010110101101011

r = 2 -> 11
r = 3 -> 011
r = 4 -> 1011

keeps only r last bits…

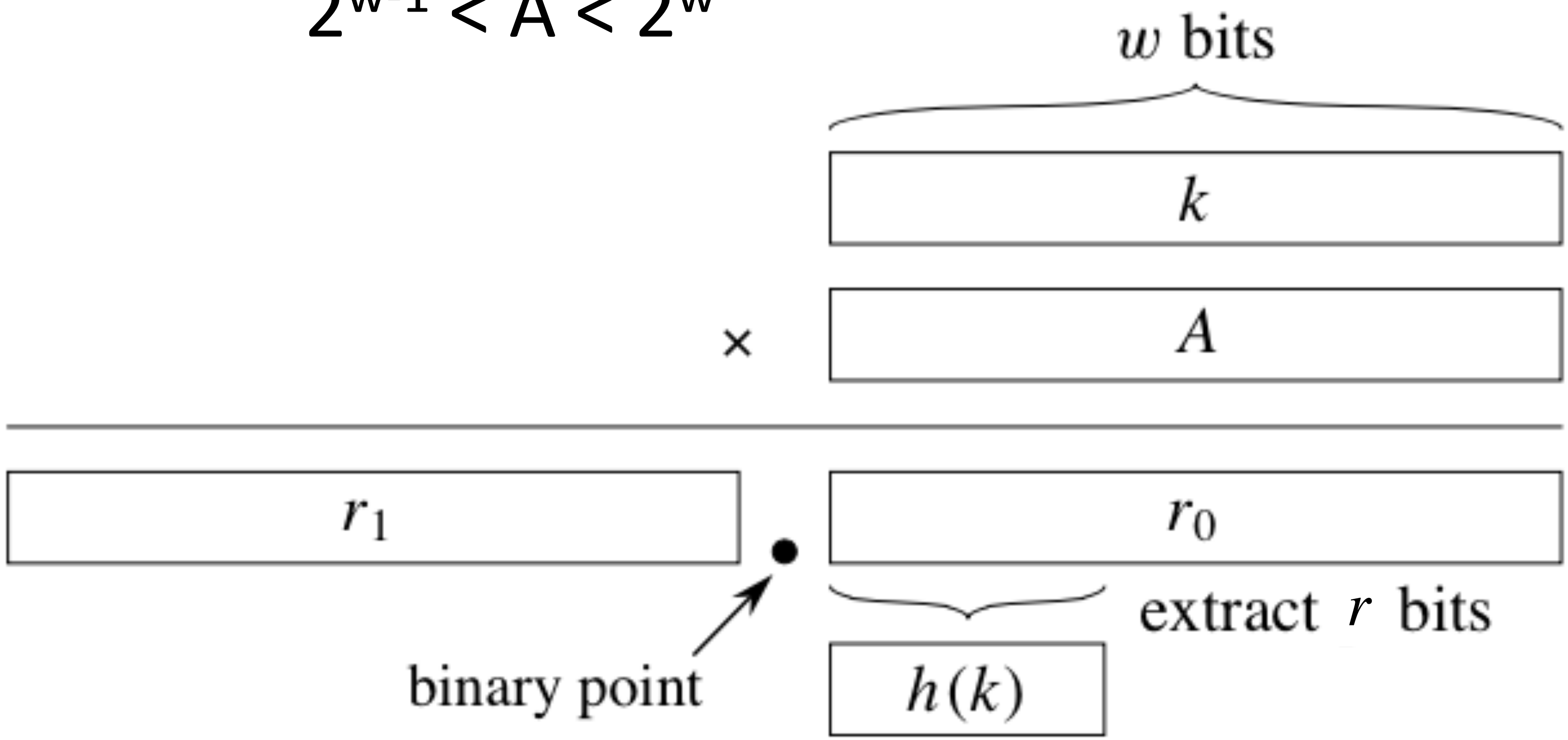
Good heuristic: Choose *r* prime not too close from a power of 2 or 10.

Note: Easy to implement, but division is slow…

Multiplication method

$$h(k) = \left(A \cdot k \bmod 2^w\right) >> (w - r)$$

$2^{w-1} < A < 2^w$



Open addressing

No storage for multiple keys on single slot (i.e. no chaining).

Idea: Probe the table.

- Insert if the slot if empty,
- Try another hash function otherwise.

$$h: U \times \{ 1, \ldots , m-1 \} \rightarrow \{ 1, \ldots , m-1 \}$$

Universe of keys   probe number   slot

Constraints:

- n < m (i.e. more slots than keys to store)
- Deletion is difficult

Challenge: How to build the hash function?

Open addressing

Illustration: Where to store key 282?

Full!

h(282,0)=3

h(282,1)=1

h(282,2)=5

✔

index	key
1	355
2	
3	567
4	233
5	282
6	799
7	

Note: Search must use the same probe sequence.

Linear & Quadratic probing

Linear probing:

$$h(k,i) = \left(h'(k) + i\right) \bmod m$$

Note: tendency to create clusters.

Quadratic probing:

$$h(k,i) = \left(h'(k) + c_1 \cdot i + c_2 \cdot i^2\right) \bmod m$$

Remarks:

- We must ensure that we have a full permutation of $\langle$ 0, … , m-1 $\rangle$.
- **Secondary clustering**: 2 distinct keys have the same h′ value, if they have the same probe sequence.

Double hashing

$$h(k,i) = \left(h_1(k) + i \cdot h_2(k)\right) \bmod m$$

Must have $h_2(k)$ be "relatively" prime to m to guarantee that the probe sequence is a full permutation of $\langle 0, 1, \ldots, m-1 \rangle$.

Examples:

- m power of 2 and h_2 returns odd numbers
- m prime number and $1 < h_2(k) < m$

Analysis of open-addressing

We assume **uniform hashing**: Each key equally likely to have anyone of the m’ permutations as its probe sequence, independently of other keys.

Theorem 1: The expected number of probes in an unsuccessful search is at most $\frac{1}{1-\alpha}$.

Theorem 2: The expected number of probes in a successful search is at most $\frac{1}{\alpha} \cdot \log\left(\frac{1}{1-\alpha}\right)$

Reminder: $\alpha = \frac{n}{m}$ is the load factor

Proof for unsuccessful searches

Initial state: *n* keys are already stored in *m* slots.

Probability that the 1st slot is occupied: n/m.

Probability that the 2nd slot is occupied: (n-1)/(m-1).

Probability that the 3rd slot is occupied: (n-2)/(m-2).

Let X be the number of unsuccessful probes.

$$E(X) = 1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(1 + \frac{n-2}{m-2}\left(\ldots\left(1 + \frac{1}{m-n}\right)\right)\right)\right)$$

$$\leq 1 + \alpha\left(1 + \alpha\left(1 + \alpha\left(\ldots(1 + \alpha)\right)\right)\right) \leq 1 + \alpha + \alpha^2 + \ldots = \sum_{i=1}^{\infty} \alpha^i = \frac{1}{1-\alpha}$$

Consequences

Corollary
The expected number of probes to insert is at most $1/(1 - \alpha)$.

Interpretation:

- If α is constant, an unsuccessful search takes $O(1)$ time.
- If $\alpha = 0.5$, then an unsuccessful search takes an average of $1/(1 - 0.5) = 2$ probes.
- If $\alpha = 0.9$, takes an average of $1/(1 - 0.9) = 10$ probes.

Proof of Theorem on successful searches: See [CLRS, 2009].

Universal Hashing

- A malicious adversary who has learned the hash function chooses keys that all map to the same slot, giving worst-case behavior.
- Defeat the adversary using **Universal Hashing**
  - Use a different random hash function each time.
  - Ensure that the random hash function is independent of the keys that are actually going to be stored.
  - Ensure that the random hash function is "good" by carefully designing a class of functions to choose from:
    - Design a universal class of functions.

Note: We solve now collision by chaining

Universal Set of Hash Functions

- A finite collection of hash functions $\mathbf{H}$ that map a universe $\mathbf{U}$ of keys into the range $\{0, 1,\ldots, m-1\}$ is **universal** if:

  for each pair of distinct keys $k, l \in \mathbf{U}$,
  the number of hash functions $h \in \mathbf{H}$
  for which $h(k)=h(l)$ is $\leq |H|/m$.

- For a hash function h chosen randomly from $\mathbf{H}$, the chance of a collision between two keys is $\leq 1/m$.

Universal hash functions give good hashing behavior.

Example of Universal Hashing

- The table size m is a prime,
- key x is decomposed into bytes s.t. $x = <x_0, \ldots, x_r>$,
- $a = <a_0, \ldots, a_r>$ denotes a sequence of $r+1$ elements randomly chosen from $\{0, 1, \ldots, m-1\}$.

The class H defined by:

$$H = \bigcup_a \{h_a\} \text{ with } h_a(x) = \sum_{i=0 \text{ to } r} a_i x_i \bmod m$$

is an universal function.

Cost of Universal Hashing

Theorem:

Using chaining and universal hashing on key k:

- If k is not in the table T, the expected length of the list that k hashes to is $\leq \alpha$.
- If k is in the table T, the expected length of the list that k hashes to is $\leq 1+\alpha$.

Proof:

X_k = # of keys ($\neq$k) that hash to the same slot as k.

$C_{kl} = I\{h(k)=h(l)\}$; $E[C_{kl}] = \Pr\{h(k)=h(l)\} \leq 1/m$.

$$X_k = \sum_{l \in T \setminus \{k\}} C_{kl}, \text{ and } E[C_k] = \mathrm{E}\left[\sum_{l \in T \setminus \{k\}} C_{kl}\right] = \sum_{l \in T \setminus \{k\}} E[C_{kl}] \leq \sum_{l \in T \wedge l \neq k} \frac{1}{m}$$

If $k \notin T, E[X_k] \leq n/m = \alpha.$

If $k \in T,\ E[X_k] + 1 \leq (n-1)/m + 1 = 1 + \alpha - 1/m < 1 + \alpha.$

Proof

Let $X = \left\langle x_0, x_1, \ldots, x_r \right\rangle$ and $Y = \left\langle y_0, y_1, \ldots, y_r \right\rangle$ be 2 distinct keys.

They differ at (at least) one position. WLOG let 0 be this position.

For how many h do X and Y collide?

$$\sum_{i=0}^{r} a_i x_i \equiv \sum_{i=0}^{r} a_i y_i (\operatorname{mod} m)$$

$$\sum_{i=0}^{r} a_i (x_i - y_i) \equiv 0 (\operatorname{mod} m)$$

$$a_0 (x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i (x_i - y_i) (\operatorname{mod} m)$$

$$a_0 \equiv \left(-\sum_{i=1}^{r} a_i (x_i - y_i) \right) \cdot (x_0 - y_0)^{-1} \ (\operatorname{mod} m)$$

For any choice of < a_1, a_2, … , a_r> there is only one choice of a_0 s.t. X and Y collide.
#{h that collide} = m x m x … x m x 1
= m^r = |H|/m

Quiz

Answer online anonymous quiz at:

https://goo.gl/forms/mi7oZwM0rRSRDok92