COMP251: Dynamic programming (3)

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Based on (Kleinberg & Tardos, 2005)

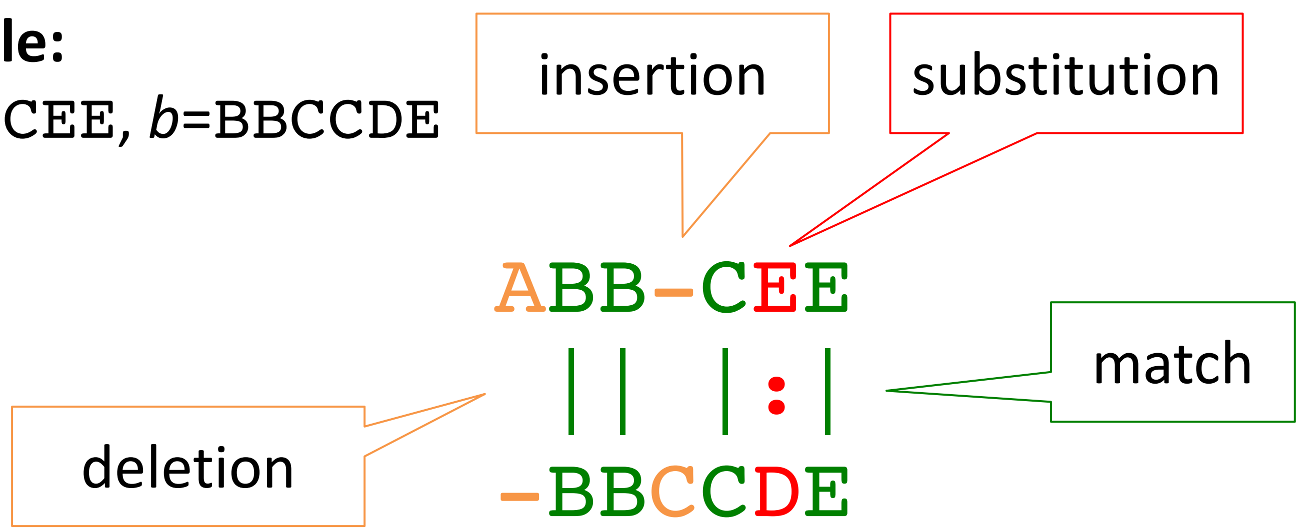
PAIRWISE SEQUENCE ALIGNMENT

Pairwise Sequence Alignment

Match: letters are identical
Substitution: letters are different
Insertion: a letter of *b* is mapped to the empty character
Deletion: a letter of *a* is mapped to the empty character

Each operation has a cost $\Longrightarrow$ find alignment with optimal score.

Example:
a=ABBCEE, *b*=BBCCDE



Needleman-Wunch Algorithm

```
for i=0 to m do
   d(i,0)=i*δ(-,-)
for j=0 to n do
   d(0,j)=j*δ(-,-)

for i=1 to m do
      for j=1 to n do
         d(i,j) = min(d(i-1,j)+δ(a_i,-),
                      d(i-1,j-1)+δ(a_i,b_j),
                      d(i,j-1)+δ(-,b_j))
return d(m,n)
```

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

		0	1	2	3	4
	d	-	A	T	T	G
0	-	0	1	2	3	4
1	C	1				
2	T	2				

d[i,j] = optimal alignment score of $a_1 \ldots a_i$ with $b_1 \ldots b_j$

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

$\binom{-}{-}$ $\binom{A}{C}$

d	-	A	T	T	G
-	0	1	2	3	4
C	1	?			
T	2				

- match/substitution: d(0,0) + δ(A,C)=0+(+1)=+1

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

$\binom{A}{-}$ $\binom{-}{C}$

d	-	A	T	T	G
-	0	1	2	3	4
C	1	?			
T	2				

- match/substitution: d(0,0) + δ(A,C)=0+(+1)=+1
- insertion: d(1,0)+δ(-,C) = 1+(+1)=+2

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

$\binom{-}{C}$ $\binom{A}{-}$

d	-	A	T	T	G
-	0	1	2	3	4
C	1	?			
T	2				

- match/substitution: d(0,0) + δ(A,C)=0+(+1)=+1
- insertion: d(1,0)+δ(-,C) = +1+(+1)=+2
- deletion: d(0,1)+δ(A,-) = +1+(+1)=+2

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

d	-	A	T	T	G
-	0	1	2	3	4
C	1	1			
T	2				

- match/substitution: d(0,0) + δ(A,C)=0+(+1)=+1
- insertion: d(1,0)+δ(-,C) = +1+(+1)=+2
- deletion: d(0,1)+δ(A,-) = +1+(+1)=+2

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2		
T	2				

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	
T	2				

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2				

Example

a=ATTG b=CT

$$\delta(x, y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2			

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1		

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	2	

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	2	3

Backtracking

How to retrieve the optimal alignment?

- Each move is associated to one edit operation
  - Vertical = insertion
  - Diagonal = match/substitution
  - Horizontal = deletion
- We use one of these 3 move to fill a cell of the array
- From the bottom-right corner (i.e. d(m,n)), find the move that has been used to determine the value of this cell.
- Apply this principle recursively.

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

		0	1	2	3	4
	d	-	A	T	T	G
0	-	0	1	2	3	4
1	C	1	1	2	3	4
2	T	2	2	1	2	3

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

		0	1	2	3	4
	d	-	A	T	T	G
0	-	0	1	2	3	4
1	C	1	1	2	3	4
2	T	2	2	1	2	3

$$\begin{pmatrix} \mathrm{ATT} \\ \mathrm{C} \end{pmatrix} \begin{pmatrix} \mathrm{G} \\ \mathrm{T} \end{pmatrix}$$

d[3,1] + δ(G,T) = 3 + 1 = 4 ✗

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

		0	1	2	3	4
	d	-	A	T	T	G
0	-	0	1	2	3	4
1	C	1	1	2	3	4
2	T	2	2	1	2	3 ↑

$\binom{ATT}{C}\binom{G}{T}$ $\binom{ATTG}{C}\binom{-}{T}$

d[4,1] + δ(-,T) = 4 + 1 = 5 ✗

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	0	1	2	3	4
d	-	A	T	T	G
0 -	0	1	2	3	4
1 C	1	1	2	3	4
2 T	2	2	1	2 ←	3

$$\binom{\text{ATT}}{\text{C}}\binom{\text{G}}{\text{T}} \qquad \binom{\text{ATTG}}{\text{C}}\binom{-}{\text{T}} \qquad \binom{\text{ATT}}{\text{CT}}\binom{\text{G}}{-}$$

d[3,2] + δ(G,-) = 2 + 1 = 3 ✓

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	2 ←	3

G

–

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	2	3

```
TG
T-
```

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1 ←	2	3	4
T	2	2	1 ↖	2 ←	3

TTG
-T-

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	2	3

```
ATTG
C-T-
```

Example

a=ATTG b=CT

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & otherwise \end{cases}$$

	-	A	T	T	G
-	0	1	2	3	4
C	1	1	2	3	4
T	2	2	1	3	3

```
ATTG   ATTG   ATTG
C-T-   CT--   -CT-
```

Analysis

Theorem: The dynamic programming algorithm computes the edit distance (and optimal alignment) of two strings of length m and n in $\Theta(mn)$ time and $\Theta(mn)$ space.

Proof:

- Algorithm computes edits distance.
- Can trace back to extract an optimal alignment.

Q. Can we avoid using quadratic space?

A. Easy to compute optimal value in $\Theta(mn)$ time and $\Theta(m+n)$ space.

- Compute OPT(i,•) from OPT(i-1,•).
- But, no longer easy to recover optimal alignment itself.

Bioinformatics

- Different cost functions, For instance:

$$\delta(x,y) = \begin{cases} 1 & if\ x = y \\ -1 & otherwise \end{cases}$$

  Cost of alignment is being maximized.

- Variants of optimal pairwise alignment algorithm:
  - Ignore trailing gaps (Smith & Waterman, 1981)

- Optimal alignment not practical for multiple sequences.

SINGLE SOURCE SHORTEST PATHS

Modeling as graphs

Input:

- Directed graph G = (V, E)
- Weight function w : E → R

Weight of path $p = \langle v_0, v_1, \ldots, v_k \rangle$

$$= \sum_{k=1}^{n} w(v_{k-1}, v_k)$$

= sum of edge weights on path *p*.

Shortest-path weight *u* to *v***:**

$$\delta(u,v) = \begin{cases} \min\left\{ w(p) : u \overset{p}{\mapsto} v \right\} & \text{If there exists a path } u \leadsto v. \\ \infty & \text{Otherwise.} \end{cases}$$

Shortest path *u* to *v* is any path *p* such that $w(p) = \delta(u,v)$.

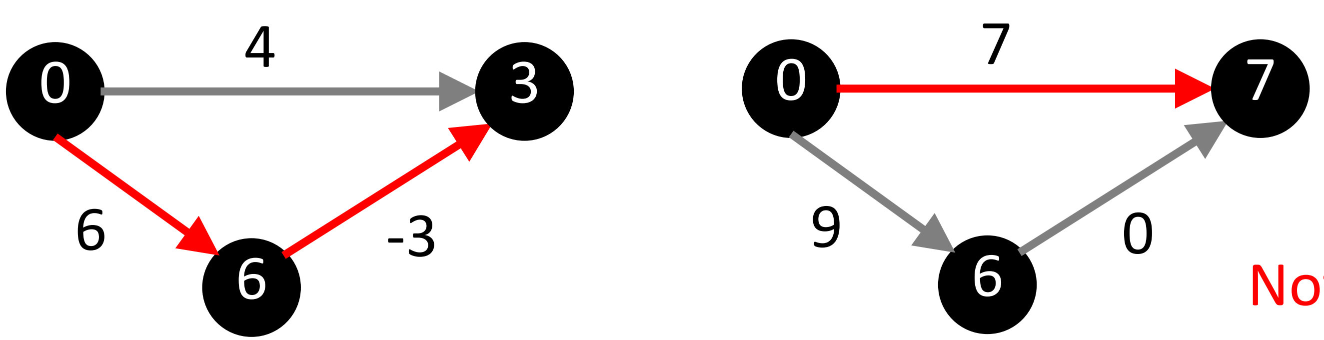
Generalization of breadth-first search to weighted graphs.

Dijkstra's algorithm

- No negative-weight edges.
- Weighted version of BFS:
  - Instead of a FIFO queue, uses a **priority queue**.
  - Keys are shortest-path weights (*d*[v]).
- Greedy choice: At each step we choose the light edge.

How to deal with negative weight edges?

- Allow re-insertion in queue? ⟹ Exponential running time…
- Add constant to each edge?



Not working…

Bellman-Ford Algorithm

- Allows negative-weight edges.
- Computes *d*[v] and π[v] for all v $\in$ *V*.
- Returns TRUE if no negative-weight cycles reachable from *s*, FALSE otherwise.

If Bellman-Ford has not converged after V(G) - 1 iterations, then there cannot be a shortest path tree, so there must be a negative weight cycle.

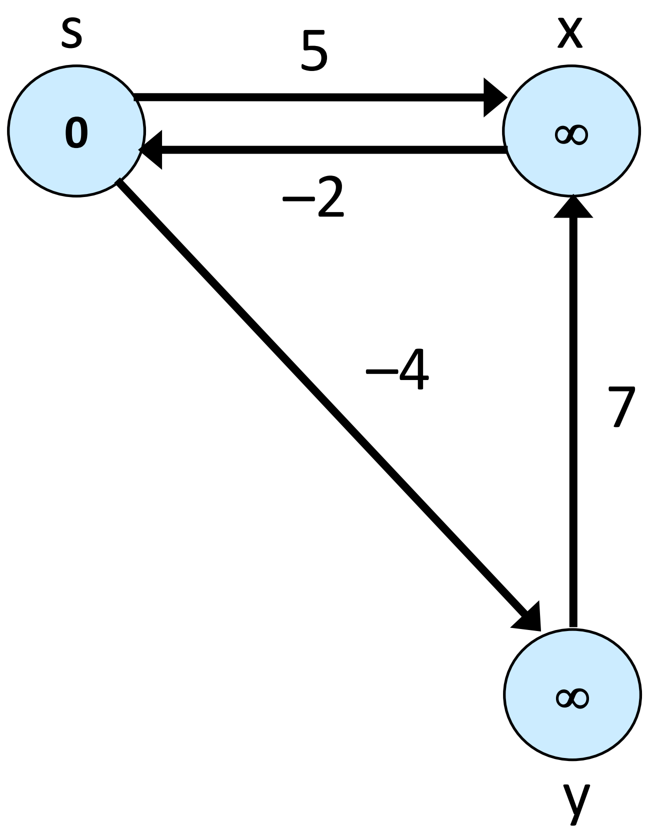
Bellman-Ford Algorithm

- Can have negative-weight edges.
- Will "detect" **reachable** negative-weight cycles.

```
Initialize(G, s);
for i := 1 to |V[G]| –1 do
    for each (u, v) in E[G] do
        Relax(u, v, w)
for each (u, v) in E[G] do
    if d[v] > d[u] + w(u, v) then
        return false
return true
```

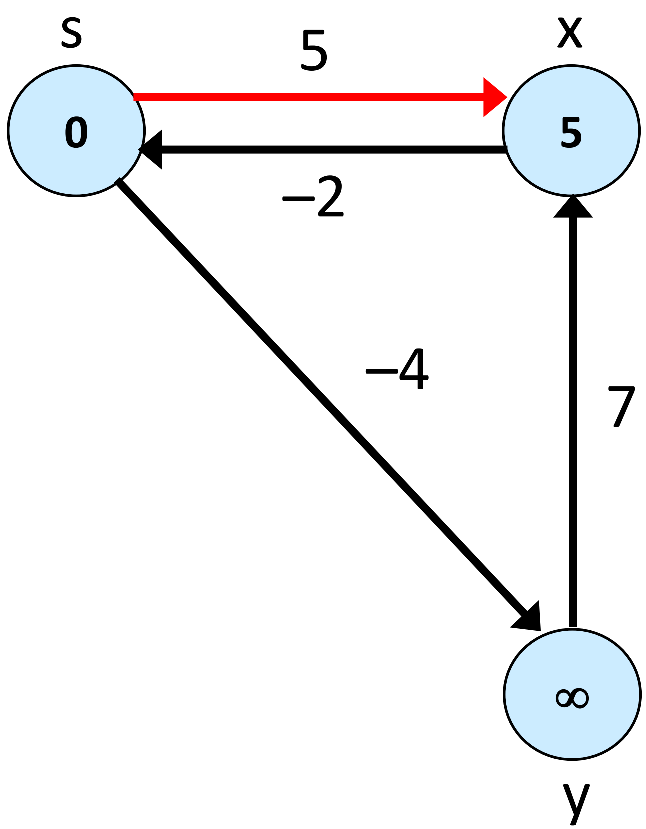
Time Complexity is O(VE).

Example



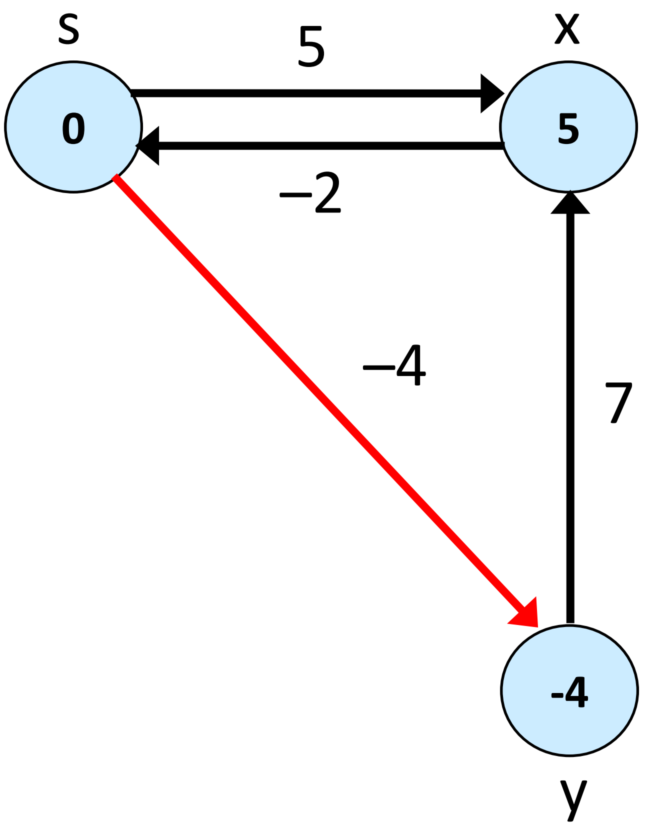
Example

Iteration 1



Example

Iteration 1



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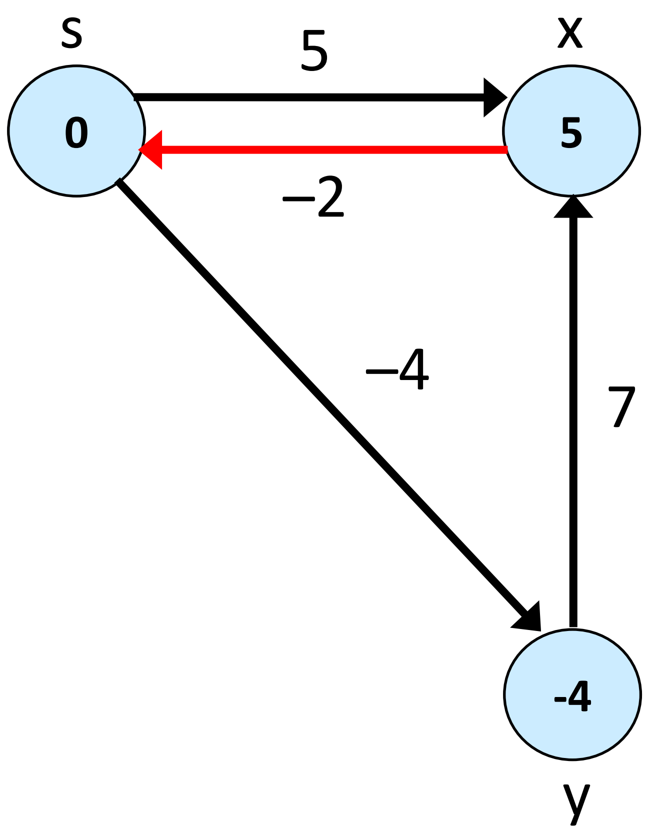
Iteration 1



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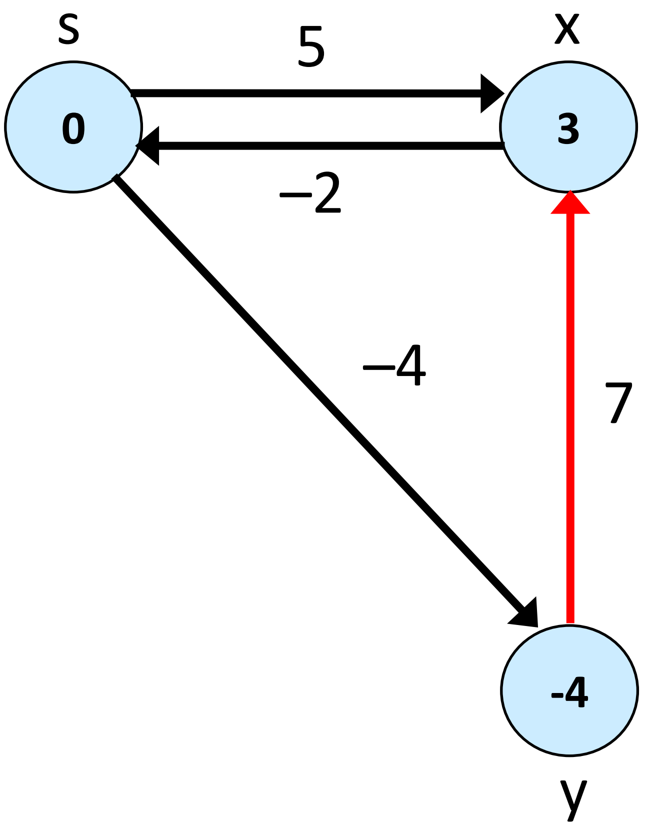
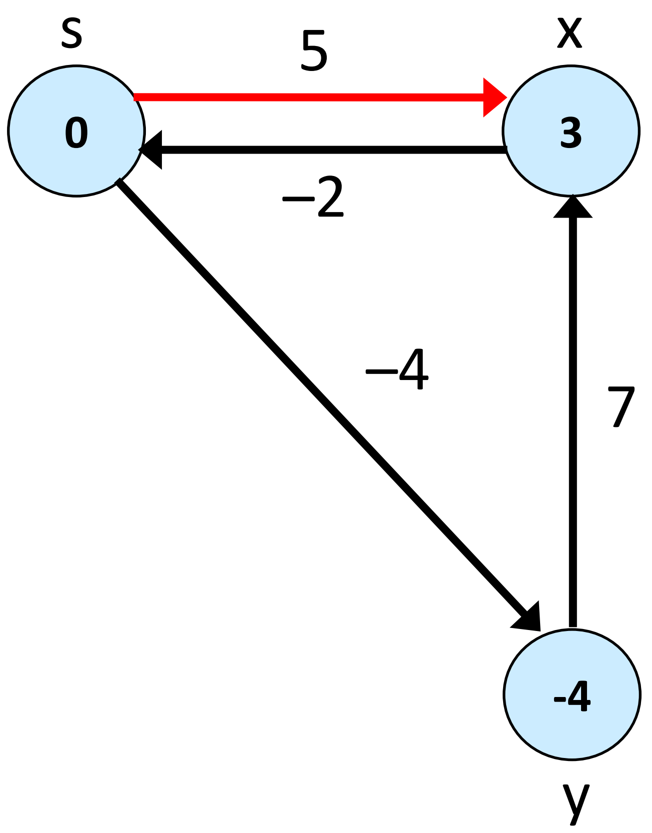
Iteration 1



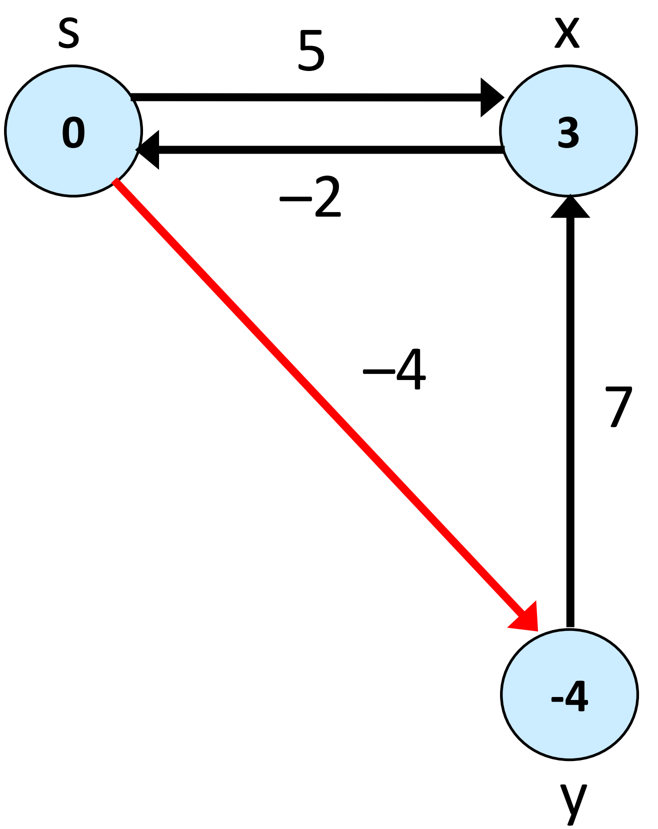
Example

Iteration 2



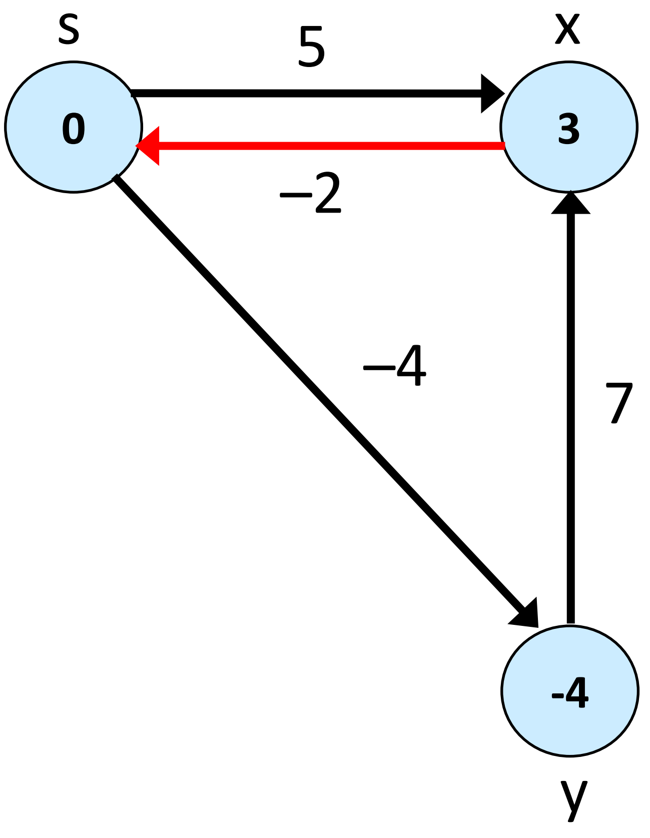
Example

Iteration 2



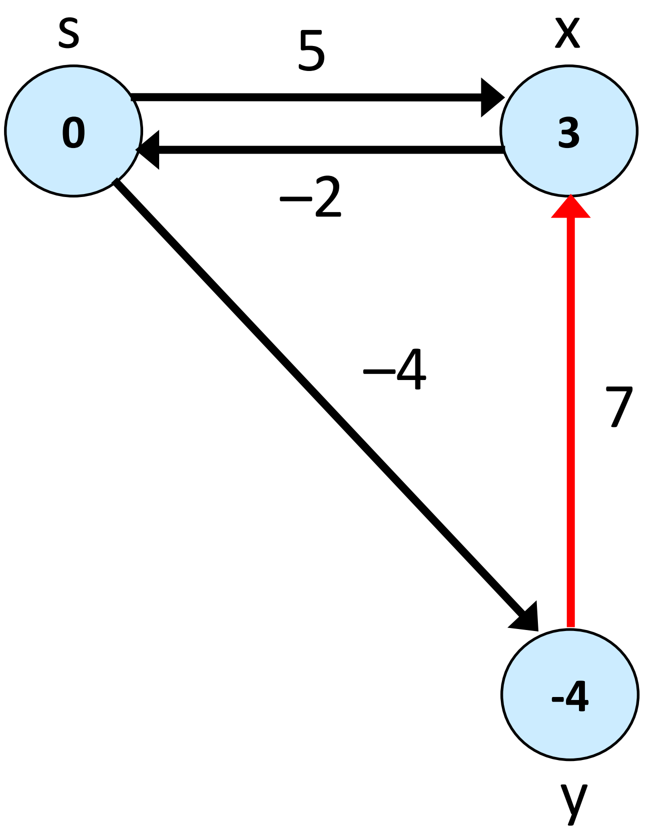
Example

Iteration 2

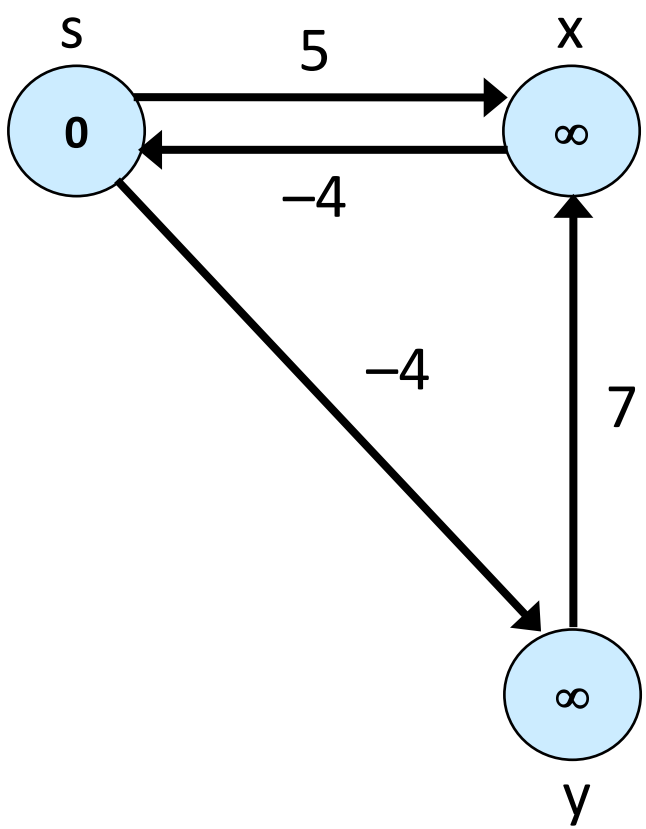


Example

Iteration 2

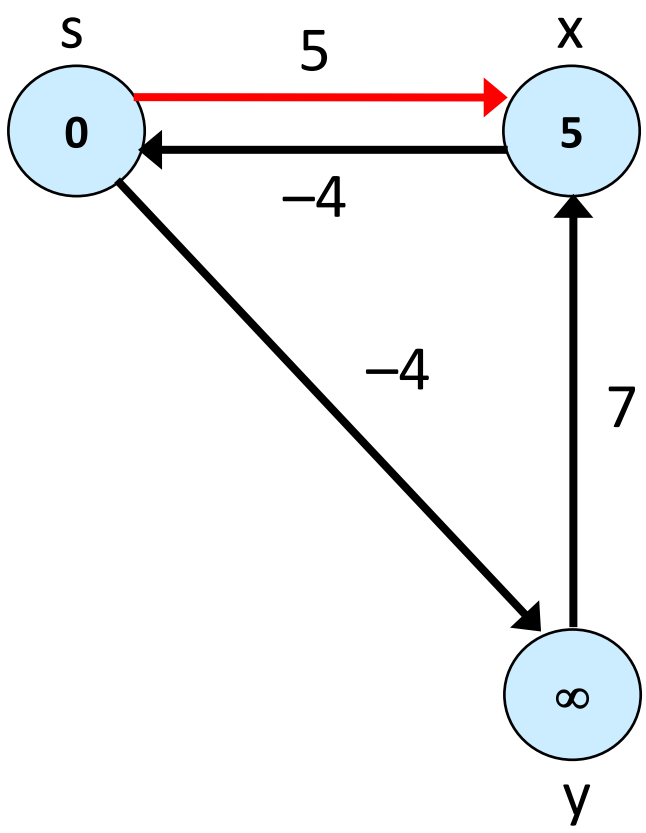


Example 2



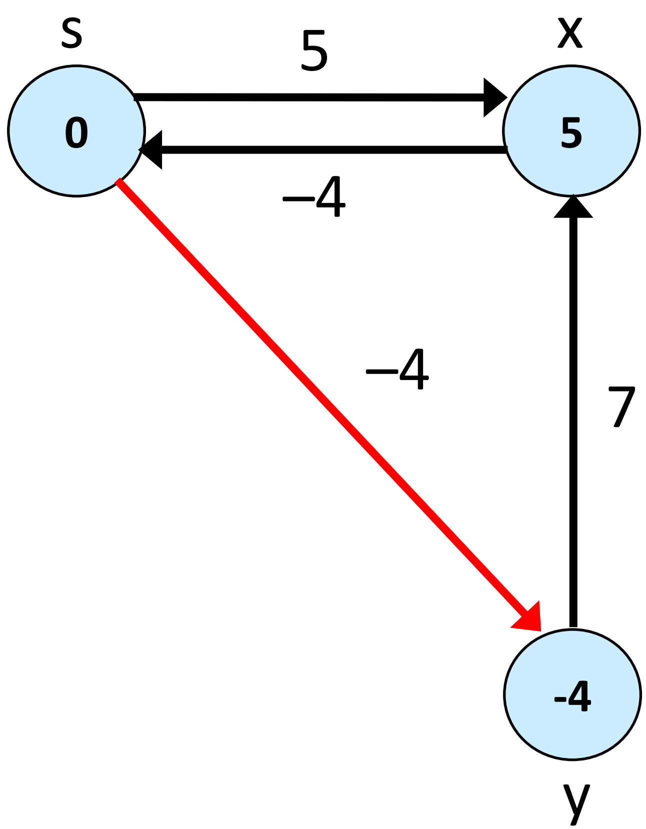
Example 2

Iteration 1



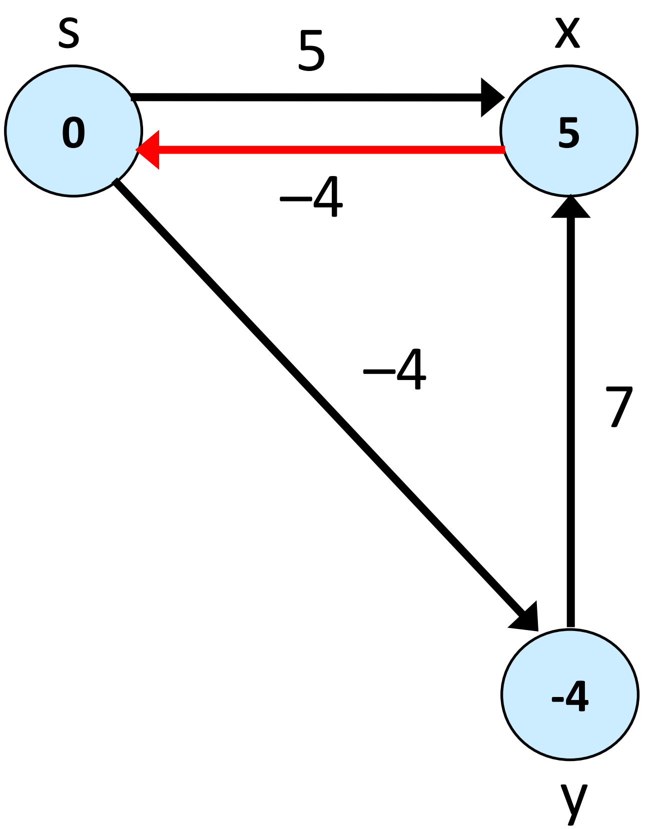
Example 2

Iteration 1



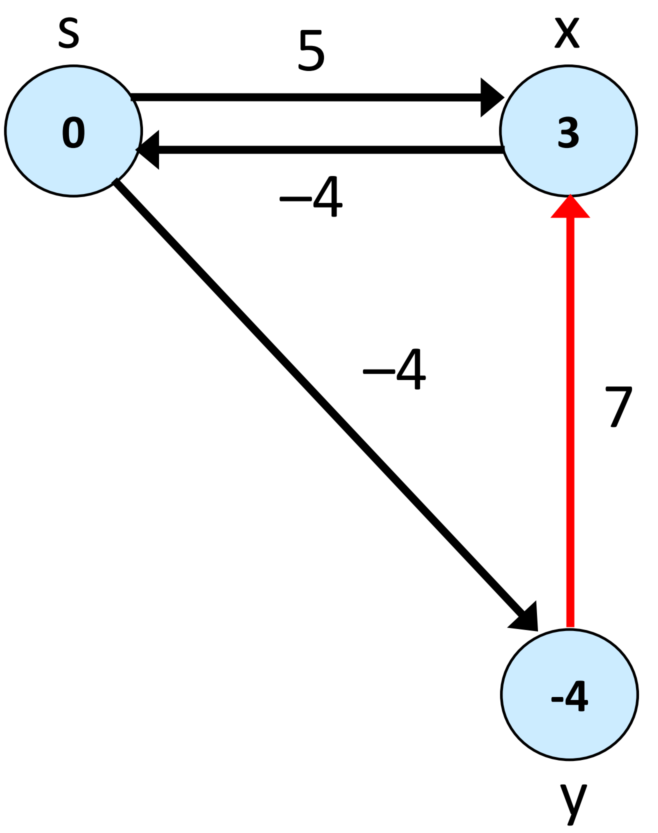
Example 2

Iteration 1



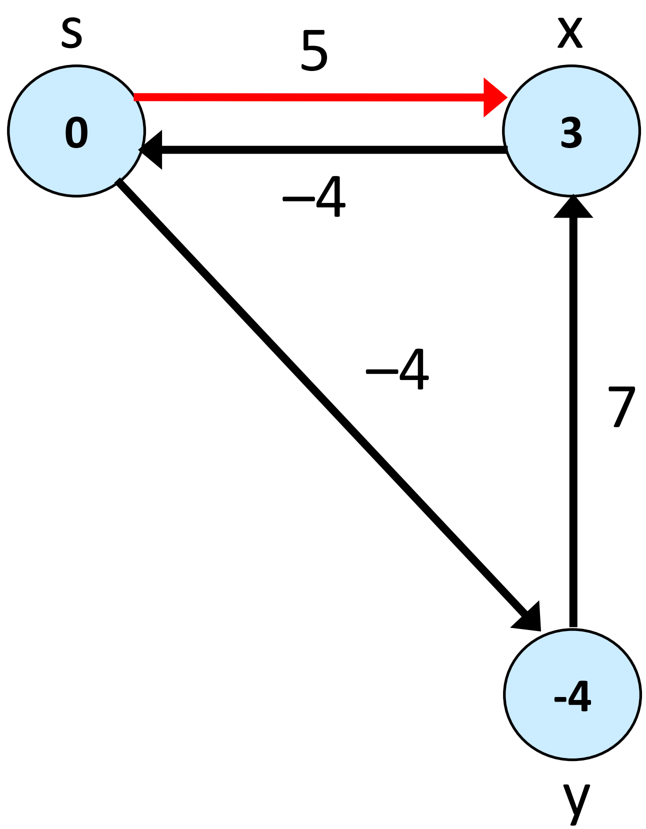
Example 2

Iteration 1



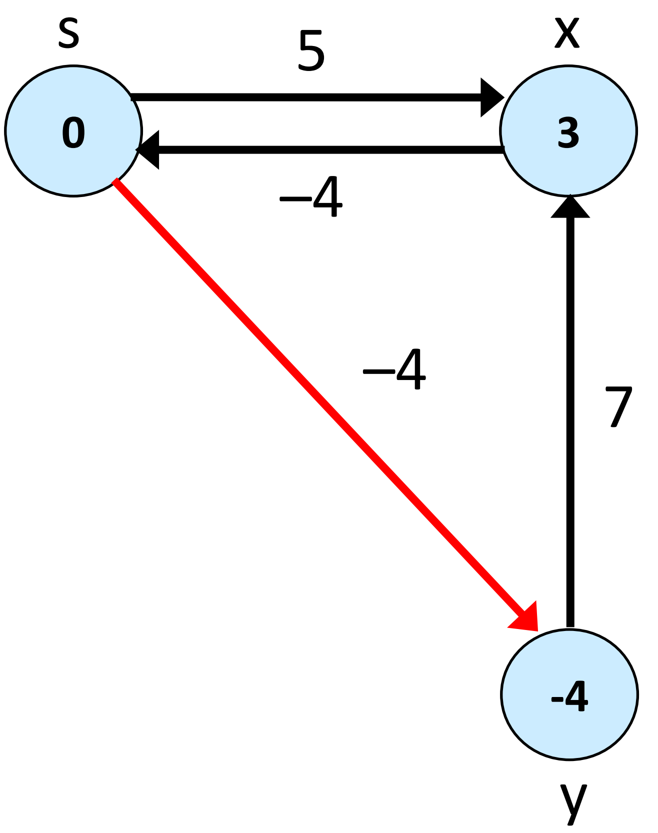
Example 2

Iteration 2



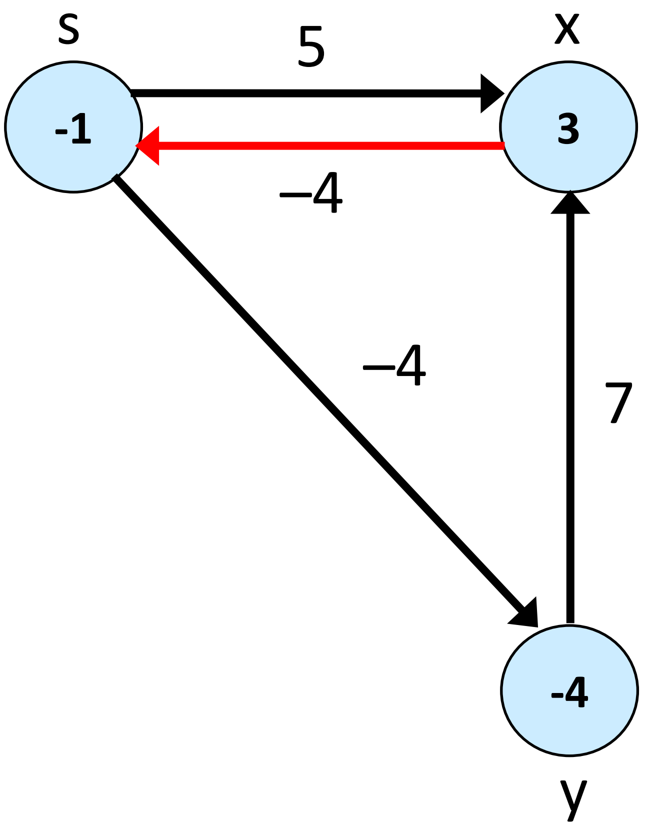
Example 2

Iteration 2



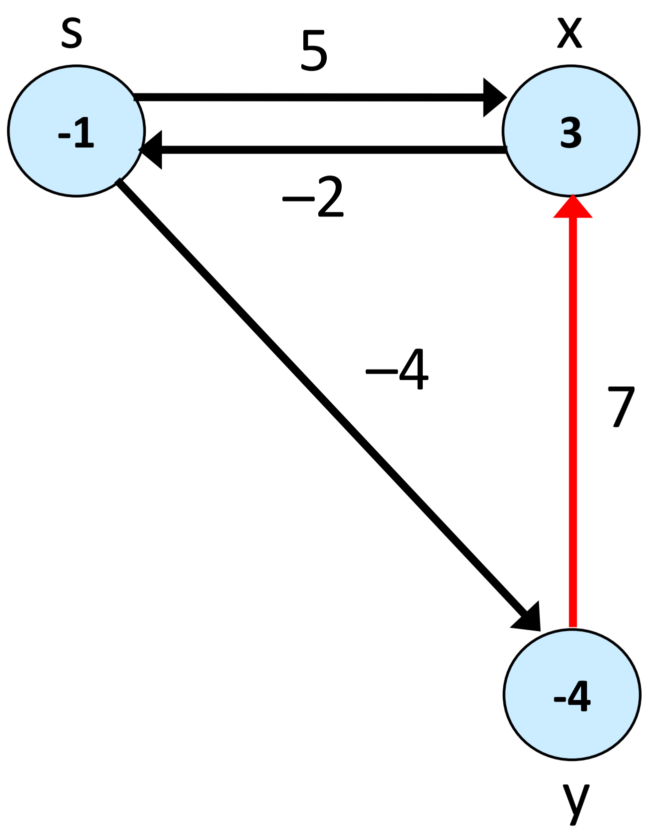
Example 2

Iteration 2



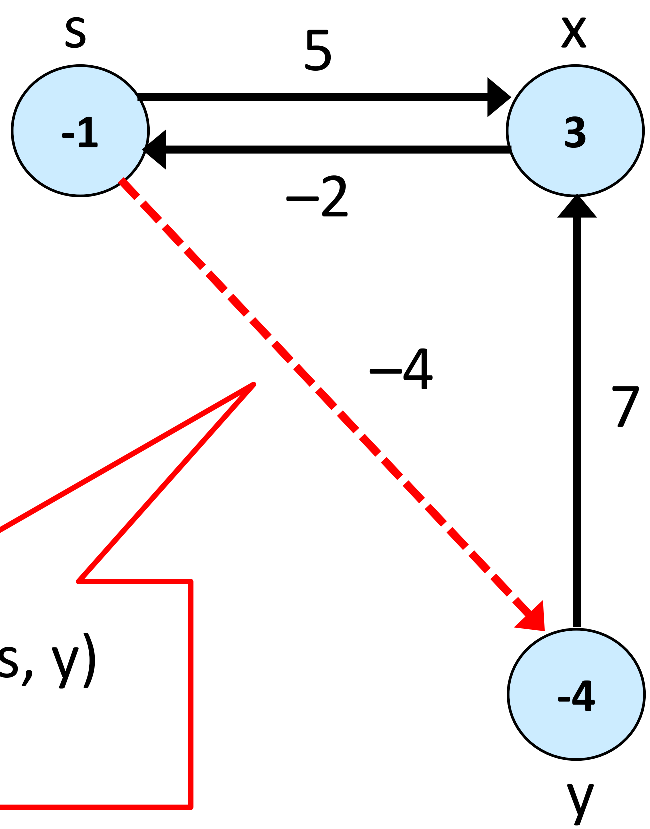
Example 2

Iteration 2



Example 2

Check



$d[y] > d[s] + w(s, y)$

$\Longrightarrow$ FALSE

Another Look at Bellman-Ford

Note: This is essentially **dynamic programming**.

Let d(i, j) = cost of the shortest path from s to i that is at most j hops.

$$d(i, j) = \begin{cases} 0 & \textbf{if } i = s \wedge j = 0 \\ \infty & \textbf{if } i \neq s \wedge j = 0 \\ \min(\{d(k, j-1) + w(k, i) : i \in Adj(k)\} \cup \{d(i, j-1)\}) & \textbf{if } j > 0 \end{cases}$$

i →, j ↓

j \ i	z (1)	u (2)	v (3)	x (4)	y (5)
0	0	∞	∞	∞	∞
1	0	6	∞	7	∞
2	0	6	4	7	2
3	0	2	4	7	2
4	0	2	4	7	–2

KNAPSACK PROBLEM

Knapsack problem

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ and has value $v_i > 0$.
- Knapsack has capacity of W.
- Goal: fill knapsack so as to maximize total value.

Ex. $\{ 1, 2, 5 \}$ has value 35.
Ex. $\{ 3, 4 \}$ has value 40.
Ex. $\{ 3, 5 \}$ has value 46 (but exceeds weight limit).

i	v_i	w_i
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

knapsack instance
(weight limit W = 11)

Greedy by value. Repeatedly add item with maximum v_i.
Greedy by weight. Repeatedly add item with minimum w_i.
Greedy by ratio. Repeatedly add item with maximum ratio v_i / w_i.

Observation. None of greedy algorithms is optimal.

False start…

Def. $OPT(i)$ = max profit subset of items $1, \ldots, i$.

Case 1. OPT does not select item i.

- OPT selects best of $\{ 1, 2, \ldots, i - 1 \}$.

optimal substructure property (proof via exchange argument)

Case 2. OPT selects item i.

- Selecting item i does not immediately imply that we will have to reject other items.
- Without knowing what other items were selected before i, we don't even know if we have enough room for i.

Conclusion. Need more subproblems!

New variable

Def. $OPT(i, w)$ = max profit subset of items $1, \ldots, i$ with weight limit w.

Case 1. OPT does not select item i.

- OPT selects best of $\{ 1, 2, \ldots, i - 1 \}$ using weight limit w.

optimal substructure property
(proof via exchange argument)

Case 2. OPT selects item i.

- New weight limit $= w - w_i$.
- OPT selects best of $\{ 1, 2, \ldots, i - 1 \}$ using this new weight limit.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{ OPT(i-1, w), \ \ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Dynamic programming algorithm

KNAPSACK $(n, W, w_1, \ldots, w_n, v_1, \ldots, v_n)$

FOR $w = 0$ TO W

  $M[0, w] \leftarrow 0.$

FOR $i = 1$ TO n

  FOR $w = 1$ TO W

  IF $(w_i > w)$ $M[i, w] \leftarrow M[i-1, w].$

  ELSE $M[i, w] \leftarrow \max \{ M[i-1, w], v_i + M[i-1, w-w_i] \}.$

RETURN $M[n, W].$

Example

i	v_i	w_i
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1, w-w_i) \,\} & \text{otherwise} \end{cases}$$

weight limit w

subset of items 1, …, i	0	1	2	3	4	5	6	7	8	9	10	11
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

Analysis

Theorem. There exists an algorithm to solve the knapsack problem with n items and maximum weight W in $\Theta(n\ W)$ time and $\Theta(n\ W)$ space.

weights are integers between 1 and W

Pf.

- Takes $O(1)$ time per table entry.
- There are $\Theta(n\ W)$ table entries.
- After computing optimal values, can trace back to find solution: take item i in $OPT(i, w)$ iff $M[i, w] < M[i-1, w]$. ▪

Remarks.

- Not polynomial in input size! ← "pseudo-polynomial"
- Decision version of knapsack problem is NP-COMPLETE. [CHAPTER 8]
- There exists a poly-time algorithm that produces a feasible solution that has value within 1% of optimum. [SECTION 11.8]