COMP251: Network flows (2)

Jérôme Waldispühl
School of Computer Science
McGill University

Based on slides from M. Langer (McGill)

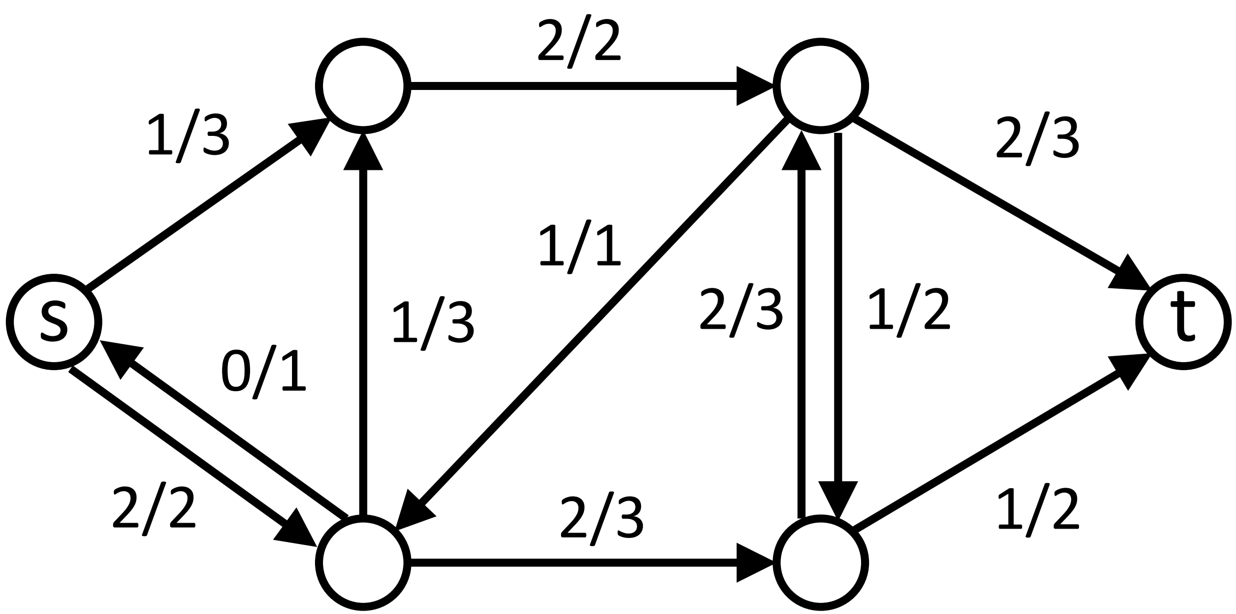
Recap Network Flows

$G = (V, E)$ directed.

Each edge (u, v) has a ***capacity*** $c(u, \mathrm{v}) \geq 0$.

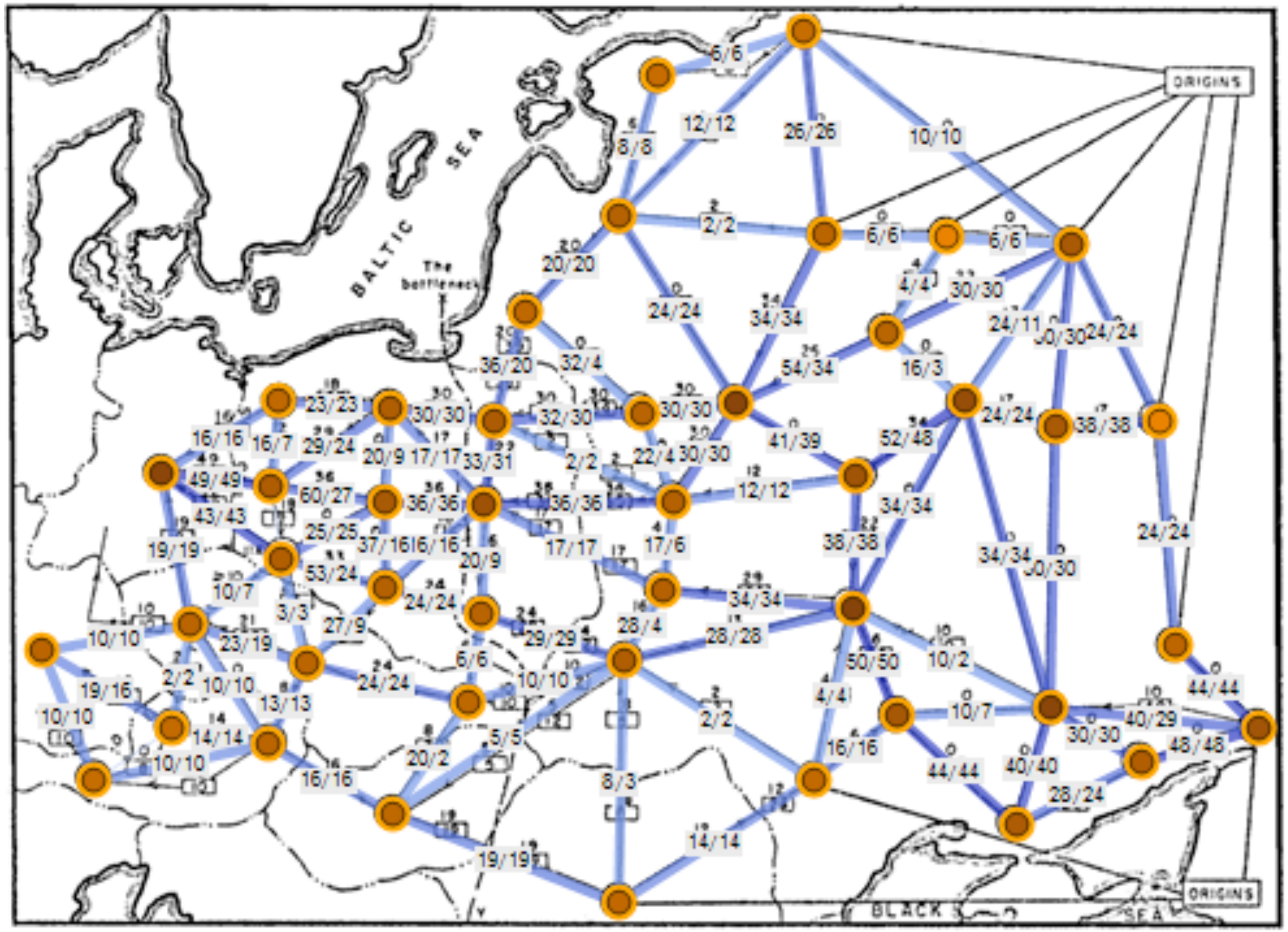
If $(u,\mathrm{v}) \notin E$, then $c(u,\mathrm{v}) = 0$.

Source vertex s, ***sink*** vertex t, assume $s \rightsquigarrow \mathrm{v} \rightsquigarrow t$ for all $\mathrm{v} \in V$.



Problem: Given G, s, t, and c, find a flow whose value is maximum.

Application

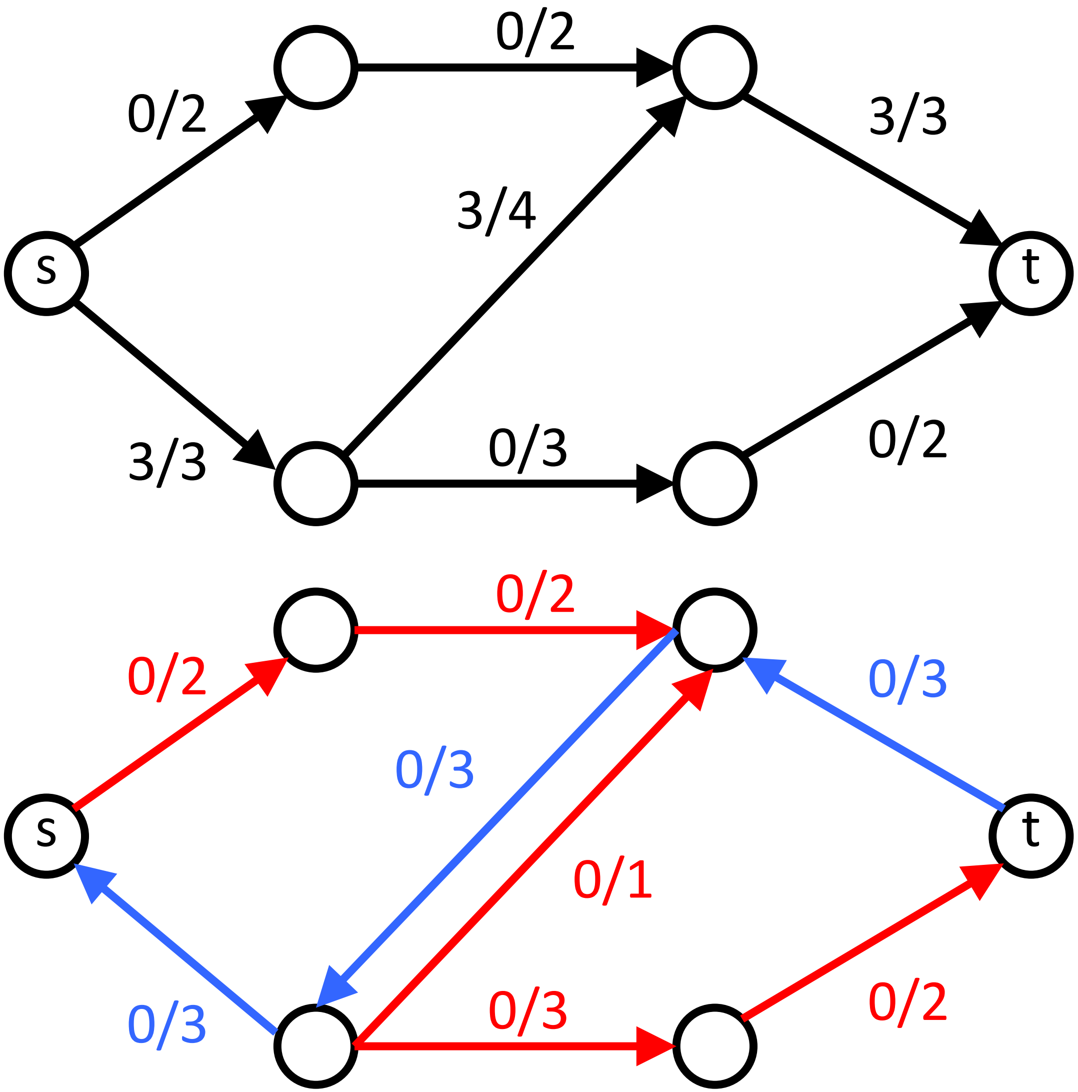


Maximize flow of supplies in eastern europe!

Recap (residual graphs)

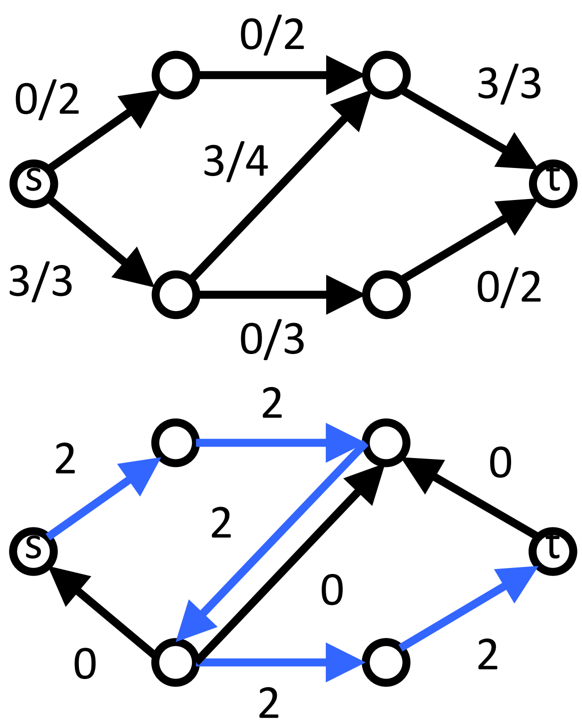
Flow

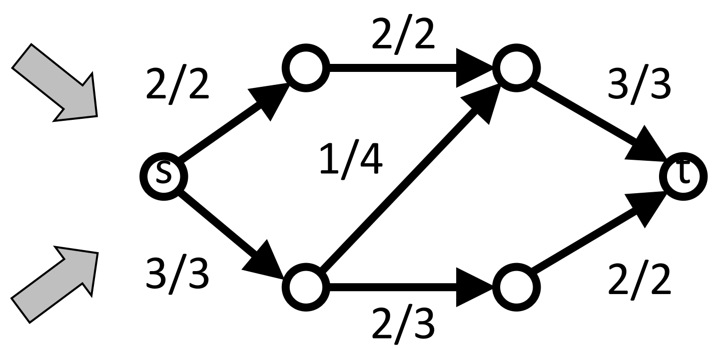
Residual graph



Recap (Ford-Fulkerson algorithm)

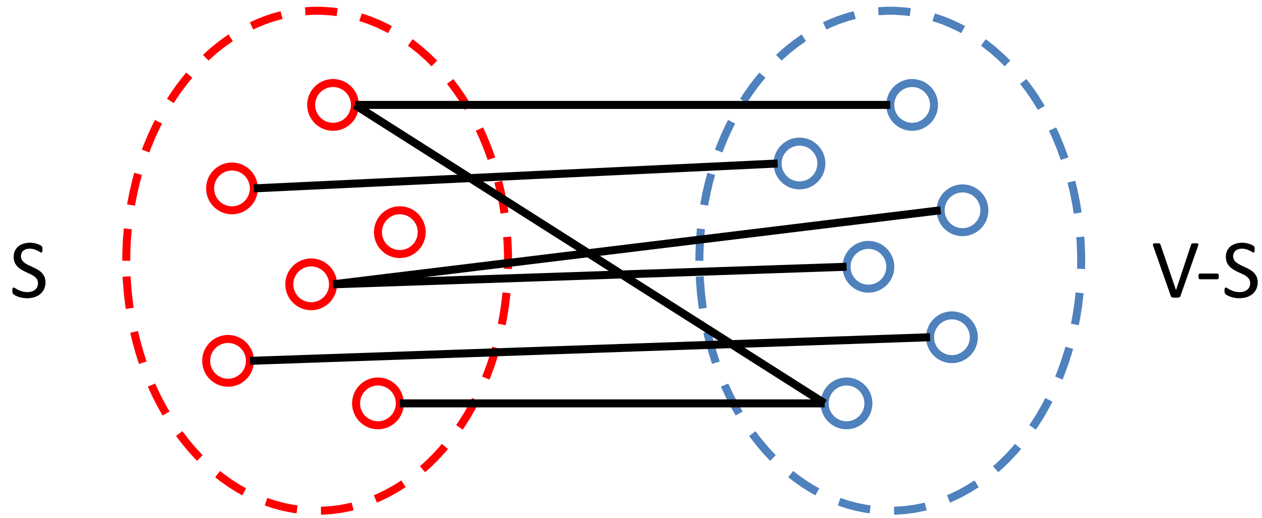
```
f ← 0
G_f ← G
while (there is a s-t path in G_f) do
    f.augment(P)
    update G_f based on new f
```





Recap graph cuts

A graph cut is a partition of the graph vertices into two sets.



The crossing edges from S to V-S are { (u,v) | $u \in S, v \in V\text{-}S$ }, also called the cut set.

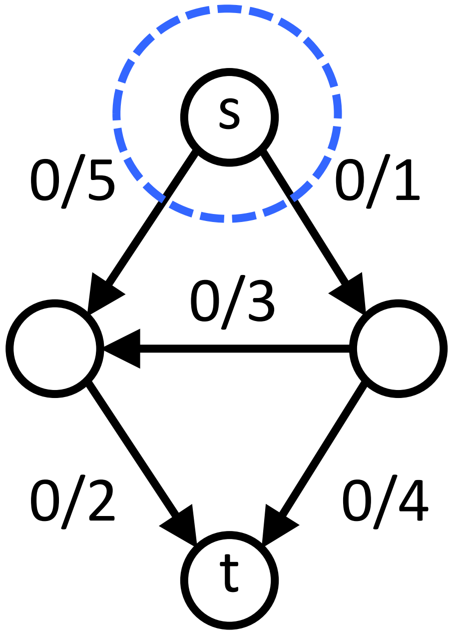
Cuts in flow networks

Definition: An s-t cut of a flow network is a cut A, B such that $s \in A$ and $t \in B$.

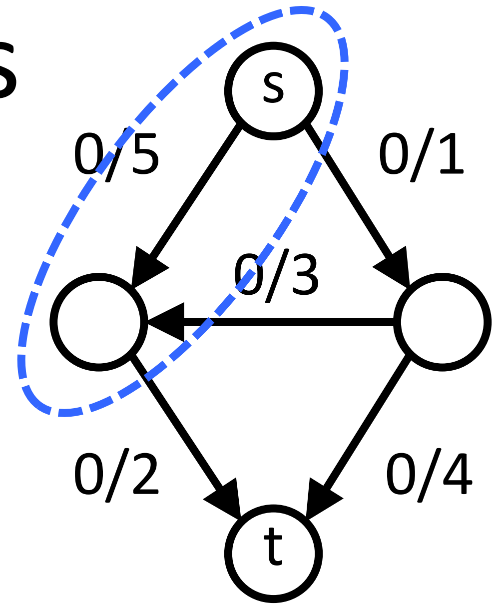
Notation: We write the cut set as cut(A,B). It is the set of edges from A to B.

Definition: The capacity of an s-t cut is $\sum_{e \in cut(A,B)} c(e)$

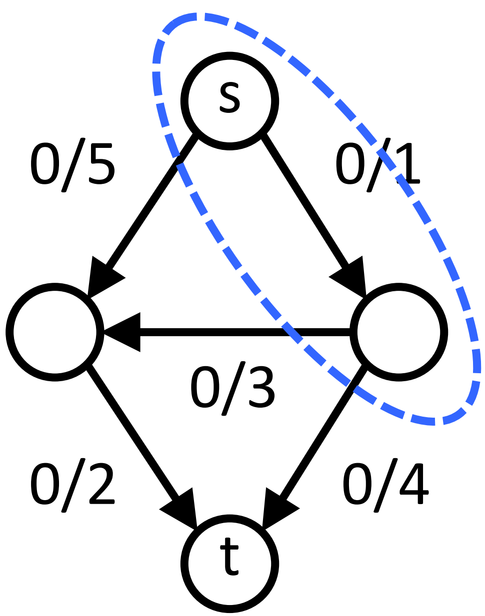
Examples



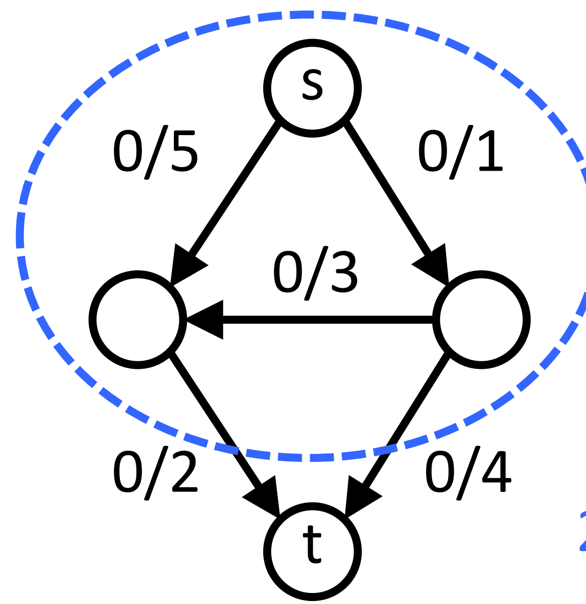
1+5=6



1+2=3



5+3+4=12



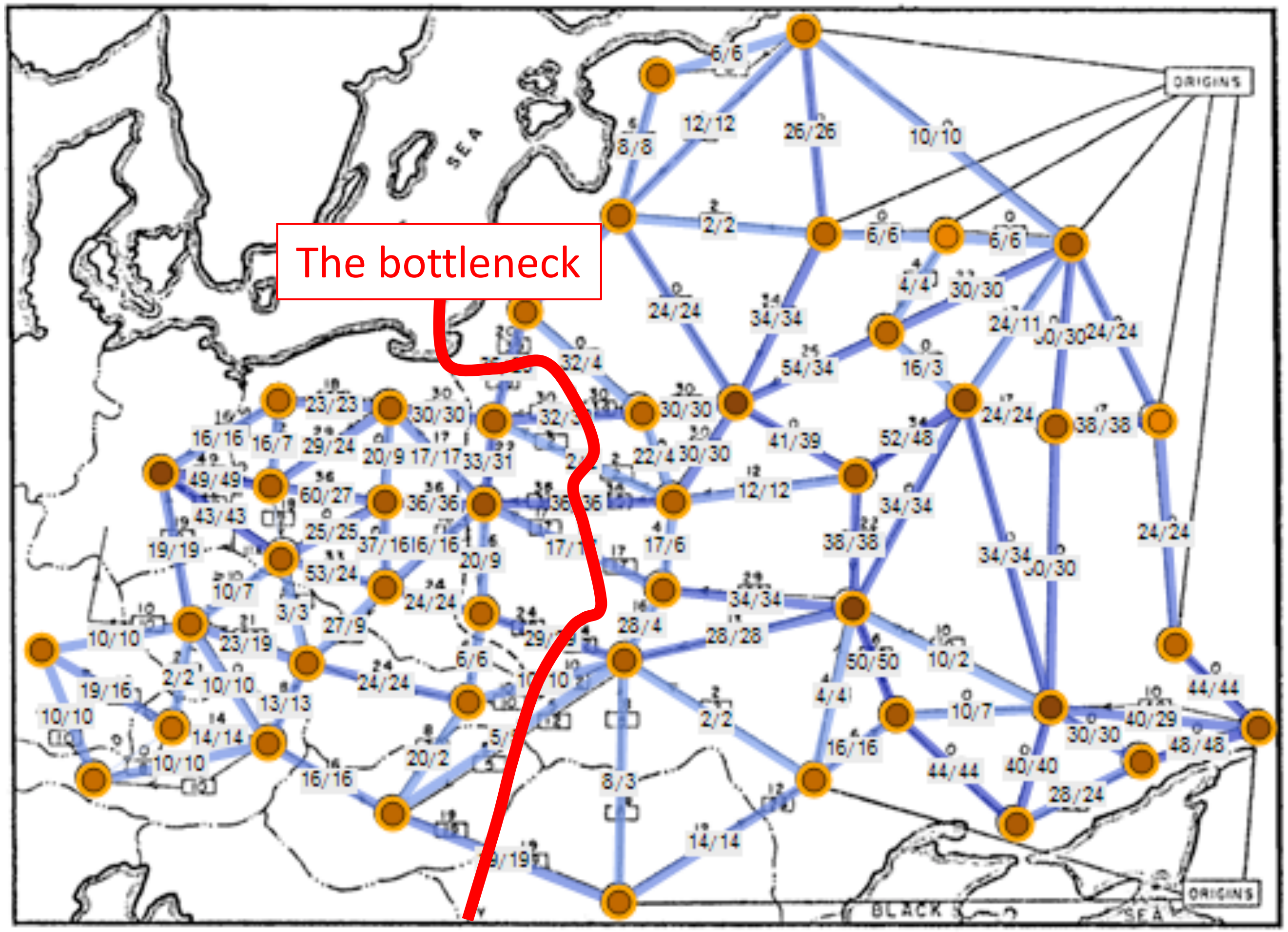
2+4=6

Objectives

For any flow network:

- Maximum value of a flow = the minimum capacity of any cut.
- Ford-Fulkerson gives the “max flow” and the “min cut”.

Application



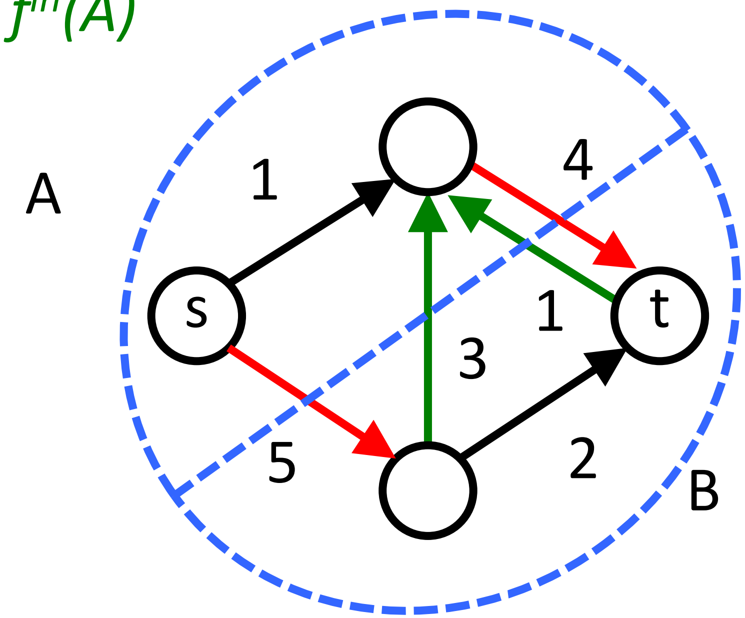
How to cut supplies if cold war turns into real war!

Flow through a cut

Claim: Given a flow network. Let f be a flow and A, B be a s-t cut. Then,

$$|f| = \sum_{e \in cut(A,B)} f(e) - \sum_{e \in cut(B,A)} f(e)$$

Notation: $|f| = f^{out}(A) - f^{in}(A)$



$|f| = 9 - 4 = 5$

Flow through a cut

Proof:

- for any $u \in V-\{s,t\}$, we have $f^{out}(u)=f^{in}(u)$.
- Summing over $u \in A-\{s\}$: $$\sum_{u\in A-\{s\}} f^{out}(u) = \sum_{u\in A-\{s\}} f^{in}(u)$$
- $$|f| = f^{out}(s) = \sum_{u\in A} f^{out}(u) - \sum_{u\in A} f^{in}(u)$$
- Each edge $e=(u,v)$ with $u,v \in A$ contributes to both sums, and can be removed (Note: $f^{in}(s) = 0$).

$$|f| = \sum_{e\in cut(A,B)} f(e) - \sum_{e\in cut(B,A)} f(e)$$

$$\equiv f^{out}(A) - f^{in}(A)$$

Upper bound on flow through cuts

Claim: For any network flow *f*, and any s-t cut(A,B)

$$|f| \leq \sum_{e \in cut(A,B)} c(e)$$

Proof:

$$\begin{aligned} |f| &= f^{out}(A) - f^{in}(A) \\ &\leq f^{out}(A) \\ &\leq \sum_{e \in cut(A,B)} c(e) \end{aligned}$$

Observations

- Some cuts have greater capacities than others.
- Some flows are greater than others.
- **But**, every flow must be ≤ capacity of every s-t cut.
- Thus, the value of the maximum flow is less than capacity of the minimum cut.

Value of flow in Ford-Fulkerson

- Ford-Fulkerson terminates when there is no augmenting path in the residual graph G_f.
- Let A be the set of vertices reachable from s in G_f, and B=V-A.
- A,B is a s-t cut in G_f.
- A,B is an s-t cut in G (G and G_f have the same vertices).
- $|f|=f^{out}(A)-f^{in}(A)$
- We want to show: $$|f| = \sum_{e \in cut(A,B)} c(e)$$
- And in particular:

1 $$f^{out}(A) = \sum_{e \in cut(A,B)} c(e)$$ 2 $$f^{in}(A) = 0$$

Value of flow in Ford-Fulkerson

(1) For any $e=(u,v) \in cut(A,B)$, $f(e)=c(e)$.

- $f(e)<c(e) \Rightarrow e=(u,v)$ would be a forward edge in the residual graph G_f with capacity $c_f(e)=c(e)-f(e)>0$.
- v reachable from s in $G_f \Rightarrow$ contradiction. ■

(2) $f^{in}(A)=0$: $\forall$ $e=(v,u) \in E$ such that $v \in B$, $u \in A$, we have $f(e)=0$.

- $f(e)>0 \Rightarrow \exists$ backward edge (u,v) in G_f such that $c_f(e)=f(e)$
- v is reachable from s in $G_f \Rightarrow$ contradiction. ■

Max flow = min cut

- Ford-Fulkerson terminates when there is no path s-t in the residual graph G_f
- This defines a cut in A,B in G (A = nodes reachable from s)
- $|f| = f^{out}(A) - f^{in}(A)$

$$= \sum_{e \in cut(A,B)} f(e) - \sum_{e \in cut(B,A)} f(e)$$

- Ford-Fulkerson flow = $\sum_{e \in cut(A,B)} c(e) - 0$

= capacity of cut(A,B)

Note: We did not proved uniqueness.

Computing the min cut

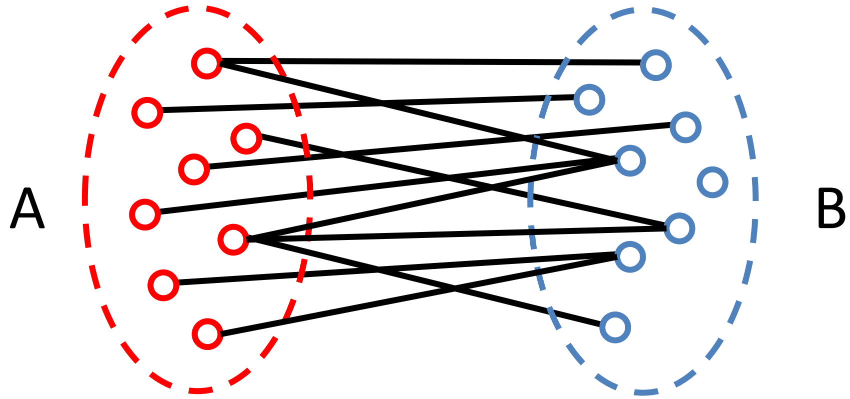
Q: Given a flow network, how can we compute a minimum cut?

Answer:

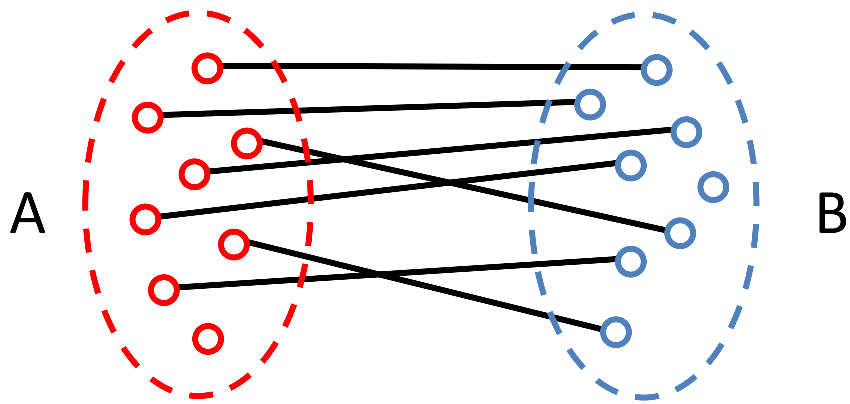
- Run Ford-Fulkerson to compute a maximum flow (it gives us G_f)
- Run BFS or DFS of s.
- The reachable vertices define the set A for the cut

Bipartite matching

Suppose we have an undirected graph bipartite graph G=(V,E).



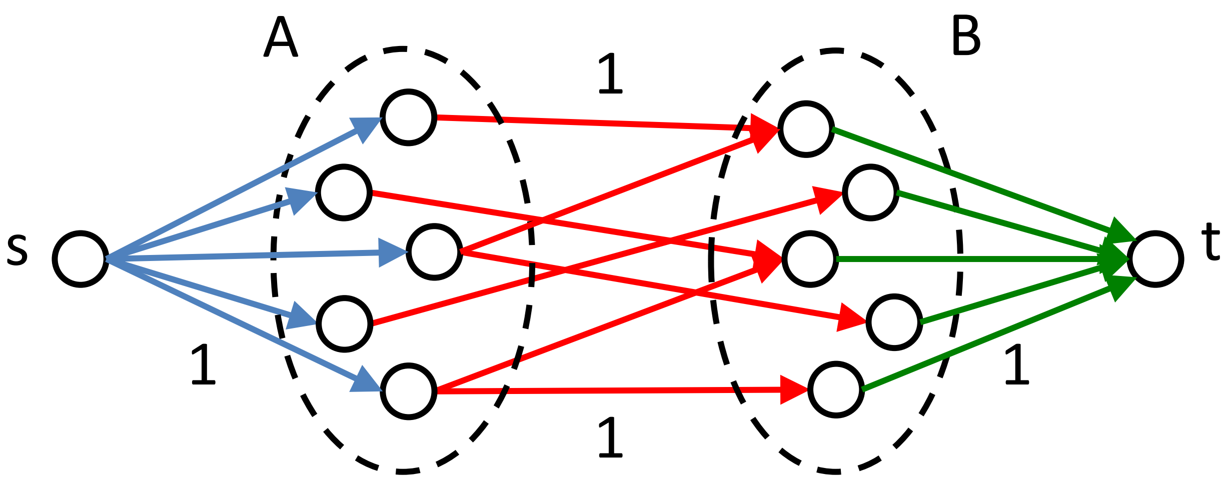
Q: How can we find the maximal matching? (Recall Lecture 11)



Bipartite matching with network flows

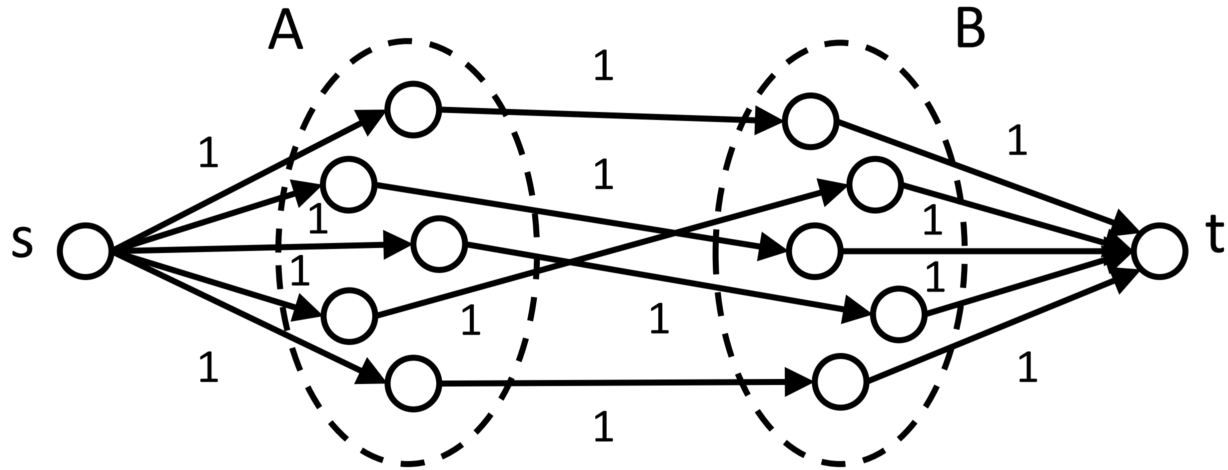
Define a flow network $G'=(V',E')$ such that:

- $V' = V \cup \{s,t\}$
- $E' = \{ (u,v) \mid u \in A, v \in B, (u,v) \in E\} \cup \{ (s,u) \mid u \in A\} \cup \{ (v,t) \mid v \in B\}$
- Capacities of every edge = 1.



Motivation: Max flow $\Rightarrow$ max matching.

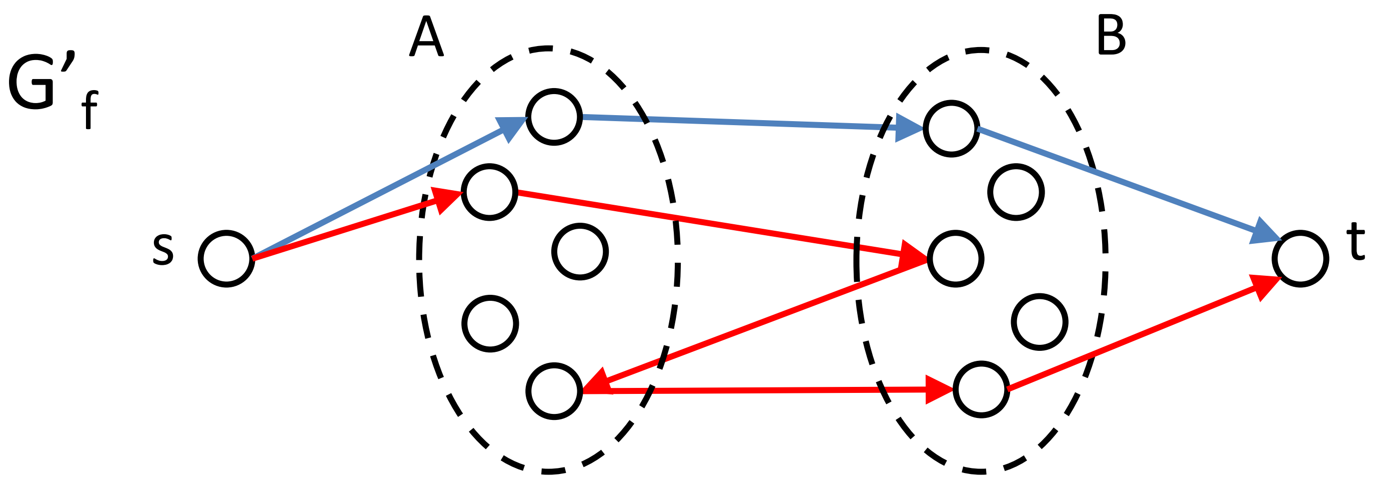
Max flow in bipartite graphs



Exercise: The maximal flow found by Ford-Fulkerson defines a maximal matching in the original graph G (the maximal set of edges (u,v) $u \in A$ & $v \in B$ such that f(u,v)=1.

Max matching with Ford-Fulkerson

Ford-Fulkerson will find an augmenting path with β=1 at each iteration. They are of the form:



Or have more than one zig-zag.

Note:

- No edge from B to A in E'. The back edges are in the residual graph.
- Edges e such that c(e)=0 are not shown.

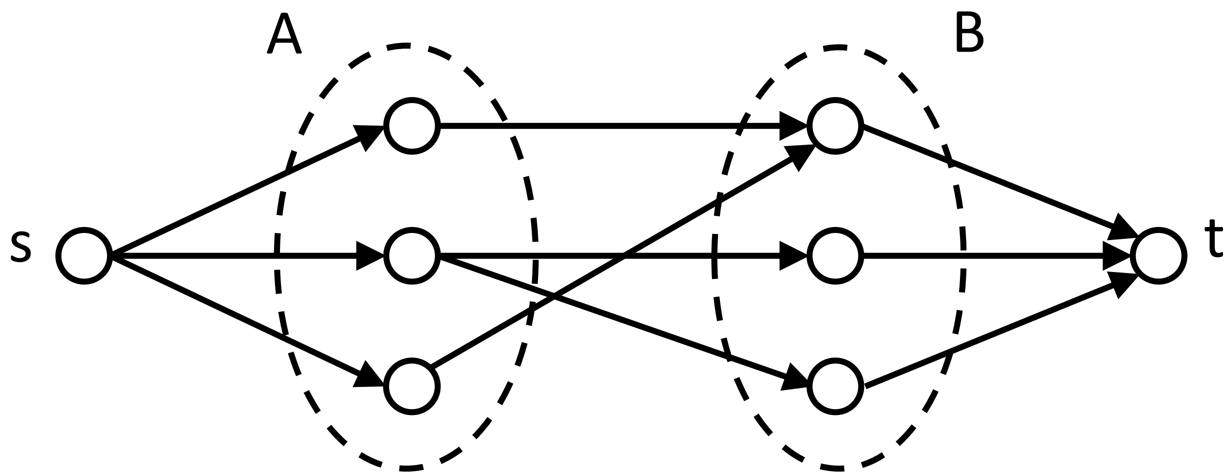
Running time

Q: How long will it take to find a maximal matching with Ford-Fulkerson?

- The general complexity of Ford-Fulkerson is O(C.|E|), where $$C = \sum_u c(s,u)$$
- Suppose |A|=|B|=n
- Then, C=|A|=n and |E'|= |E|+2n = m+2n (Assume m>n)
- Thus, C . |E'| = n . (m + 2n)
- Running time is O(nm)

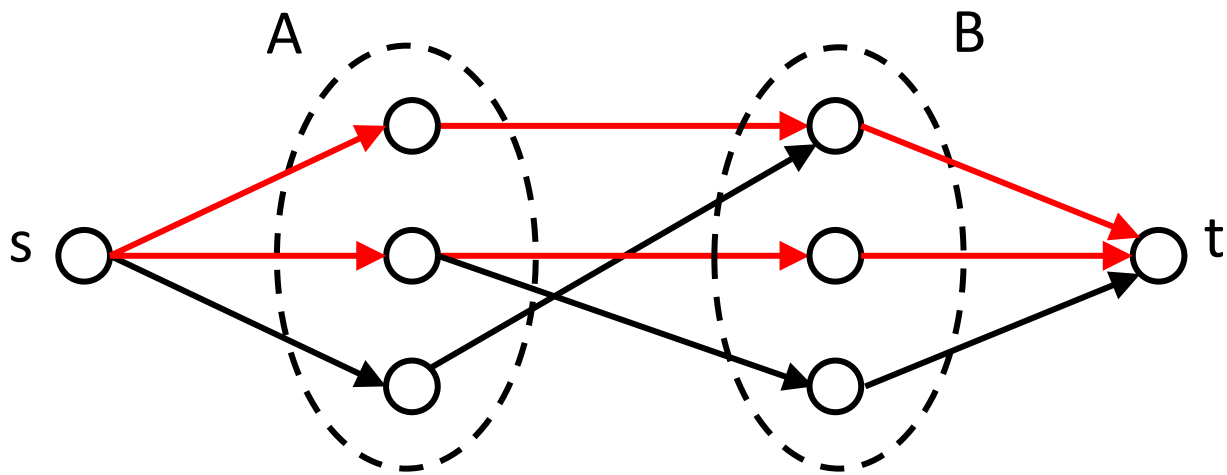
Example

What is max flow? What is min cut?



Example

What is max flow? What is min cut?



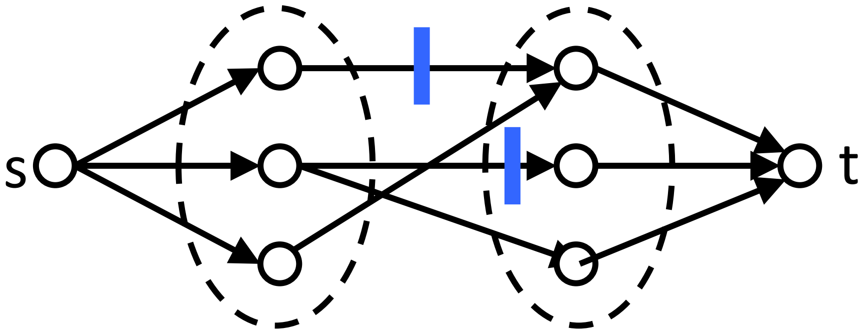
Max flow $|f|=2$.

There are other flows having $|f|=2$.

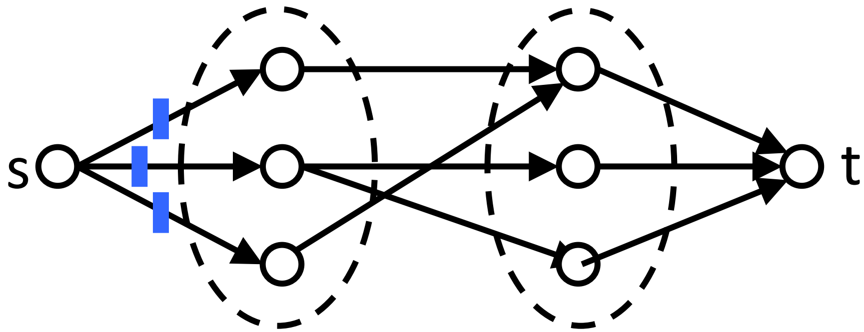
What is the minimum cut?

Example

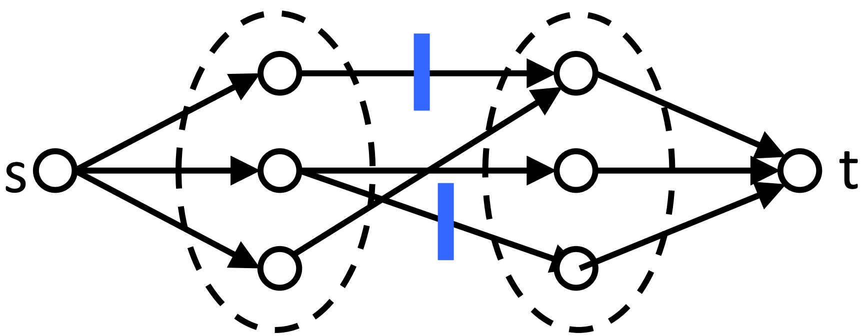
Find any min cut with capacity 2.



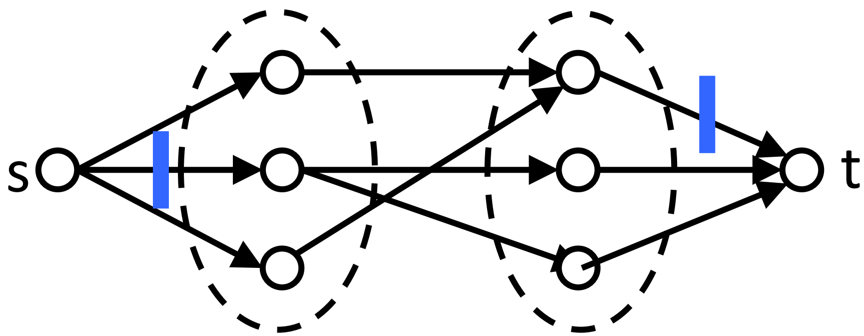
Not a cut!



Not a min cut!



Not a cut!

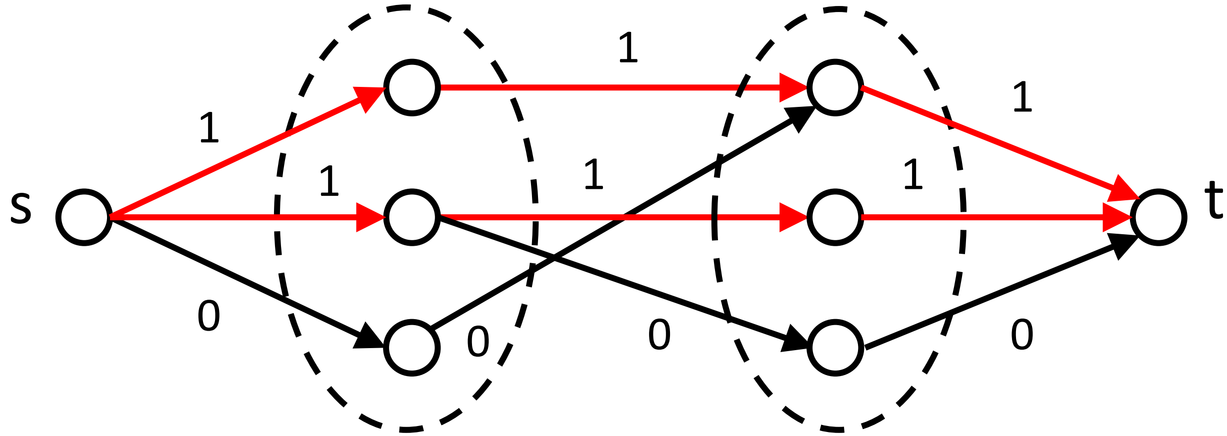


min cut!

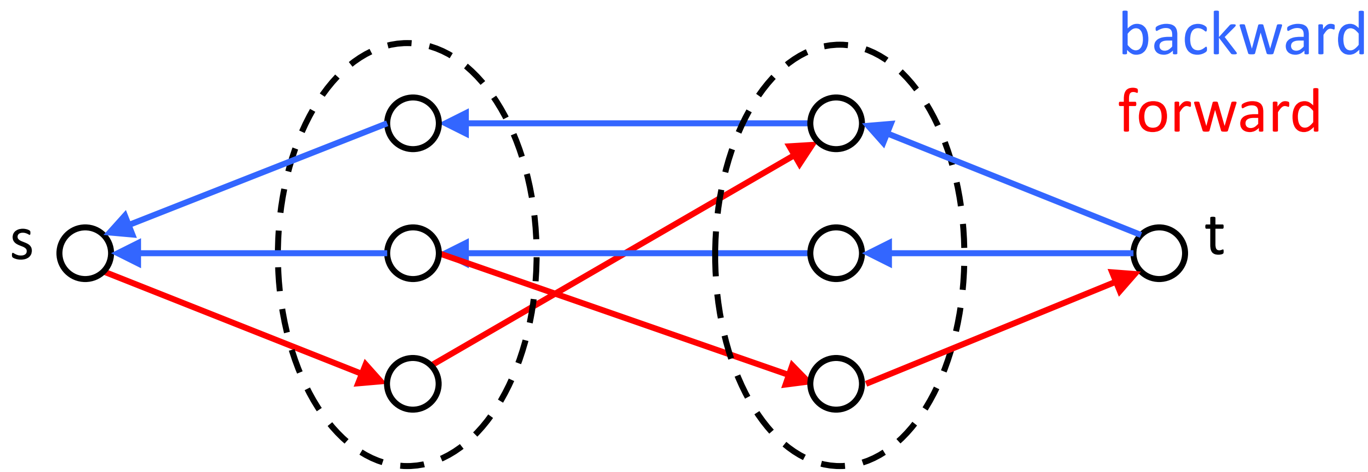
Example

To find a min cut compute a max flow.

Flow



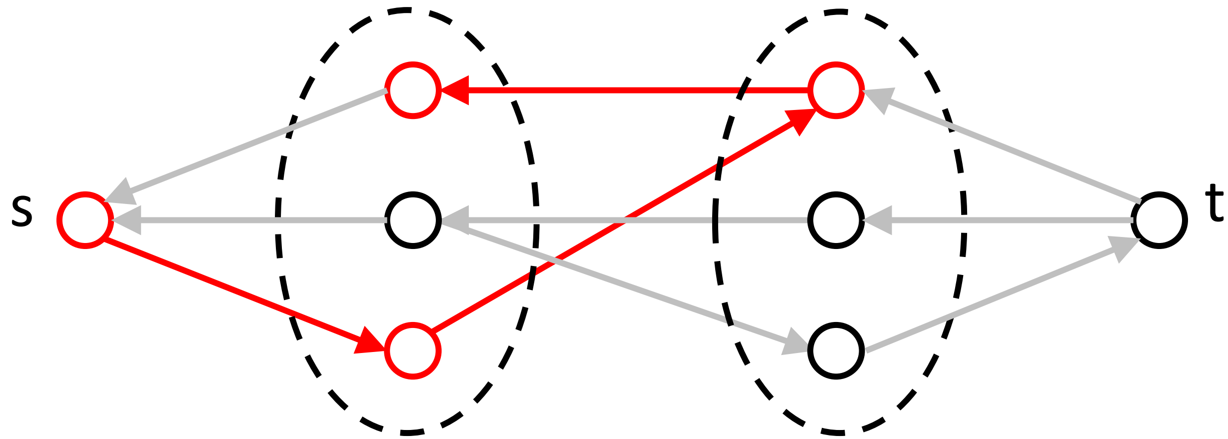
Residual graph



Example

To find the cut run BFS (or DFS) from s on the residual graph.
The reachable vertices define the (min) cut.

Residual graph with DFS



Min cut (in G!)