COMP251: Network flows (1)

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Based on slides from M. Langer (McGill) & (Cormen *et al.*, 2009)

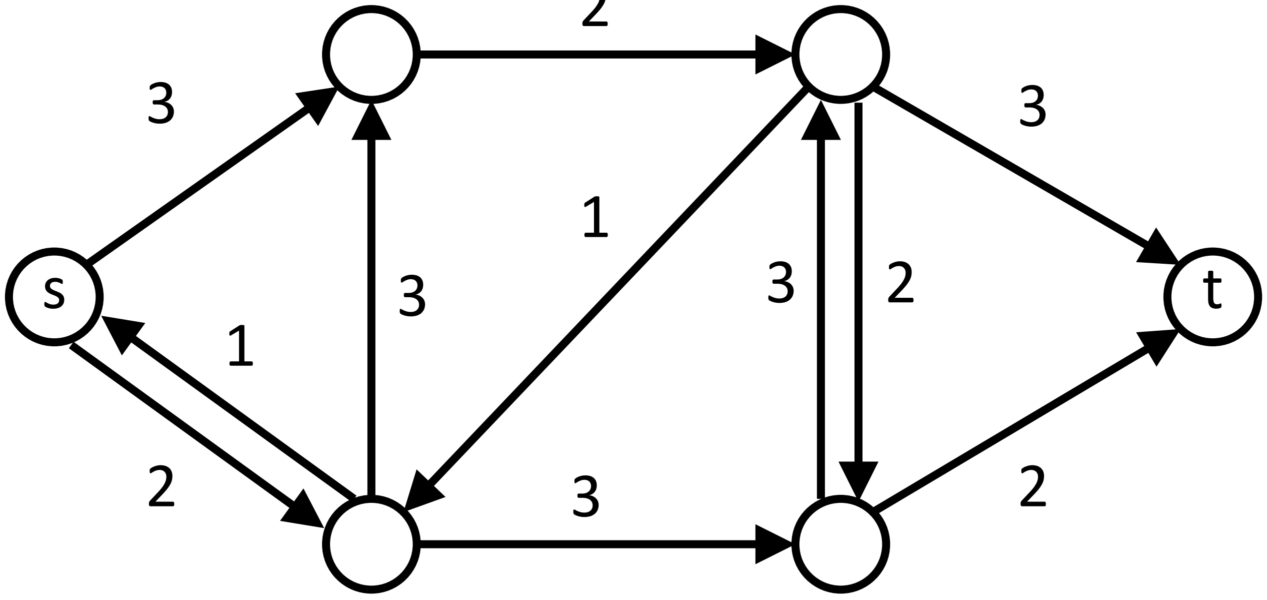
Flow Network

$G = (V, E)$ directed.

Each edge (u, v) has a ***capacity*** $c(u, \mathrm{v}) \geq 0$.

If $(u,\mathrm{v}) \notin E$, then $c(u,\mathrm{v}) = 0$.

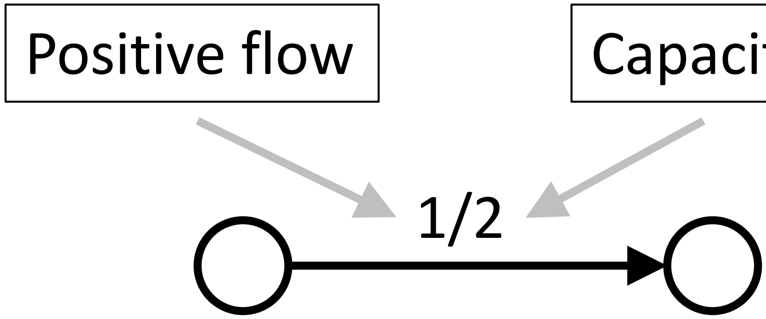
Source vertex s, ***sink*** vertex t, assume $s \rightsquigarrow \mathrm{v} \rightsquigarrow t$ for all $\mathrm{v} \in V$.



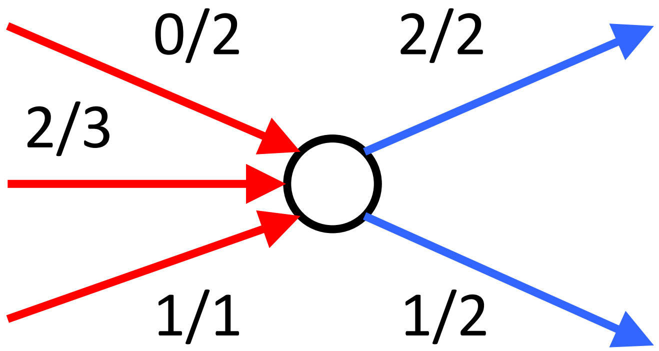
Definitions

Positive flow: A function p : V × V → **R** satisfying.

Capacity constraint: For all u, v ∈ V, 0 ≤ p(u, v) ≤ c(u, v), ⍰⍰



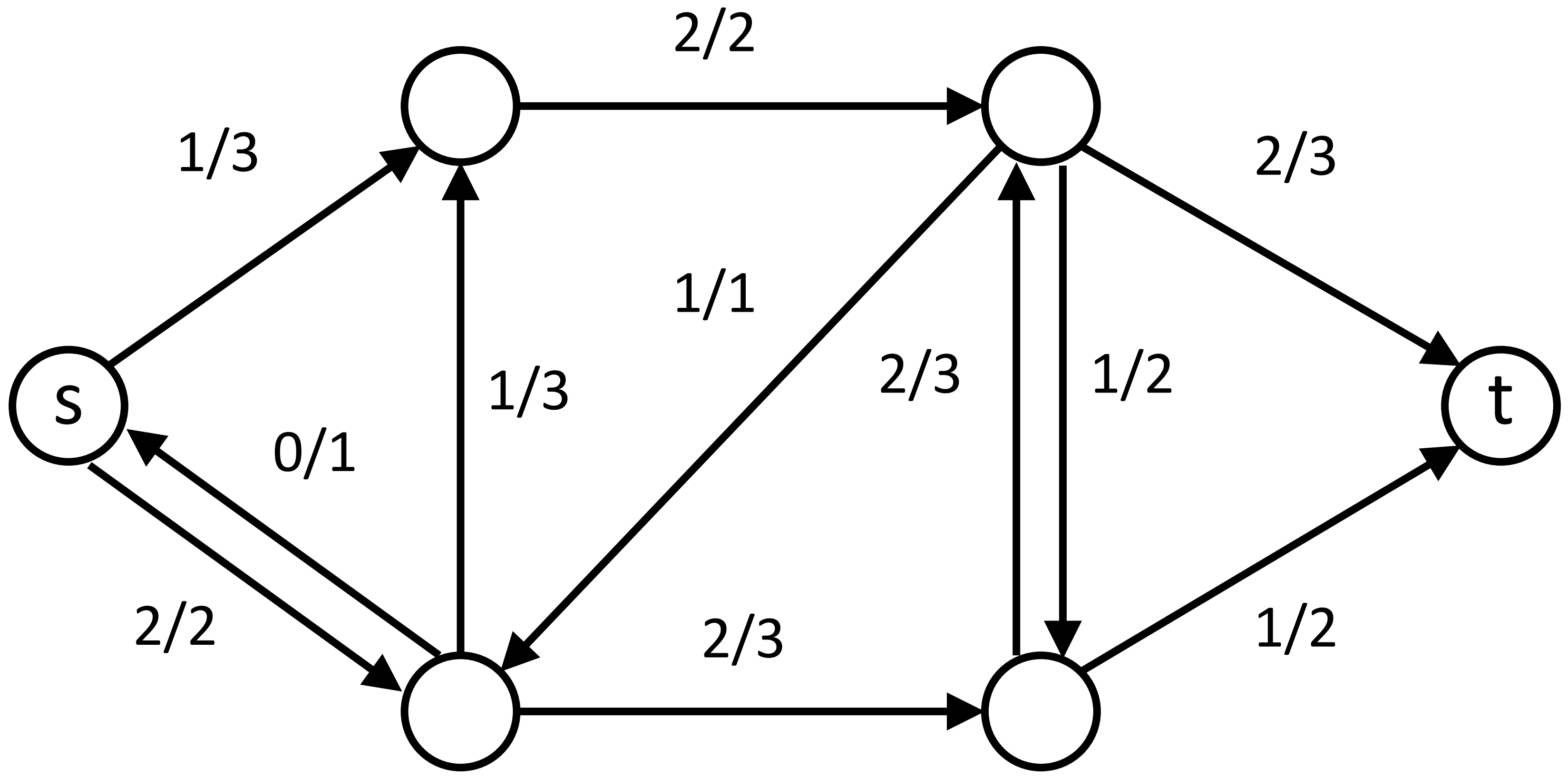
Flow conservation: For all u ∈ V – {s, t}, $\underbrace{\sum_{v\in V} p(v,u)}_{\text{Flow into u}} = \underbrace{\sum_{v\in V} p(u,v)}_{\text{Flow out of u}}$



Flow in: 0 + 2 + 1 = 3

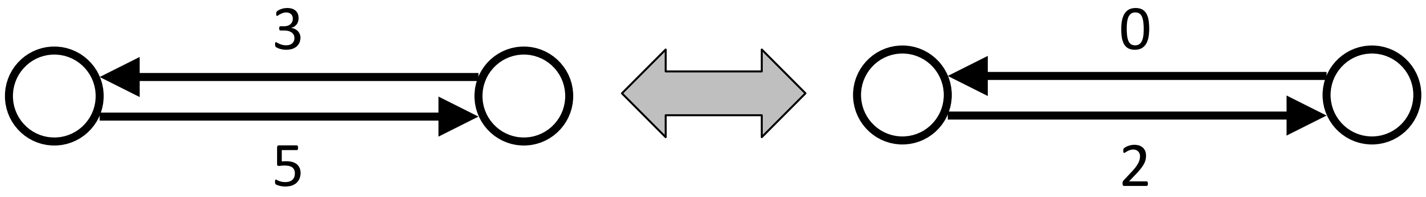
Flow out: 2 + 1 = 3

Example



Cancellation with positive flows

- Without loss of generality, can say positive flow goes either from *u* to v or from v to *u*, but not both.
- In the above example, we can "cancel" 1 unit of flow in each direction between *x* and *z*.



- Capacity constraint is still satisfied.
- Flow conservation is still satisfied.

Net flow

A function $f : V \times V \to \mathbf{R}$ satisfying:

- ***Capacity constraint:*** For all u, v $\in V$, $f(u, \text{v}) \leq c(u, \text{v})$,
- ***Skew symmetry:*** For all u, v $\in V$, $f(u, \text{v}) = -f(\text{v}, u)$, v$\in V$
- ***Flow conservation:*** For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u,v) = 0$

$$\underbrace{\sum_{v \in V; f(v,u) > 0} f(v,u)}_{\text{Total positive flow entering u}} = \underbrace{\sum_{v \in V; f(u,v) > 0} f(u,v)}_{\text{Total positive flow leaving u}}$$

Positive vs. Net flows

Define net flow in terms of positive flow:

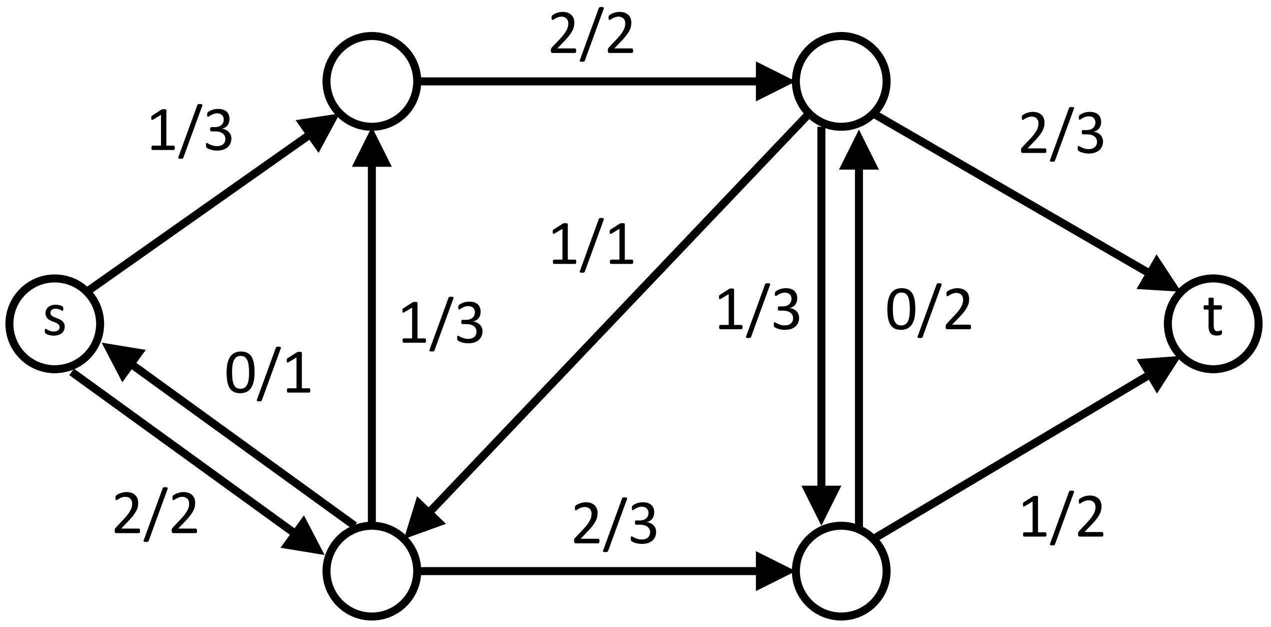
$$f(u,\mathrm{v}) = p(u,\mathrm{v}) - p(\mathrm{v},u).$$

The differences between positive flow p and net flow f :

- $p(u,\mathrm{v}) \geq 0$,
- f satisfies skew symmetry.

Values of flows

Definition: $f = |f| = \sum_{v \in V} f(s,v)$ = total flow out of source.

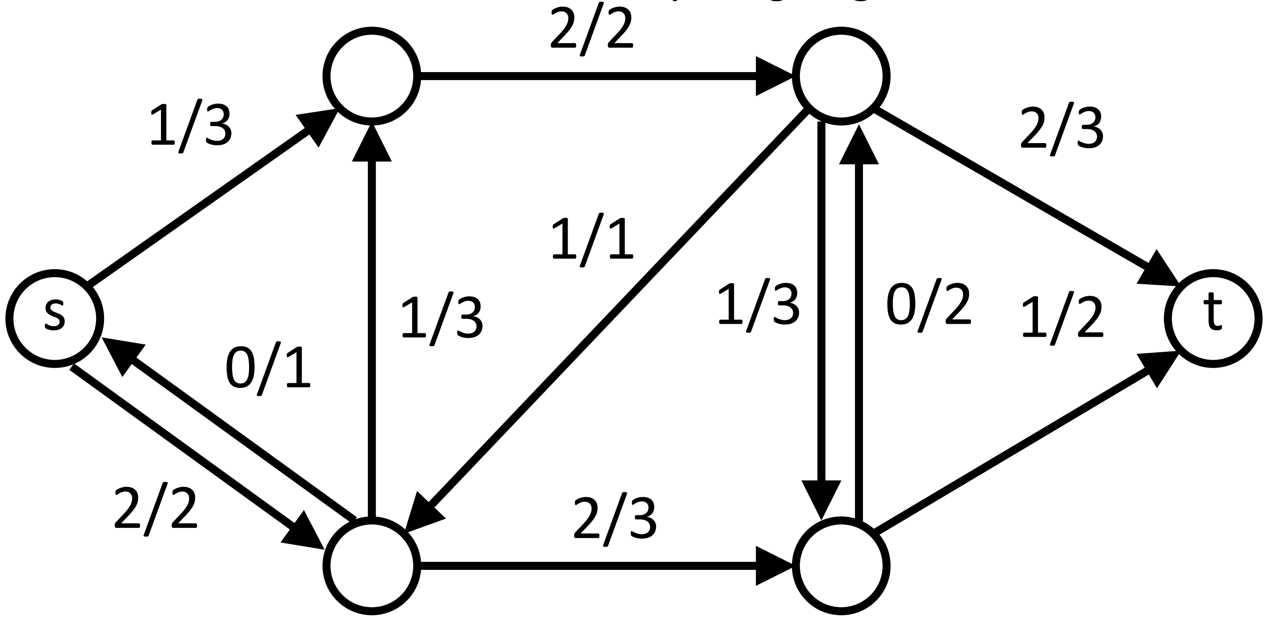


Value of flow $f = |f| = 3$.

Flow properties

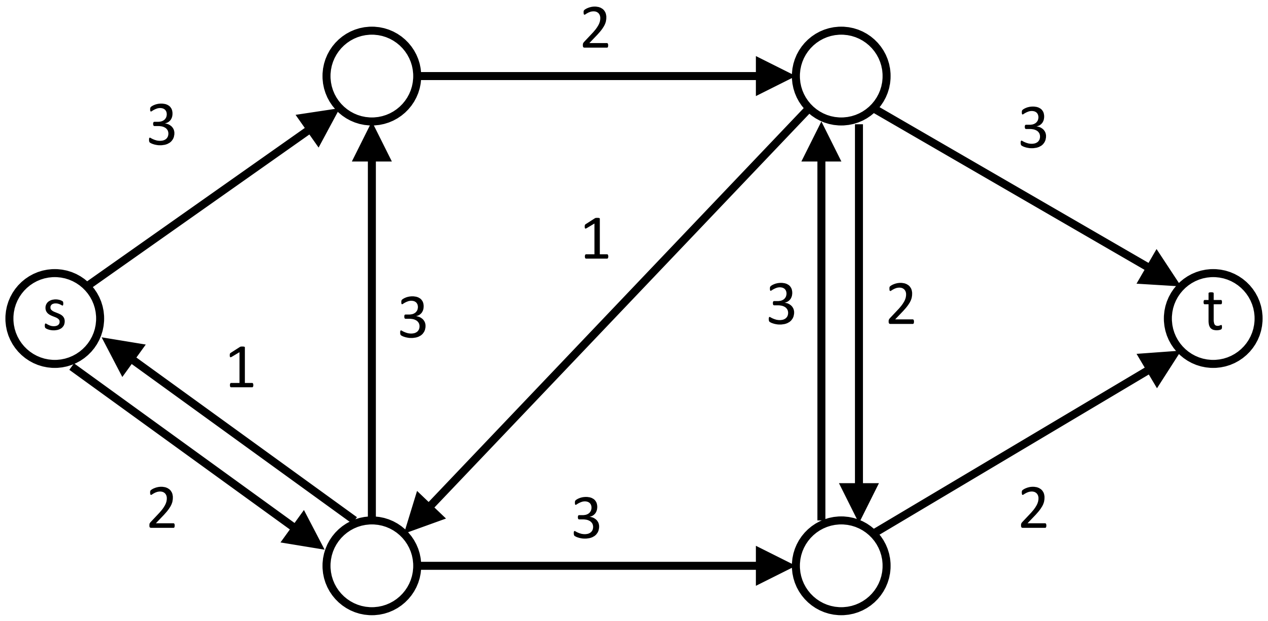
- Flow in == Flow out
- Source *s* has outgoing flow
- Sink *t* has ingoing flow
- Flow out of source *s* == Flow in the sink *t*

- Source s has only outgoing flow

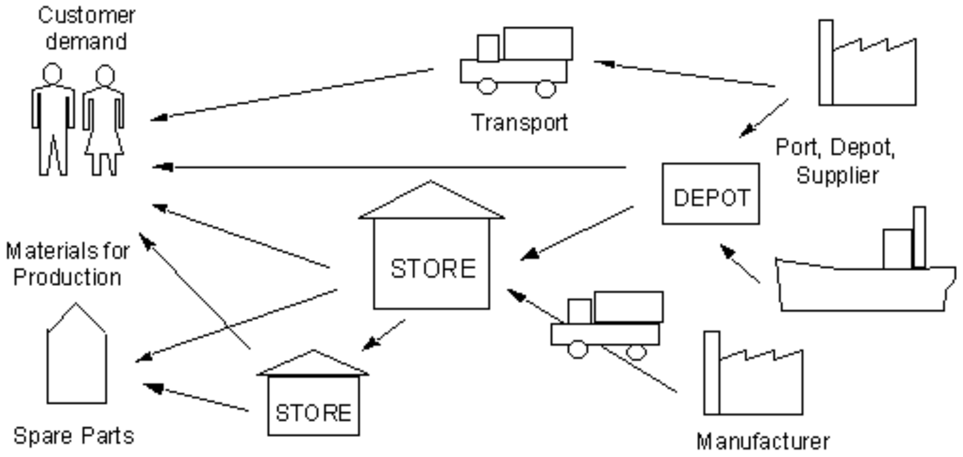


Maximum-flow problem

Given *G*, *s*, *t*, and *c*, find a flow whose value is maximum.



Applications



(https://ais.web.cern.ch/ais/)

(http://driverlayer.com)

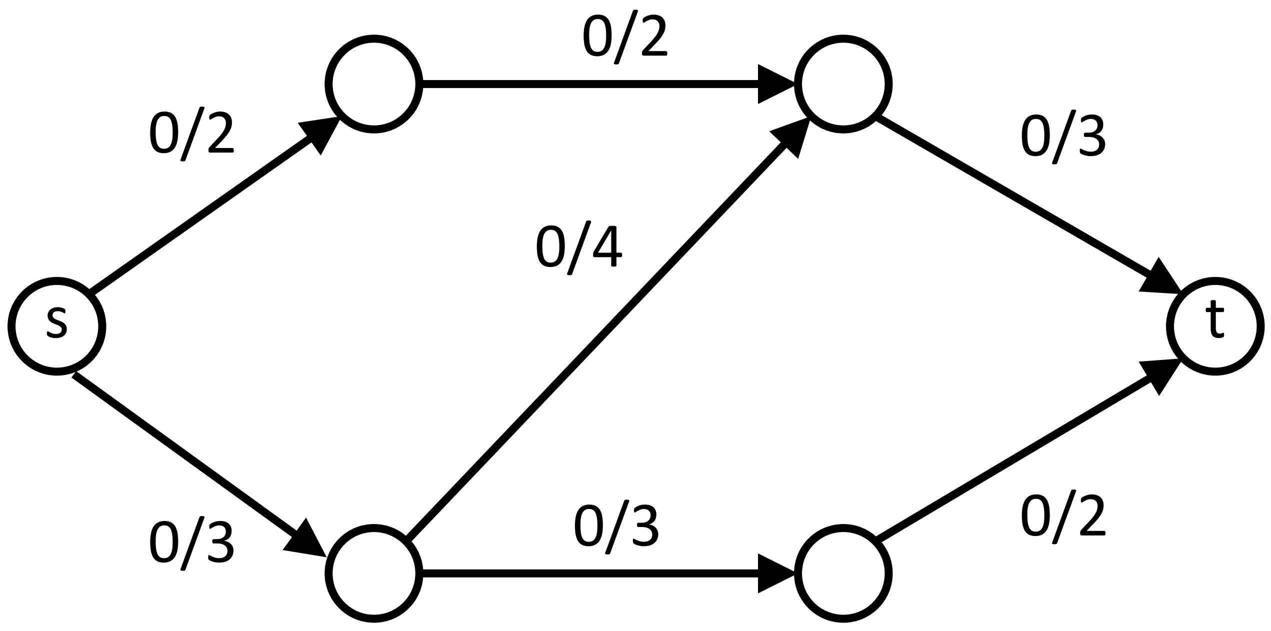
Naïve algorithm

```
Initialize f = 0
While true {
   if (∃ path P from s to t such that all
edges have a flow less than capacity)
   then
      increase flow on P up to max capacity
   else
      break
}
```

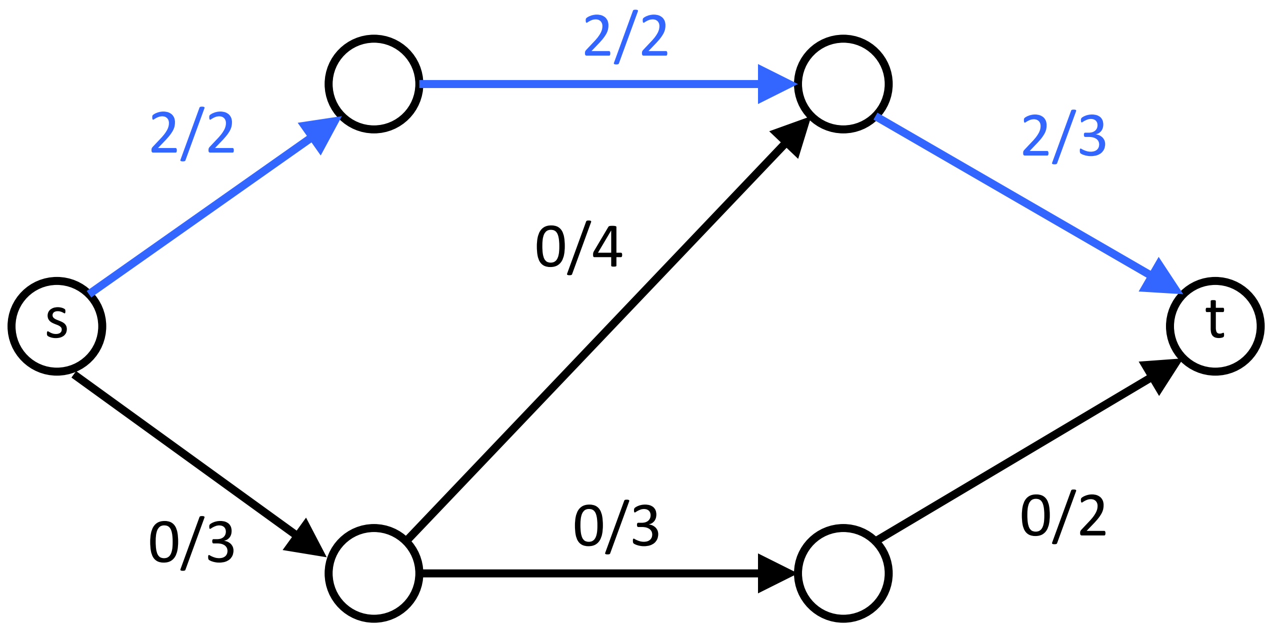
Naïve algorithm

```
Initialize f = 0
While true {
   if ( ∃ a path P from s to t  s.t. all edges e ∈ P f(e) < c(e) )
   then {
      β = min{ c(e)-f(e) | e ∈ P}
      for all e ∈ P { f(e) += β }
   } else { break }
}
```

Example where algorithm works

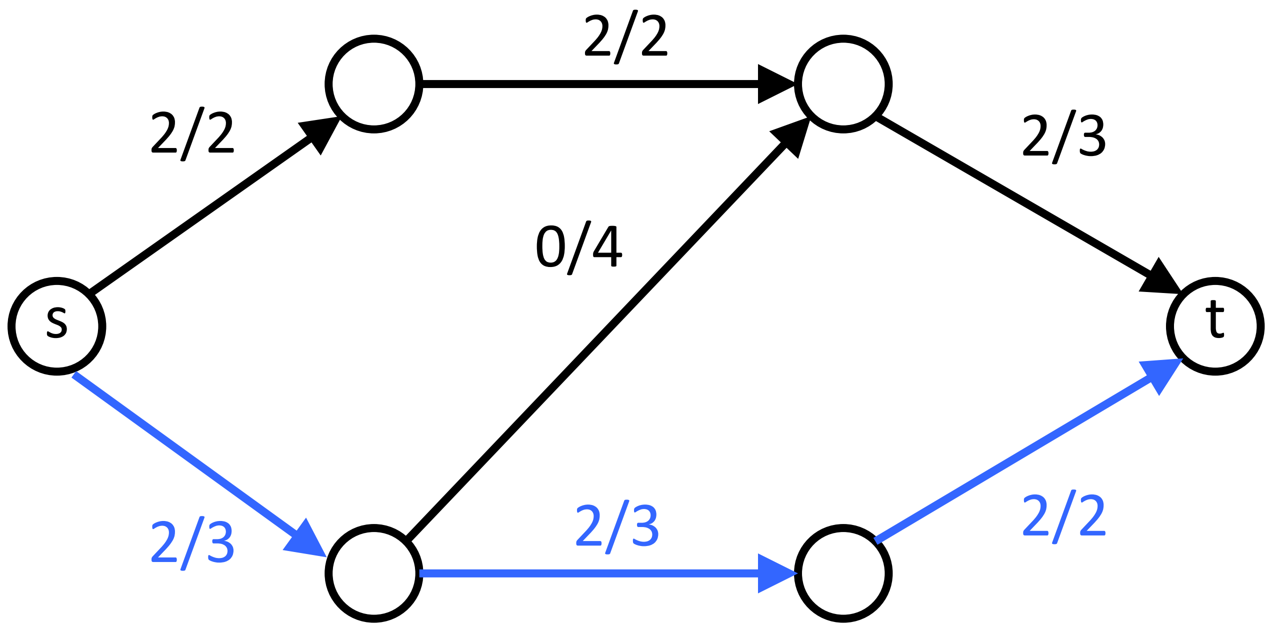


Example where algorithm works



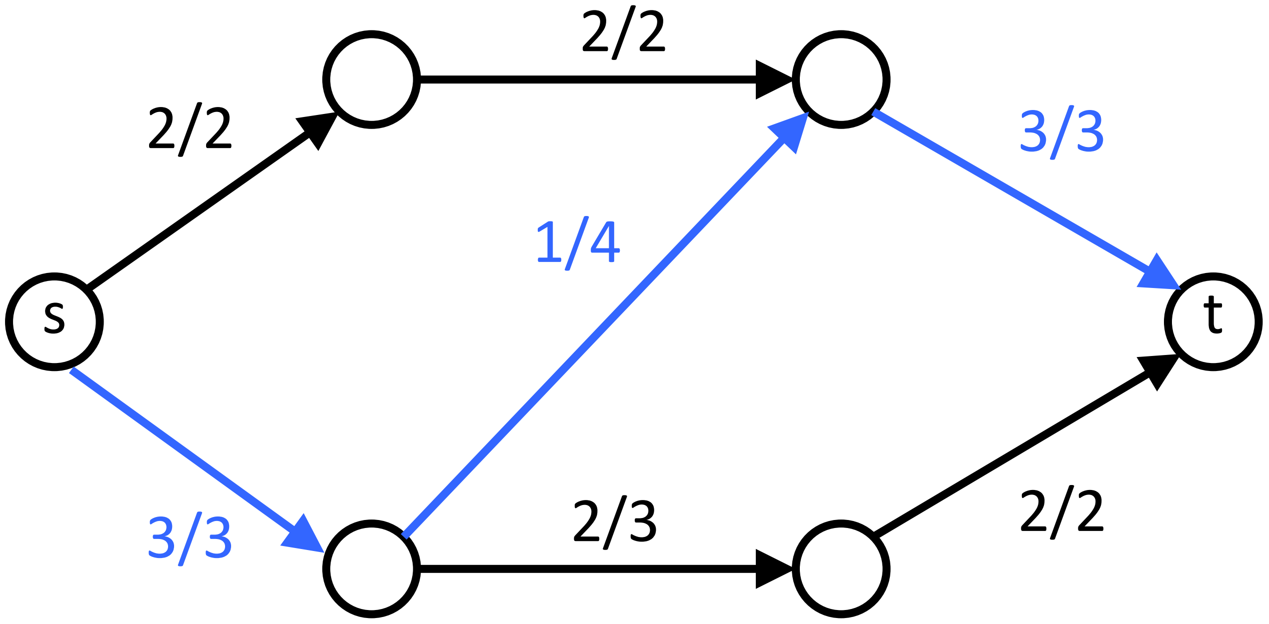
|f|=2

Example where algorithm works



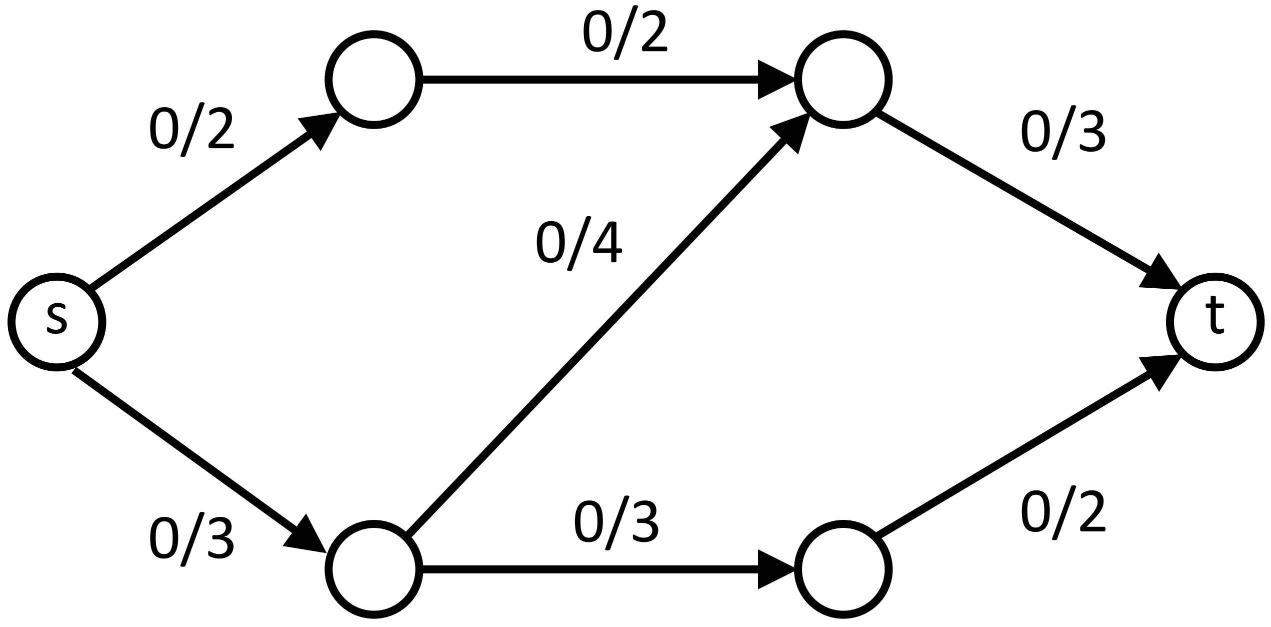
|f|=4

Example where algorithm works

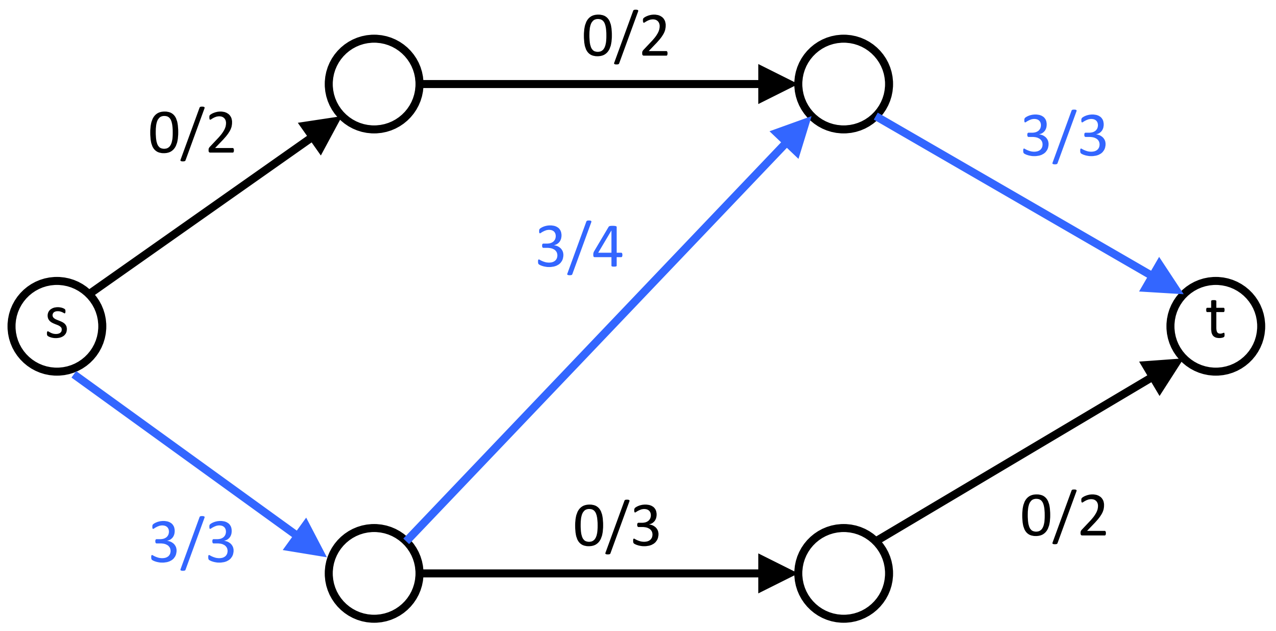


|f|=5

Example where algorithm fail!



Example where algorithm fail!



|f|=3 And terminates…

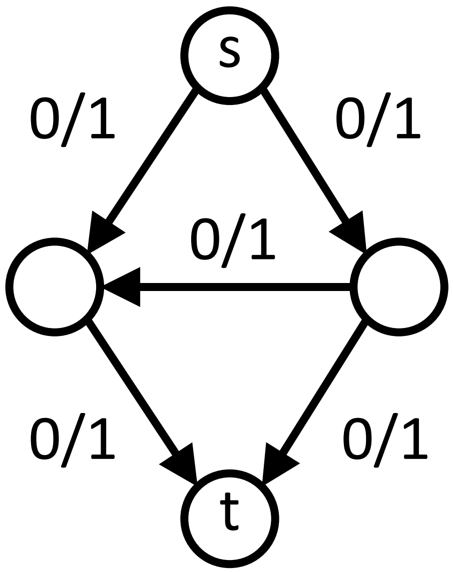
Challenges

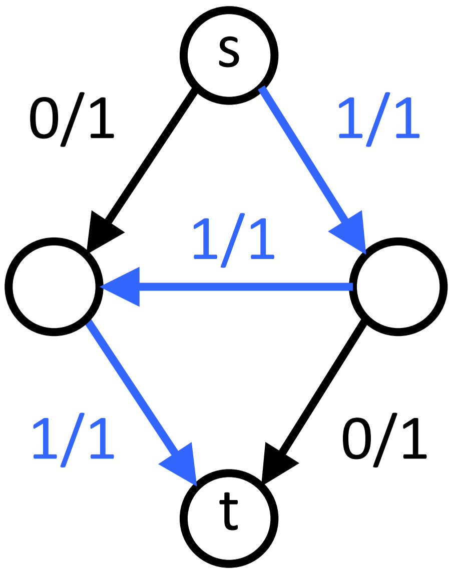
How to choose paths such that:

- We do not get stuck
- We guarantee to find the maximum flow
- The algorithm is efficient!

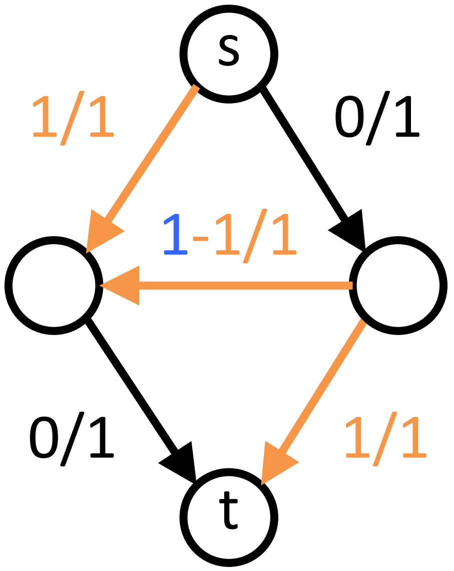
A better algorithm

Motivation: If we could subtract flow, then we could find it.

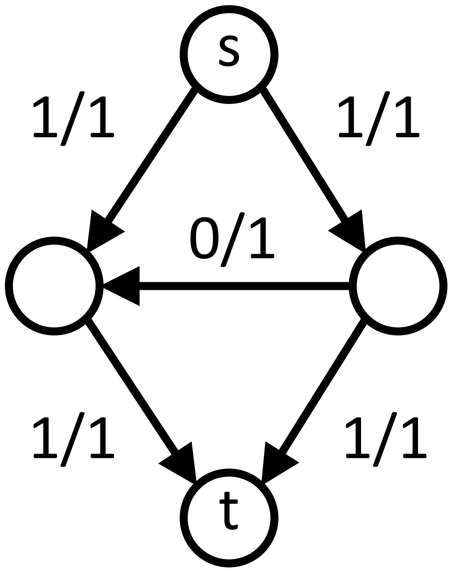




Algo 1
terminates
here…



Negative value
on edge that
does not satisfy
the definition



Residual graphs

Given a flow network G=(V,E) with edge capacities c and a given flow f, define the *residual graph G_f* as:

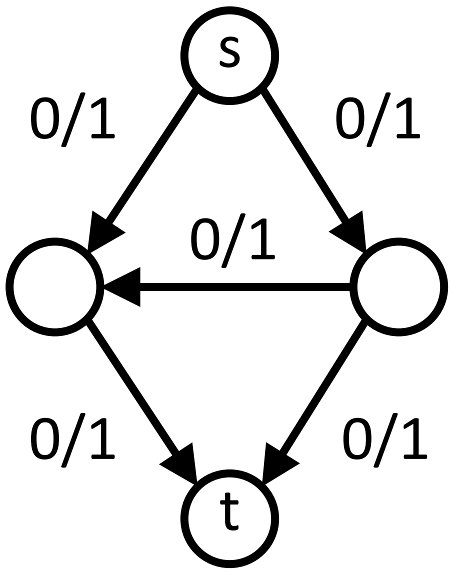
- G_f has the same vertices as G
- The edges E_f have capacities c_f (called *residual capacities*) that allow us to change the flow f, either by:
  1. Adding flow to an edge e $\in$ E
  2. Subtracting flow from an edge $\in$ E

Residual graphs

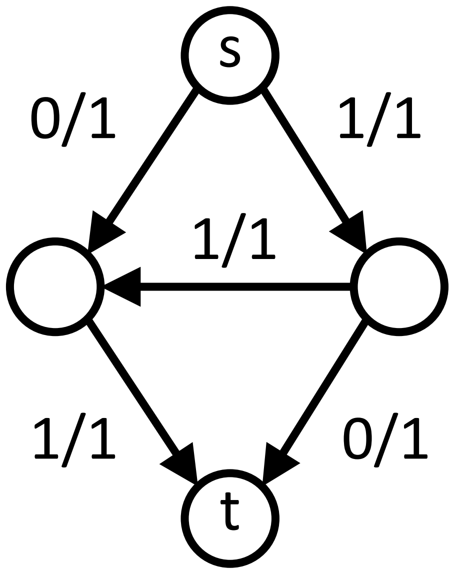
```
for each edge e = (u, v) $\in$ E
   if f(e) < c(e)
   then {
      put a **forward edge** (u,v) in $E_f$
      with residual capacity $c_f(e)=c(e)-f(e)$
   }
   if f(e)>0
   then {
      put a **backward edge** (v,u) in $E_f$
      with residual capacity $c_f(e) = f(e)$
   }
}
```

Example 1/3

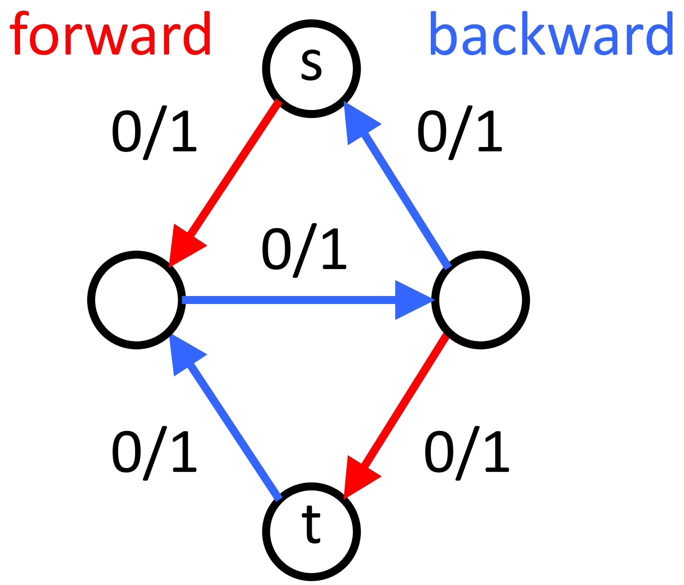
Flow network



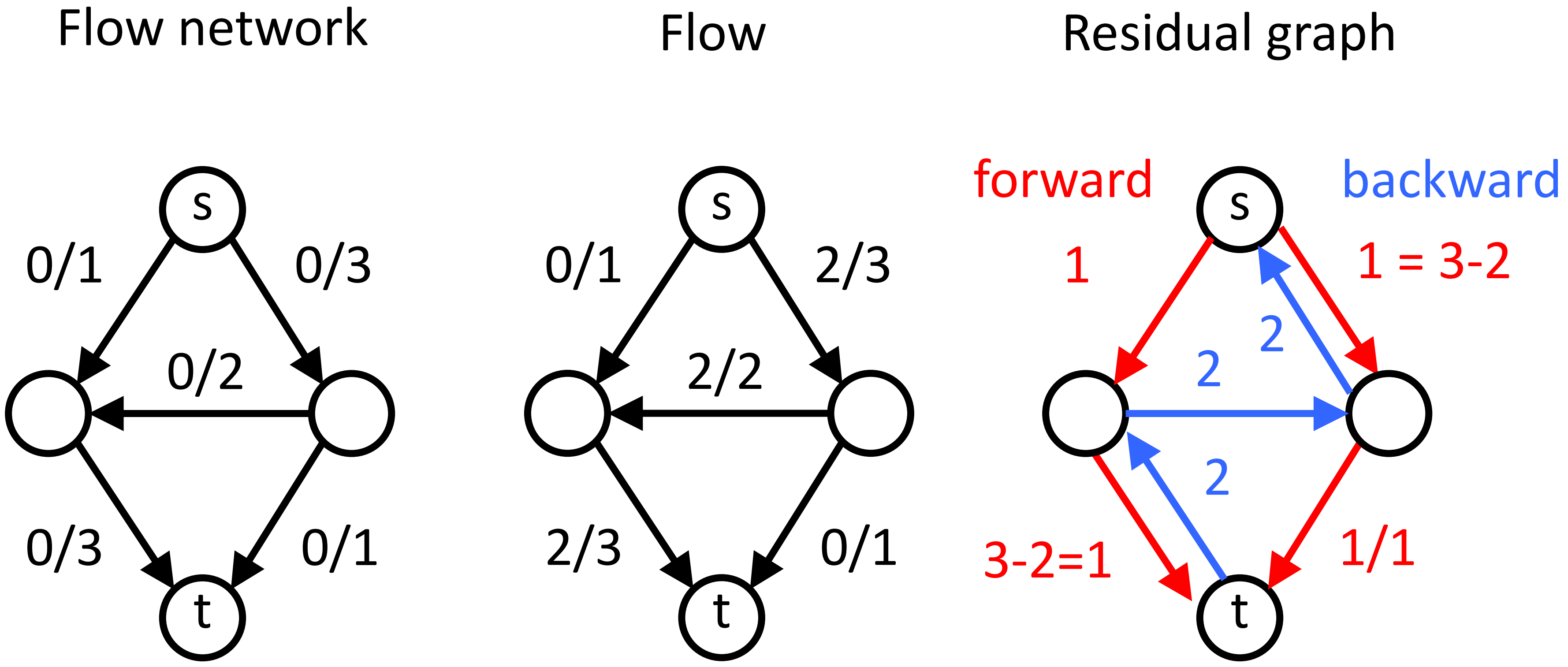
Flow



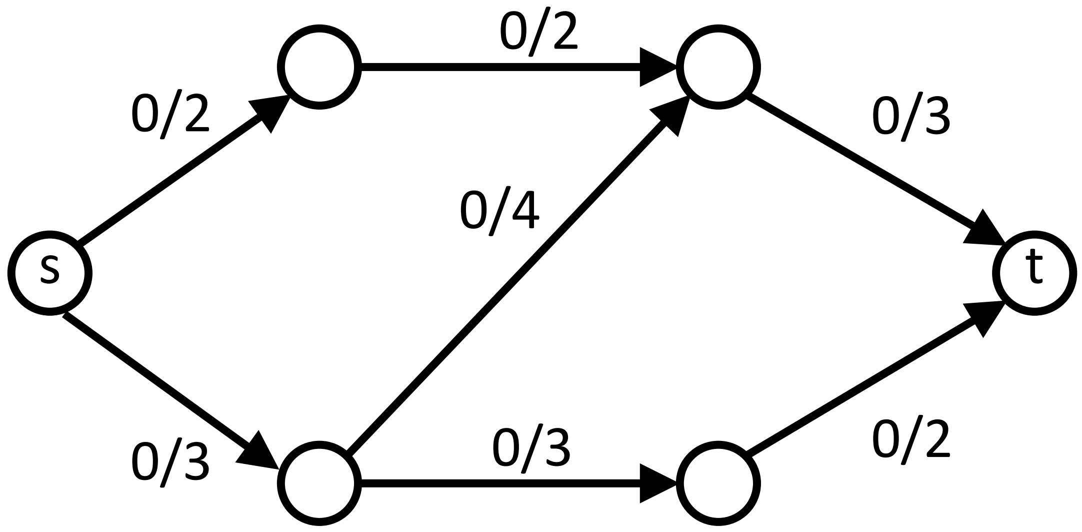
Residual graph

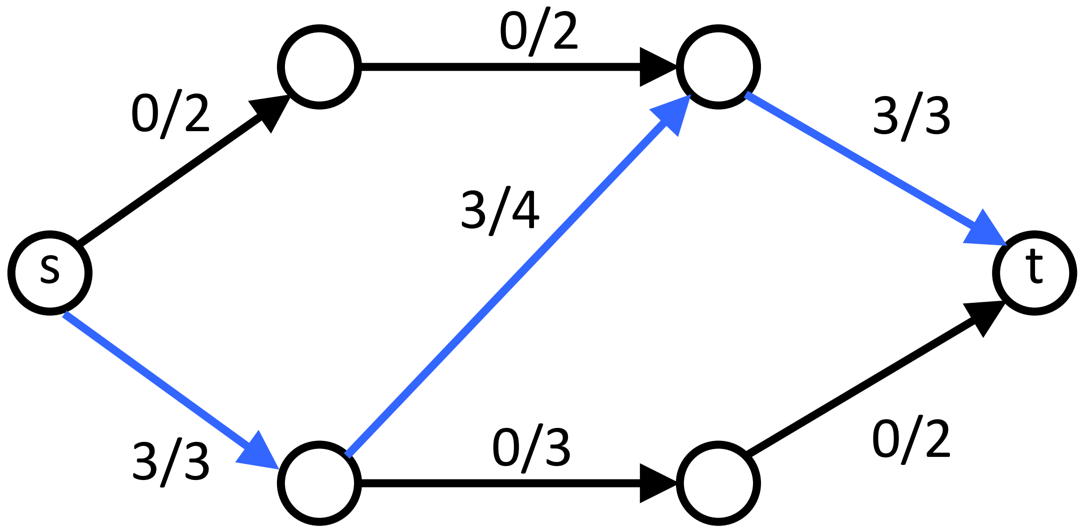


Example 2/3


Flow network
s
0/1
0/3
0/2
0/3
0/1
t
Flow
s
0/1
2/3
2/2
2/3
0/1
t
Residual graph
forward
s
backward
1
1 = 3-2
2
2
2
3-2=1
1/1
t

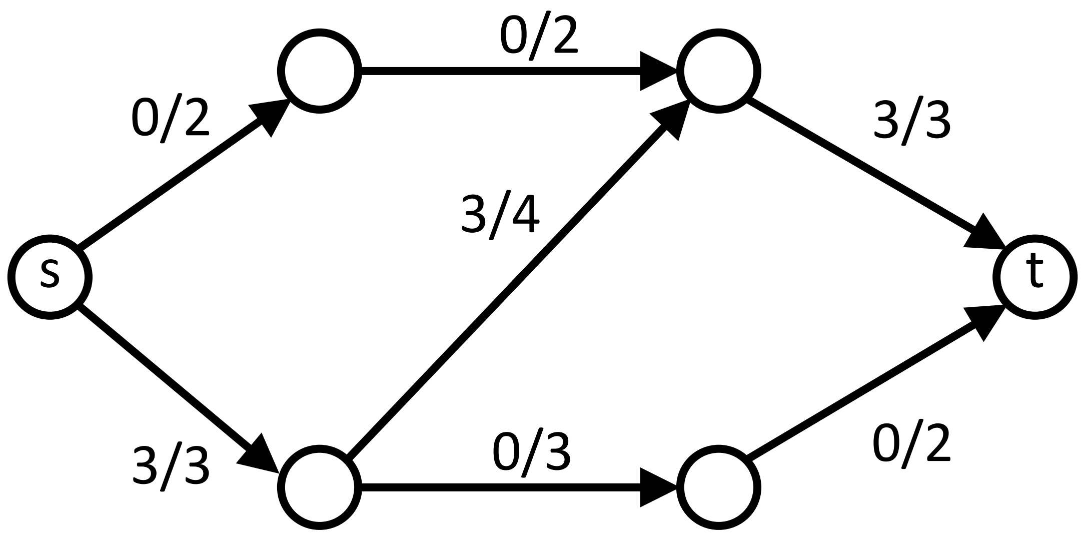
Example 3/3



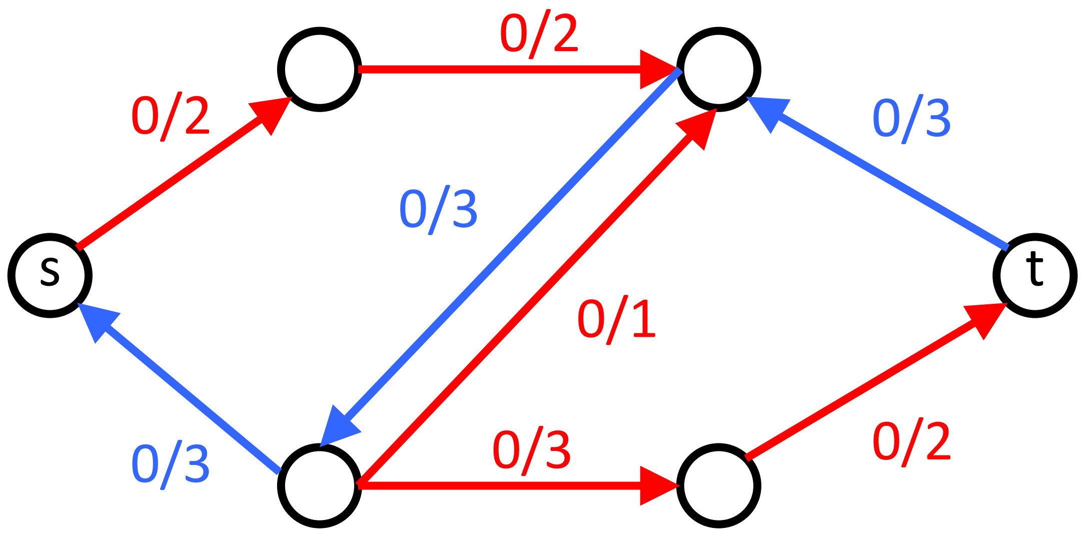


Example 3/3

Flow

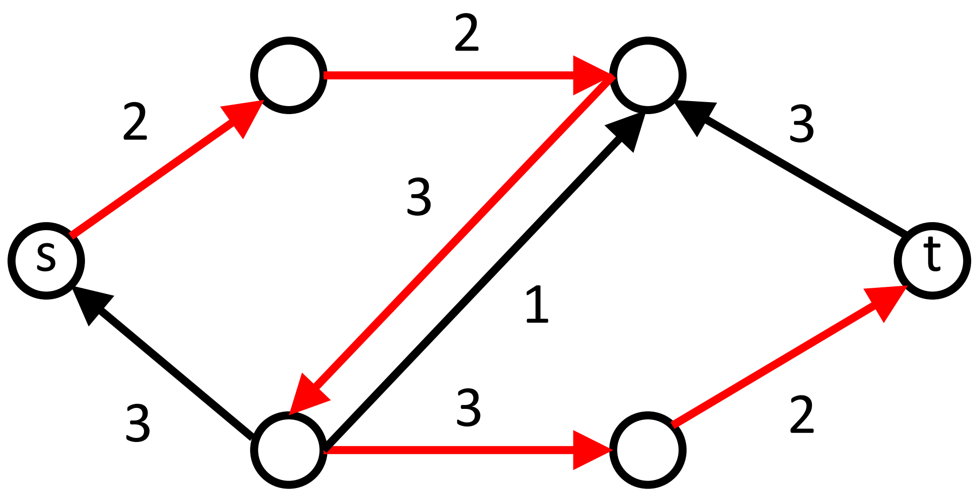


Residual graph



Augmenting path

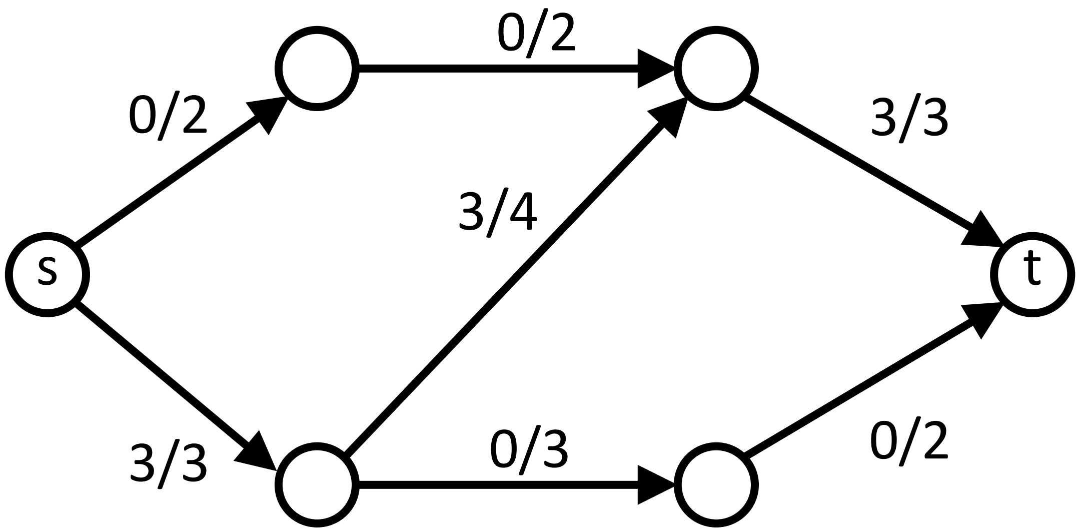
An augmenting path is a path from the source *s* to the sink *t* in the residual graph G_f that allows us to increase the flow.



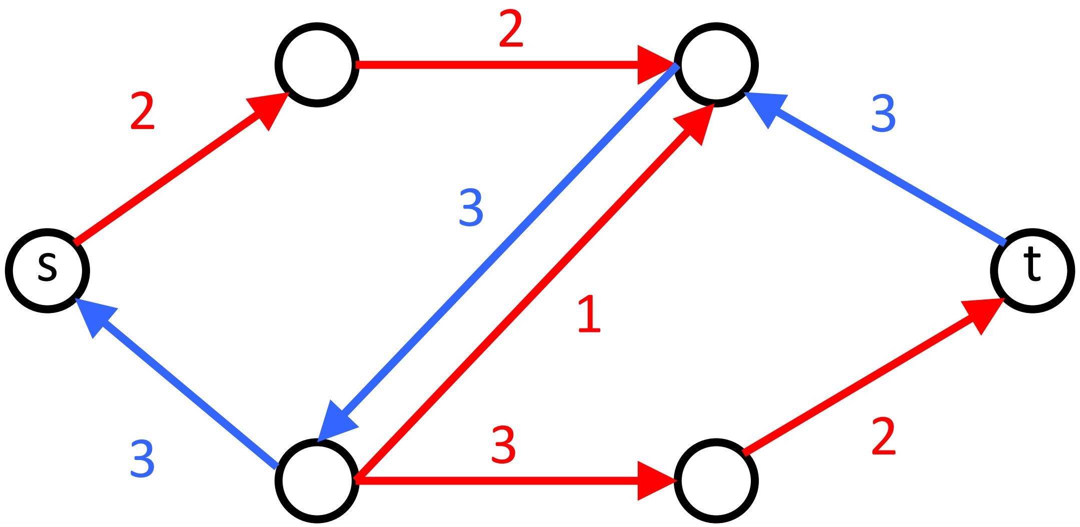
Q: By how much can we increase the flow using this path?

Example

Flow in G

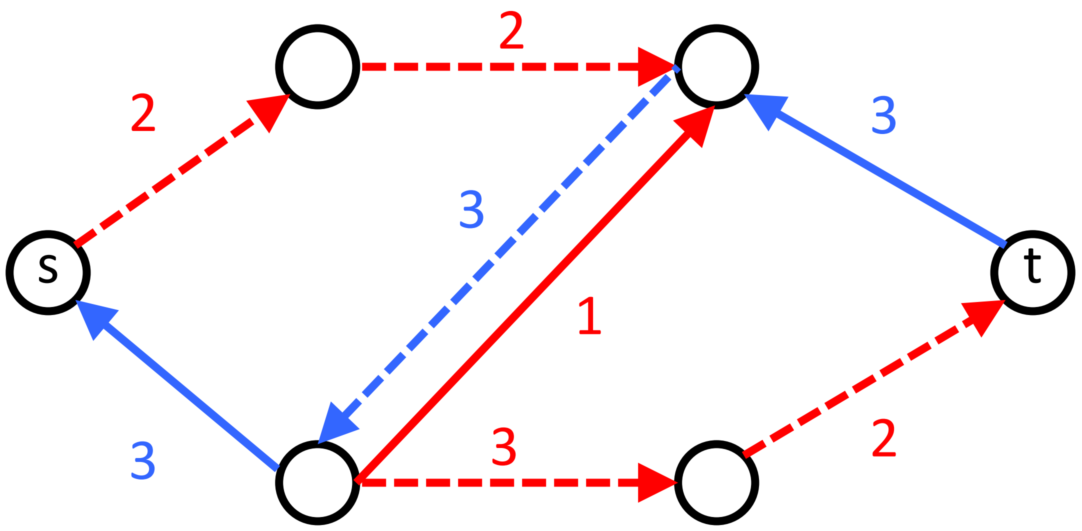


Residual graph G_f

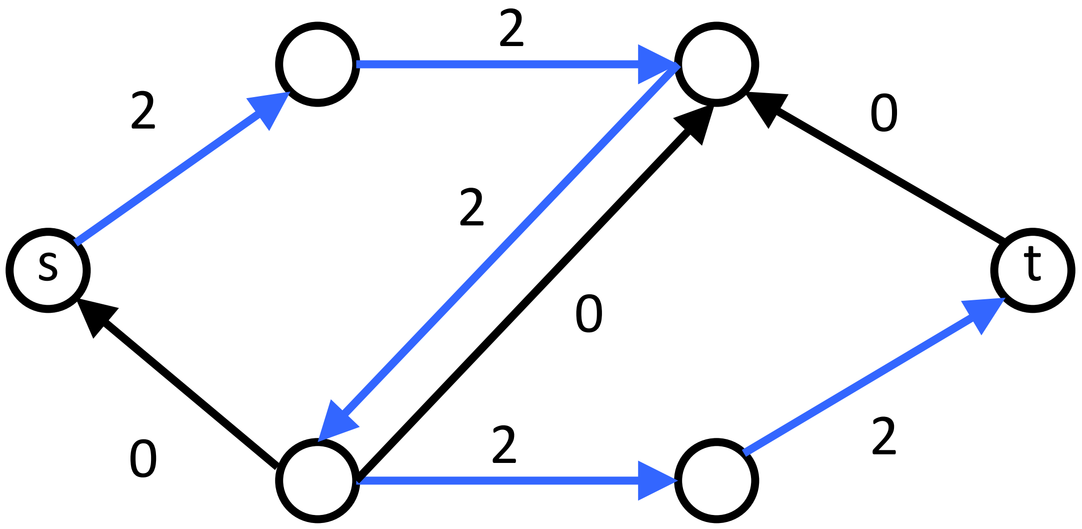


Example

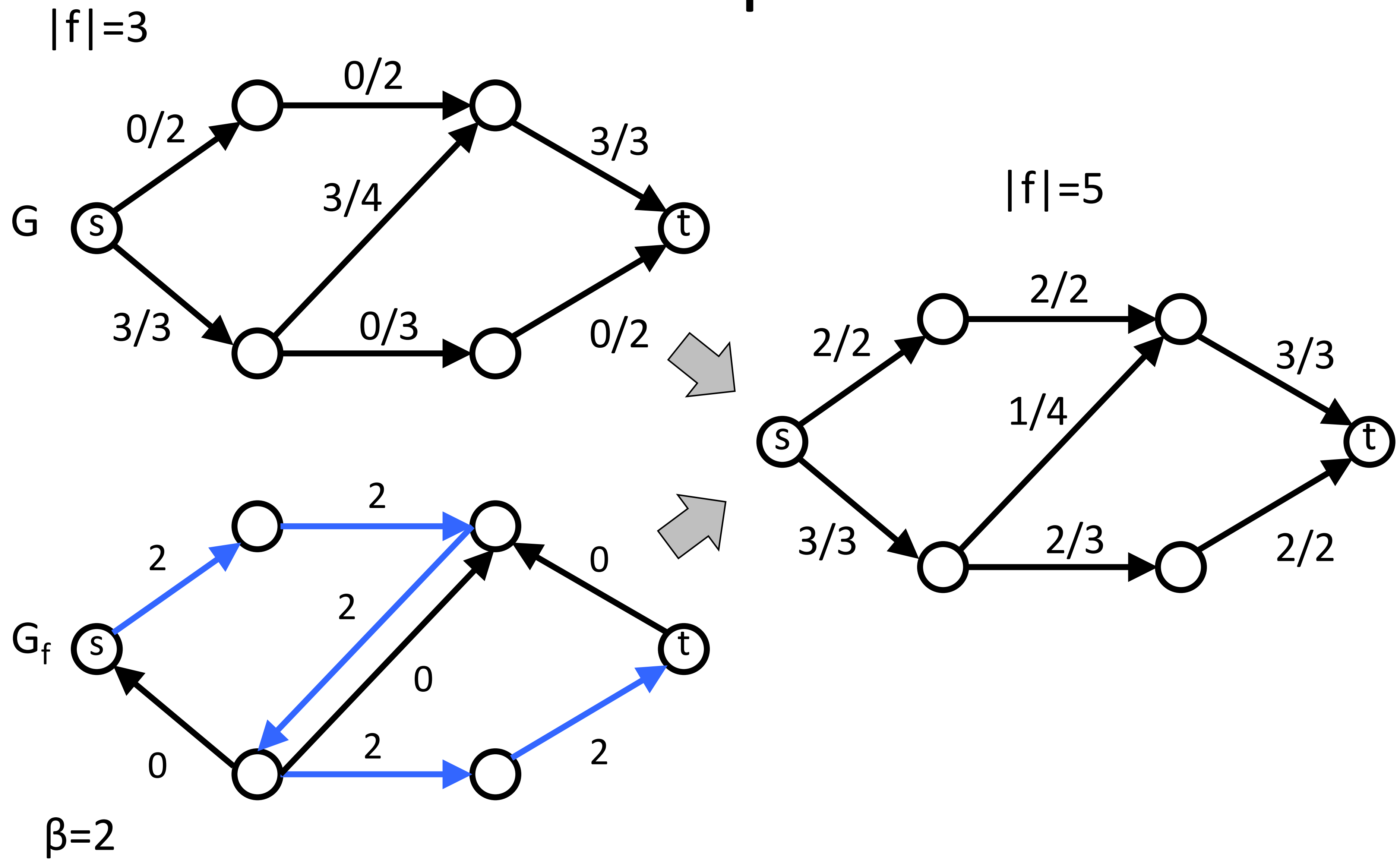
Residual graph G_f



Flow in G_f



Example



|f|=3
G
s
t
0/2
0/2
3/3
3/4
3/3
0/3
0/2
Gf
s
t
2
2
2
0
0
0
2
2
β=2
|f|=5
s
t
2/2
2/2
3/3
1/4
3/3
2/3
2/2

Methodology

- Compute the residual graph G_f
- Find a path P
- Augment the flow f along the path P
    1. Let β be the bottleneck (smallest residual capacity $c_f(e)$ of edges on P)
    2. Add β to the flow f(e) on each edge of P.

Q: How do we add β into G?

Augmenting a path

```
f.augment(P) {
    $\beta$ = min { c(e)-f(e) | e $\in$ P }
    for each edge e = (u,v) $\in$ P {
        if e is a forward edge {
            f(e) += $\beta$
        } else { // e is a backward edge
            f(e) -= $\beta$
        }
    }
}
```

Ford-Fulkerson algorithm

```
$f \leftarrow 0$
$G_f \leftarrow G$
while (there is a s-t path in $G_f$) {
    f.augment(P)
    update $G_f$ based on new f
}
```

Correctness (termination)

Claim: The Ford-Fulkerson algorithm terminates.

Proof:

- The capacities and flows are strictly positive integers.
- The sum of capacities leaving s is finite.
- Bottleneck values β are strictly positive integers.
- The flow increase by β after each iteration of the loop.
- The flow is an increasing sequence of integers that is bounded.

Complexity (Running time)

- Let $C = \sum_{\substack{e \in E \\ outgoing \\ from\ s}} c(e)$
- Finding an augmenting path from *s* to *t* takes O(|E|) (e.g. BFS or DFS).
- The flow increases by at least 1 at each iteration of the main while loop.
- The algorithm runs in O(*C* . |E|)