COMP251: Single source shortest paths

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Based on (Cormen *et al.*, 2002)

Which assertions are true?

- A light edge crosses the cut. ✓
- A light edge is unique. ✗
- A MST is unique. ✗
- A graph A that respects the cut has no light edge. ✓

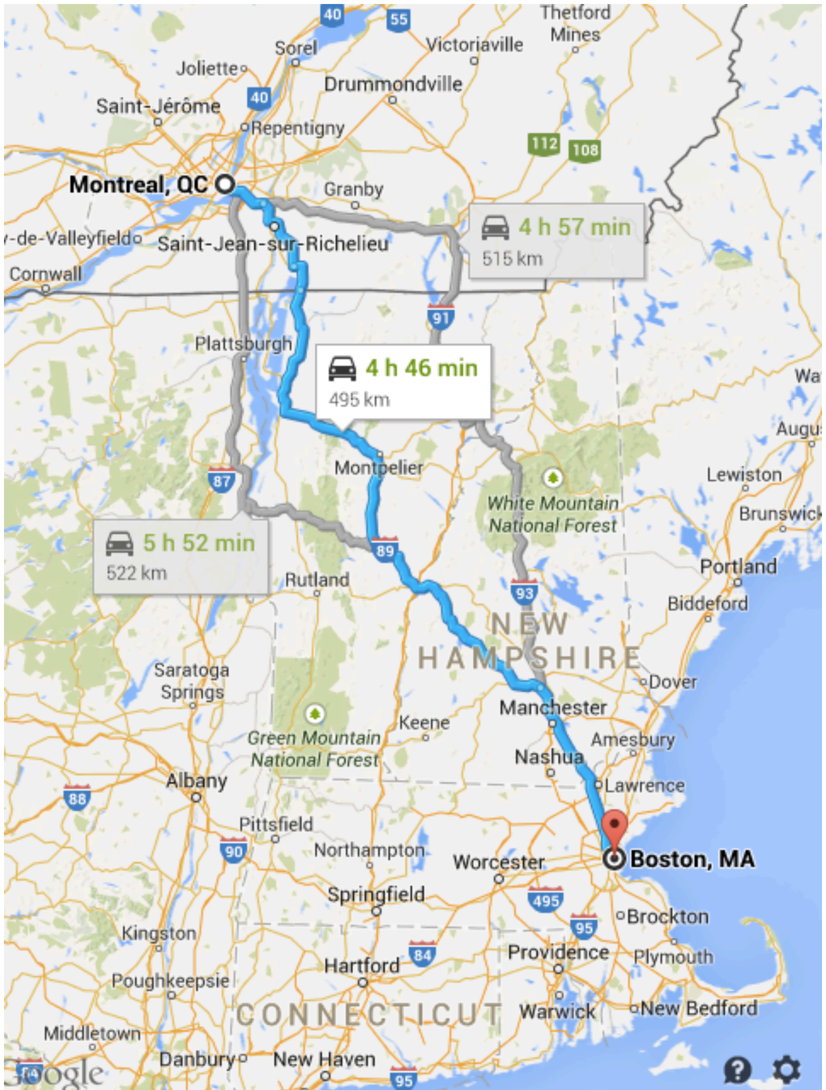
How do we decide if an edge (i,j) belongs to a MST during the execution of the Kruskal's algorithm?

- When this edge connects two sets of vertices that are not connected. ✓
- When the weight of (i,j) is the lowest among all candidate edges. ✗
- When the vertices i and j have not been used in the solution under construction. ✗

Let G=(V,E) be a connected undirected weighted graph on which we run the Prim's algorithm to compute a MST. How many iterations the main loop of the algorithm will do?

- |E| ✗
- |V| - 1 ✓
- |E| + |V| ✗
- |V| ^ 2 ✗

Problem



What is the shortest road to go from one city to another?

Example: Which road should I take to go from Montréal to Boston (MA)?

Variants:

- What is the fastest road?
- What is the cheapest road?

Modeling as graphs

Input:

- Directed graph G = (V,E)
- Weight function w: E$\rightarrow\mathbb{R}$

Weight of path *p* = $\langle$ v_0, v_1, … , v_k $\rangle$

= $$\sum_{k=1}^{n} w(v_{k-1}, v_k)$$

= sum of edges weights on path *p*

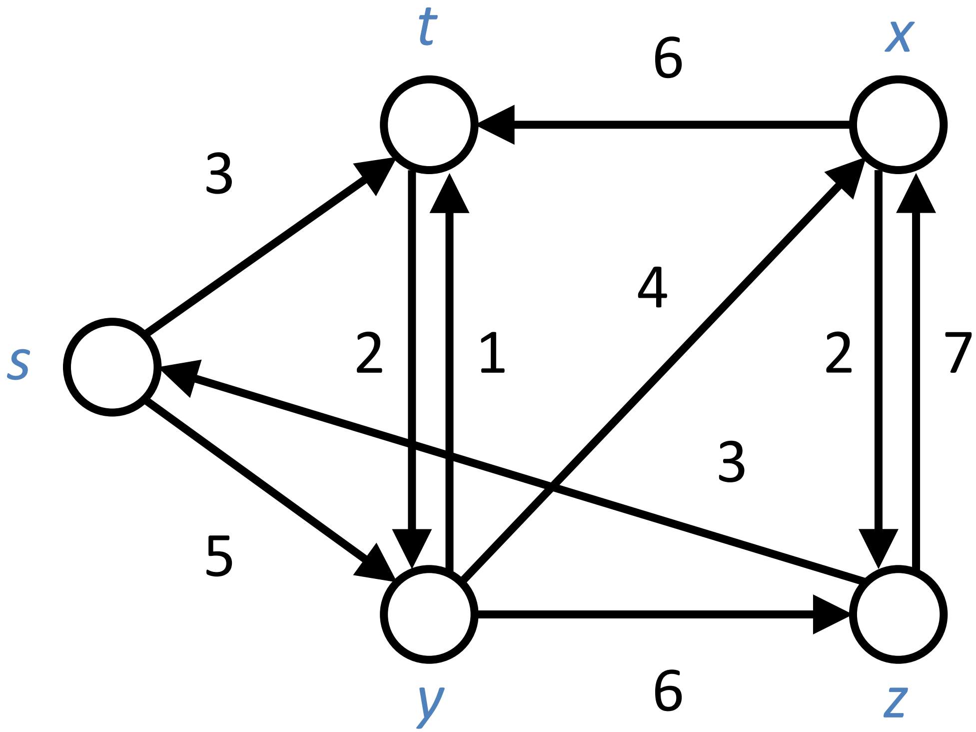
Shortest-path weight *u* to *v*:

$$\delta(u,v) = \begin{cases} \min\left\{ w(p) : u \overset{p}{\mapsto} v \right\} & \text{If there exists a path } u \rightsquigarrow v. \\ \infty & \text{Otherwise.} \end{cases}$$

Shortest path *u* to *v* is any path *p* such that w(*p*) = δ(*u,v*)

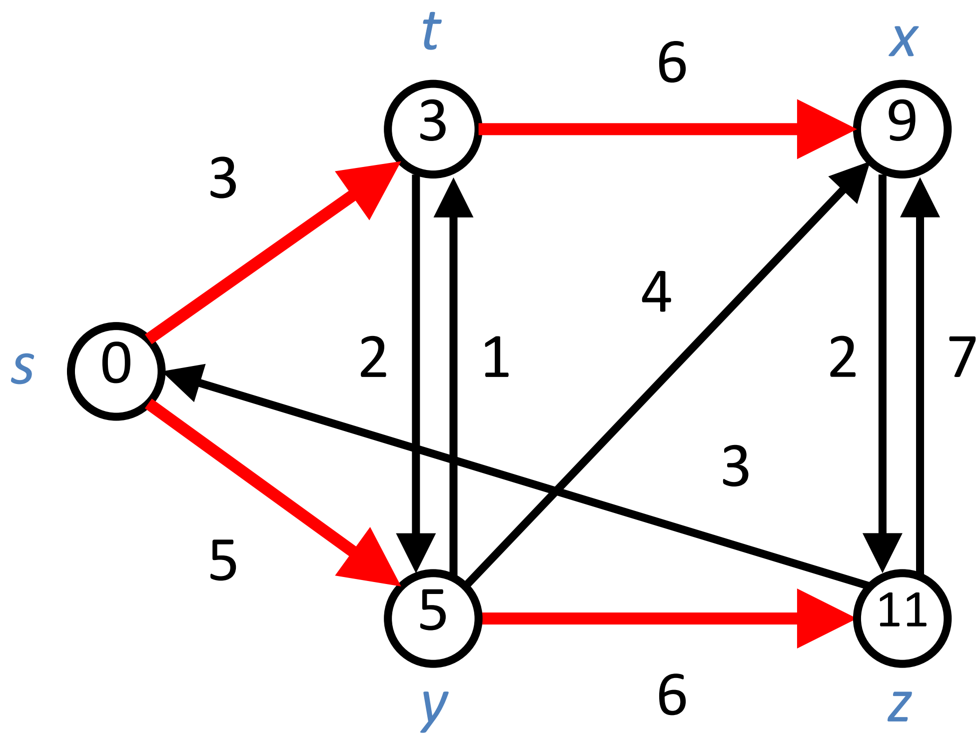
Generalization of breadth-first search to weighted graphs

Example



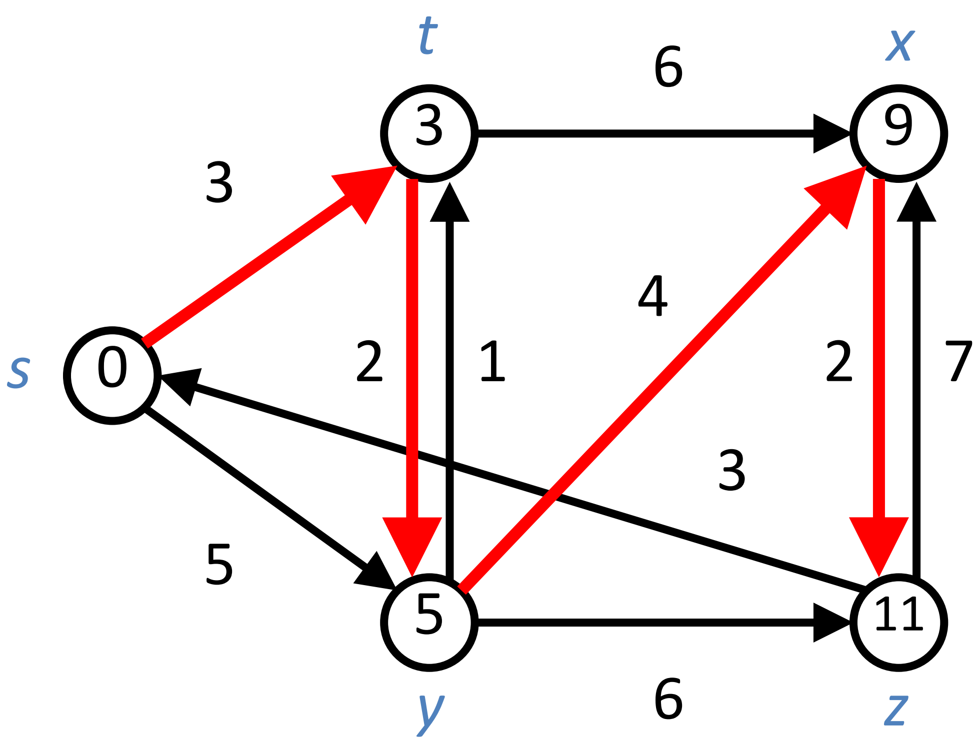
Shortest path from s?

Example



Shortest paths are organized as a tree.
Vertices store the length of the shortest path from s.

Example



Shortest paths are not necessarily unique!

Variants

- **Single-source:** Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.
- **Single-destination:** Find shortest paths to a given destination vertex.
- **Single-pair:** Find shortest path from *u* to *v*.

  Note: No way to known that is better in worst case than solving the single-source problem!

- **All-pairs:** Find shortest path from *u* to *v* for all $u, v \in V$.

Negative weight edges

Negative weight edges can create issues.

Why? If we have a negative-weight cycle, we can just keep going around it, and get $w(s, v) = -\infty$ for all v on the cycle.

When? If they are reachable from the source. Corollary: OK, if the negative-weight cycles is not reachable from the source.

Who? Some algorithms work only if there are no negative-weight edges in the graph. We must specify when they are allowed and not.

Cycles

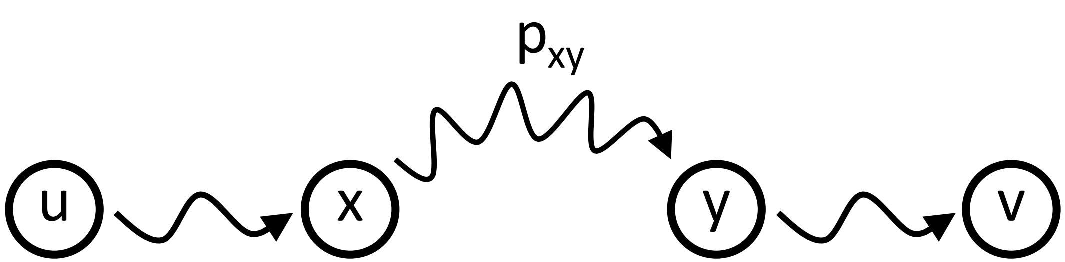
Shortest paths cannot contain cycles:

- Negative-weight: Already ruled out.
- Positive-weight: we can get a shorter path by omitting the cycle.
- Zero-weight: no reason to use them $\Rightarrow$ assume that our solutions will not use them.

Optimal substructure

> ***Lemma***
> Any subpath of a shortest path is a shortest path.

Proof: (cut and paste)



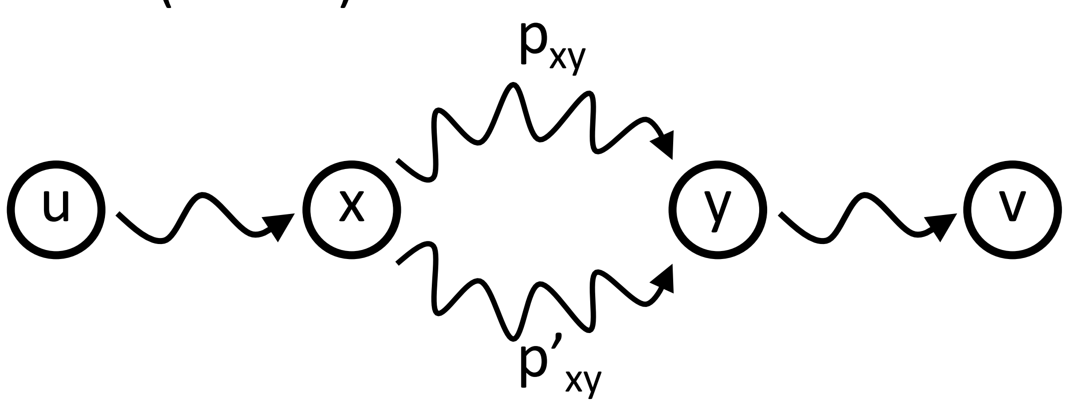
Suppose this path *p* is a shortest path from *u* to *v*.

Then $\delta(u,v) = w(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv})$.

Optimal substructure

> ***Lemma***
> Any subpath of a shortest path is a shortest path.

Proof: (cont'd)



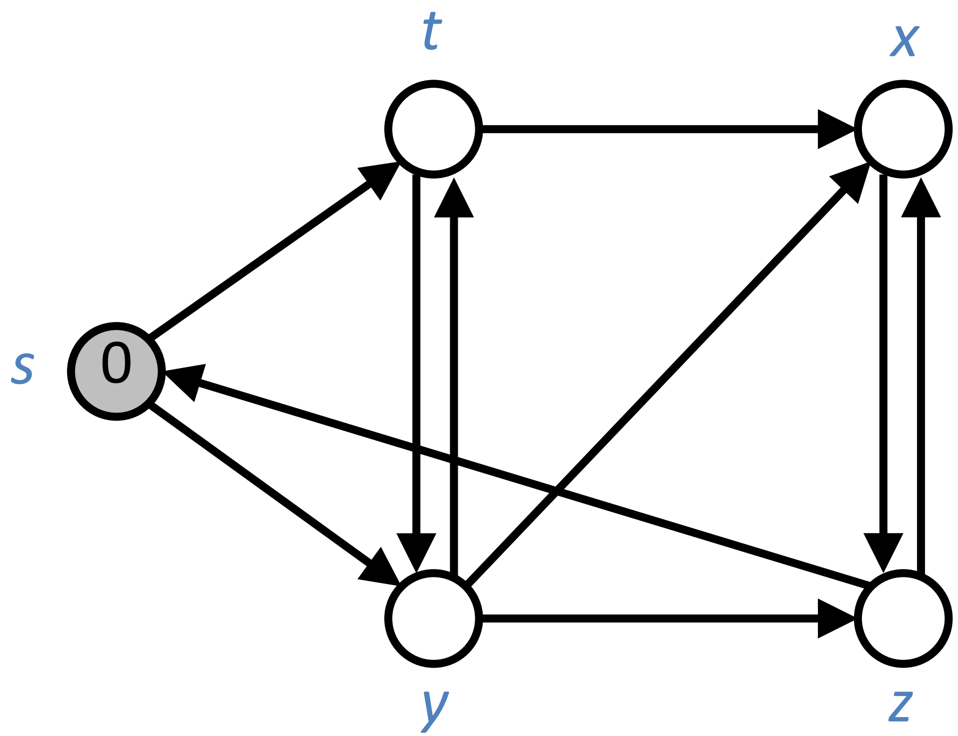
Now suppose there exists a shorter path $x \overset{p'_{xy}}{\rightsquigarrow} y$.

Then $w(p'_{xy}) < w(p_{xy})$.

$w(p') = w(p_{ux}) + w(p'_{xy}) + w(p_{yv}) < w(p_{ux}) + w(p_{xy}) + w(p_{yv}) = w(p)$.

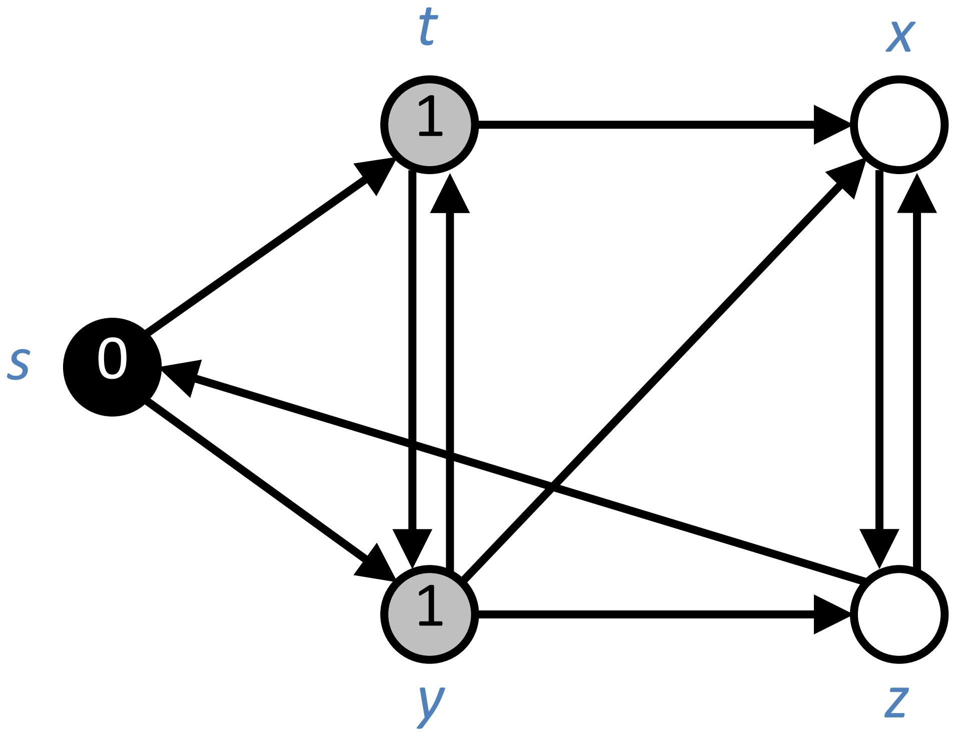
Contradiction of the hypothesis that p is the shortest path!

Customized breadth-first search

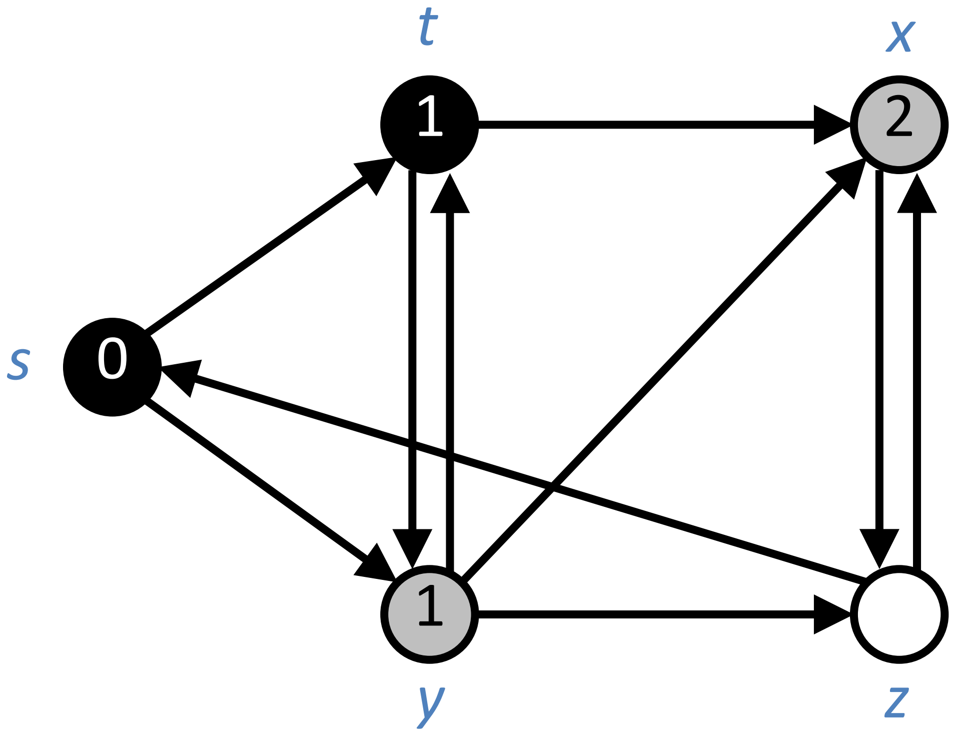


Vertices count the number of edges used to reach them.

Customized breadth-first search



Customized breadth-first search



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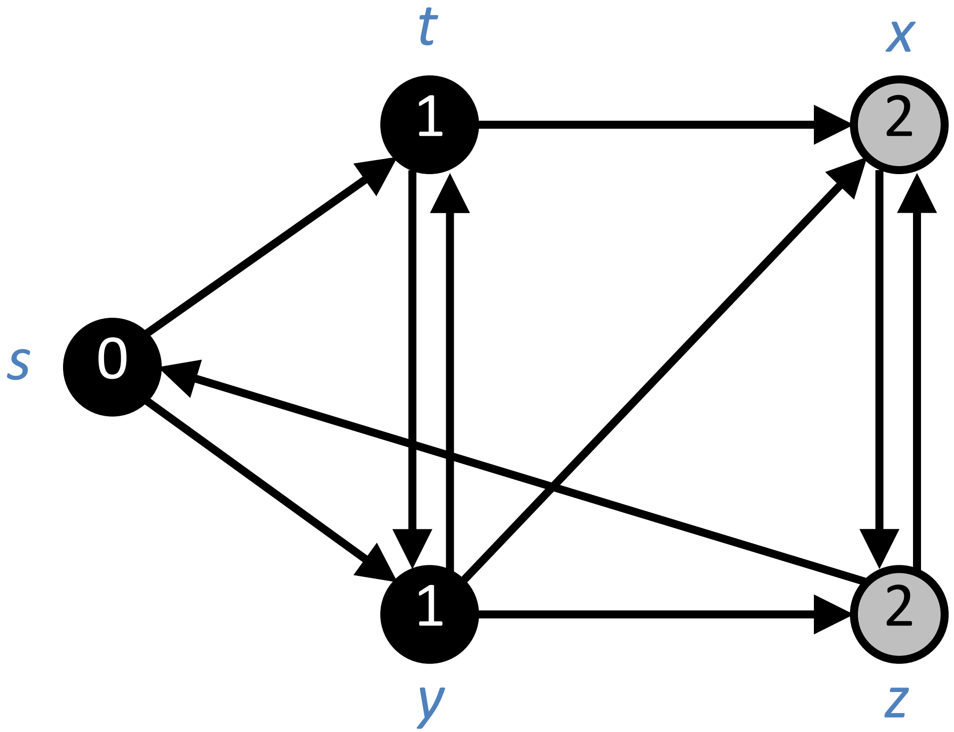
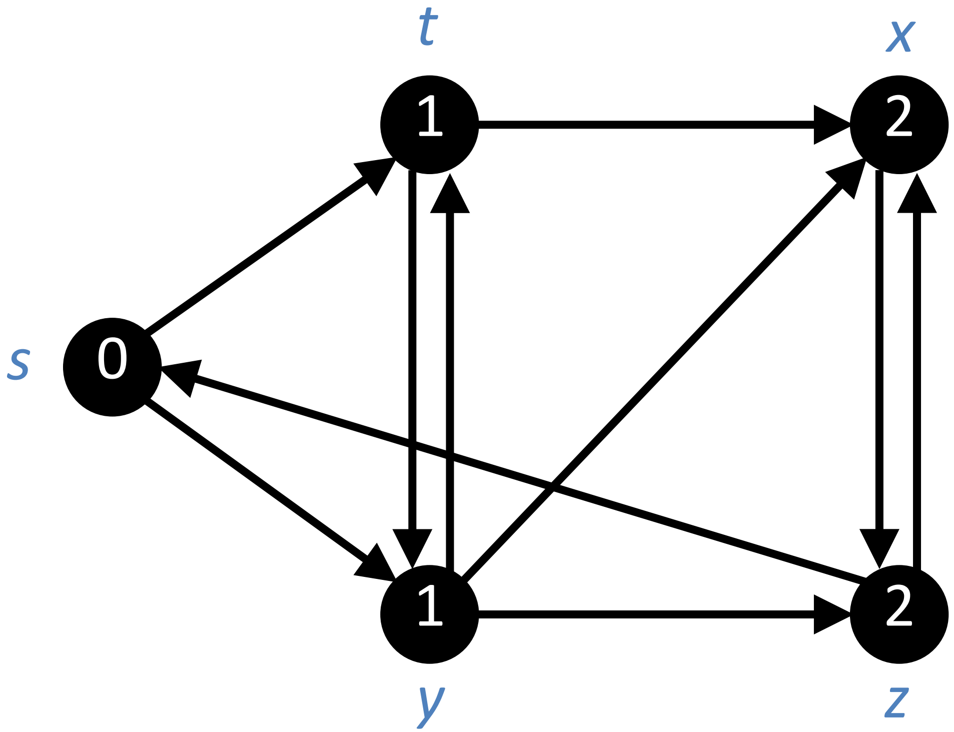


Customized breadth-first search



Can we generalize BFS to use edge weights?

Output of single-source shortest-path algorithm

For each vertex $v \in V$:

- $d[v] = \delta(s,v)$.
  - Initially, $d[v] = \infty$.
  - Reduces as algorithms progress, but always maintain $d[v] \geq \delta(s,v)$.
  - Call $d[v]$ a **shortest-path estimate**.
- $\pi[v]$ = predecessor of v on a shortest path from *s*.
  - If no predecessor, $\pi[v]$ = NIL.
  - π induces a tree - **shortest-path tree** (see proof in textbook).

Algorithm structure

1. Initialization
2. Scan vertices and relax edges

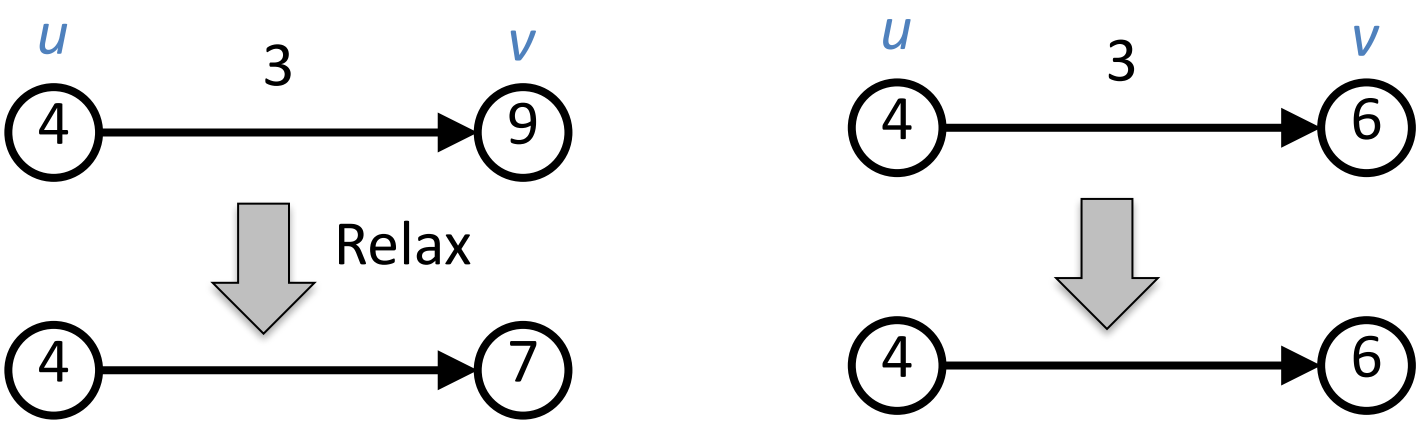
The algorithms differ in the order and how many times they relax each edge.

Initialization

```
INIT-SINGLE-SOURCE(V,s)
for each v ∈ V do
    d[v]← ∞
    π[v]← NIL
d[s]←0
```

Relaxing an edge

```
RELAX(u,v,w)
if d[v]>d[u]+w(u,v) then
   d[v] ← d[u]+w(u,v)
   π[v]←u
```



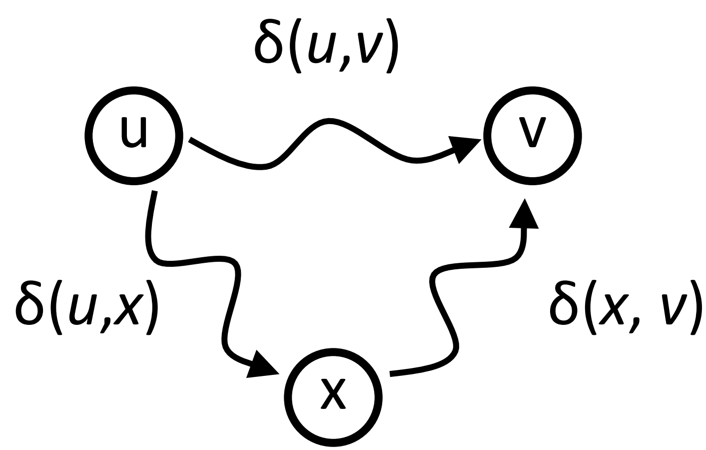
Triangle inequality

For all $(u,v) \in E$, we have $\delta(u,v) \leq \delta(u,x) + \delta(x,v)$.

Proof:

Weight of shortest path $u \rightsquigarrow v$ is ≤ weight of any path $u \rightsquigarrow v$.

Path $u \rightsquigarrow x \rightsquigarrow v$ is a path $u \rightsquigarrow v$, and if we use a shortest path $u \rightsquigarrow x$ *and* $x \rightsquigarrow v$, its weight is $\delta(u, x) + \delta(x, v)$.



Upper bound property

$\boxed{\begin{array}{l}\text{Always have } \delta(s, \text{v}) \le d[\text{v}] \text{ for all v.}\\ \text{Once } d[\text{v}] = \delta(s, \text{v}) \text{, it never changes.}\end{array}}$

Proof:

Initially true.

Suppose there exists a vertex such that $d[\text{v}] < \delta(s, \text{v})$.

Assume v is first vertex for which this happens.

Let u be the vertex that causes $d[\text{v}]$ to change.
Then $d[\text{v}] = d[u] + \delta(u, \text{v})$.

$$d[\text{v}] < \delta(s,\text{v}) \le \underbrace{\delta(s, u) + \delta(u, \text{v})}_{\text{(triangle inequality)}} \le \underbrace{d[u] + \delta(u, \text{v})}_{\text{(v is first violation)}} \Rightarrow d[\text{v}] < d[u] + \delta(u,\text{v}).$$

Contradicts $d[\text{v}] = d[u] + \delta(u, \text{v})$.

No-path property

$\boxed{\text{If } \delta(s, \text{v}) = \infty\text{, then } d[\text{v}] = \infty \text{ always.}}$

Proof: $d[\text{v}] \geq \delta(s,\text{v}) = \infty \Rightarrow d[\text{v}] = \infty$.

Convergence property

If:

1. $s \rightsquigarrow u \to v$ is a shortest path,
2. $d[u] = \delta(s,u)$,
3. we call RELAX(u,v,w),

then $d[v] = \delta(s,v)$ afterward.

Proof:

After relaxation:

$d[v] \le d[u] + w(u,v)$ (RELAX code)

$= \delta(s, u) + w(u, v)$

$= \delta(s, v)$ (lemma-optimal substructure)

Since $d[v] \ge \delta(s, v)$, must have $d[v] = \delta(s, v)$.

Path-relaxation property

Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from $s = v_0$ to v_k. If we relax, *in order*, (v_0,v_1), (v_1,v_2),..., (v_{k-1},v_k), even intermixed with other relaxations, then $d[v_k] = \delta(s,v_k)$.

Proof:

Induction to show that $d[v_i]=\delta(s,v_i)$ after (v_{i-1},v_i) is relaxed.

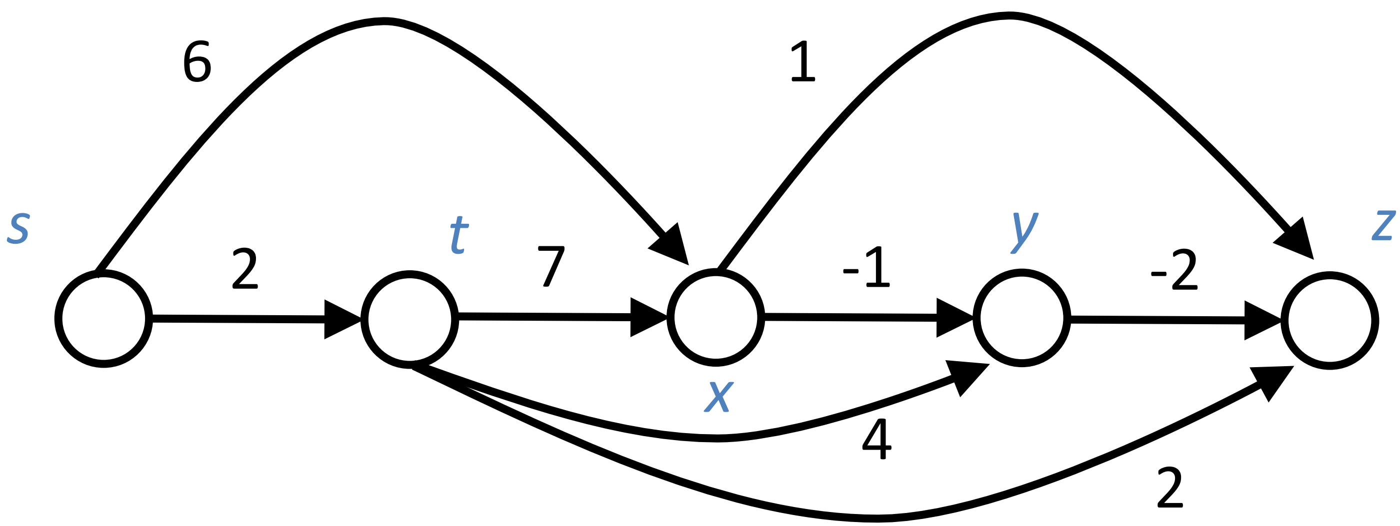
Basis: $i = 0$. Initially, $d[v_0] = 0 = \delta(s, v_0) = \delta(s,s)$.

Inductive step: Assume $d[v_{i-1}] = \delta(s,v_{i-1})$. Relax (v_{i-1},v_i). By convergence property, $d[v_i] = \delta(s, v_i)$ afterward and $d[v_i]$ never changes.

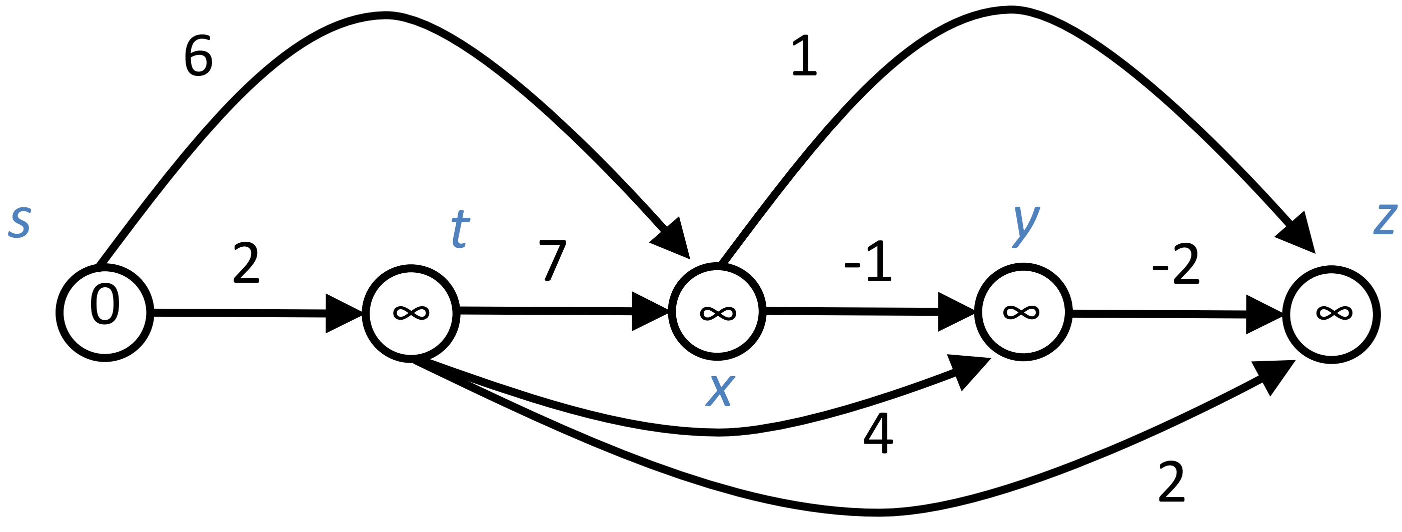
Single-source shortest paths in a DAG

Since a DAG, we are guaranteed no negative-weight cycles.

```
DAG-SHORTEST-PATHS(V,E,w,s)
topologically sort the vertices
INIT-SINGLE-SOURCE(V,s)
for each vertex u in topological order do
   for each vertex v∈Adj[u] do
      RELAX(u,v,w)
```



Example



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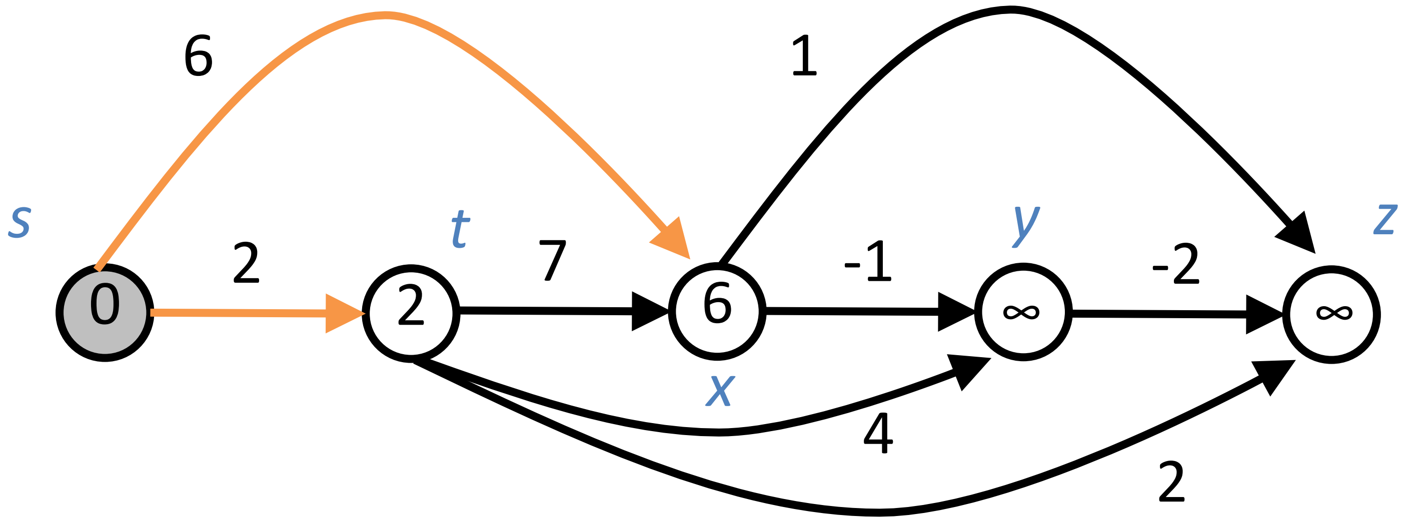

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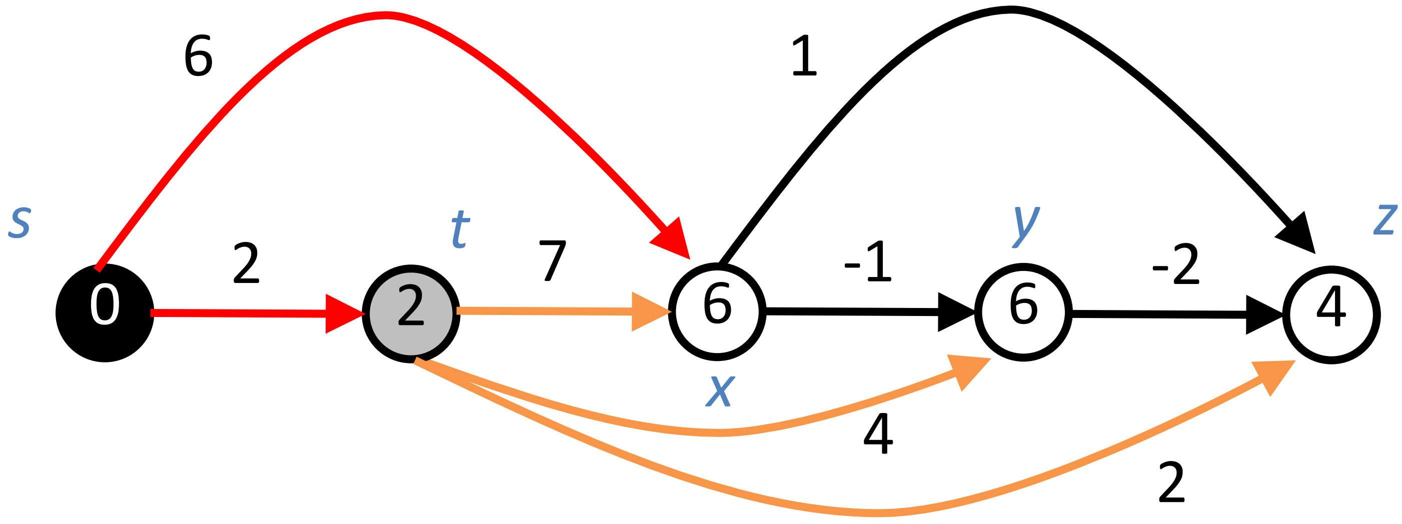

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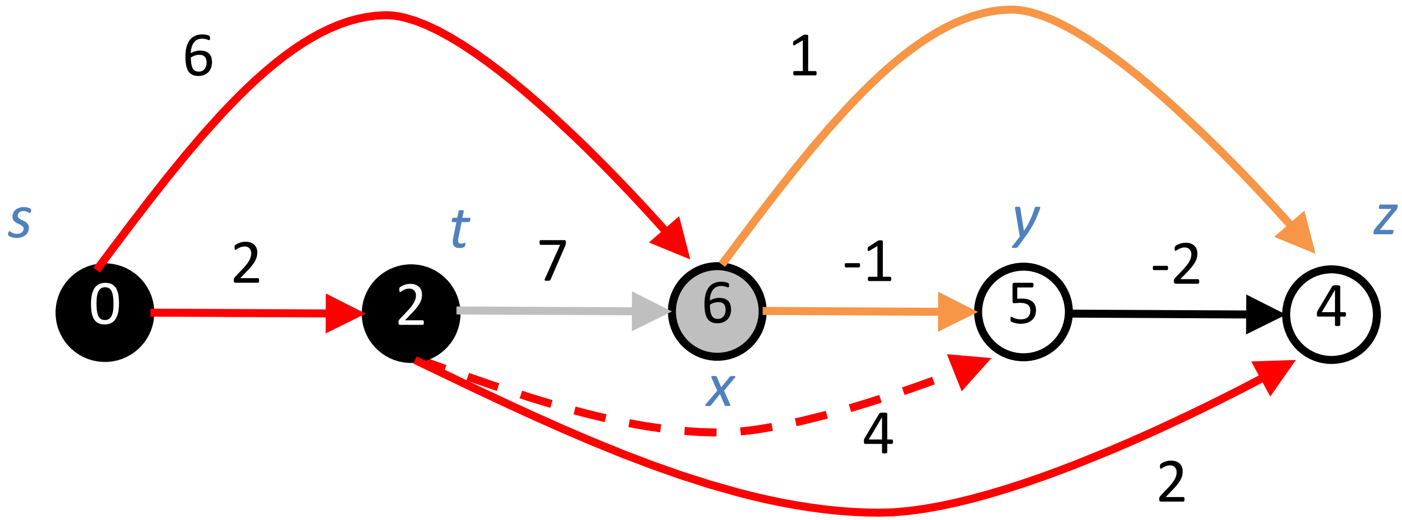

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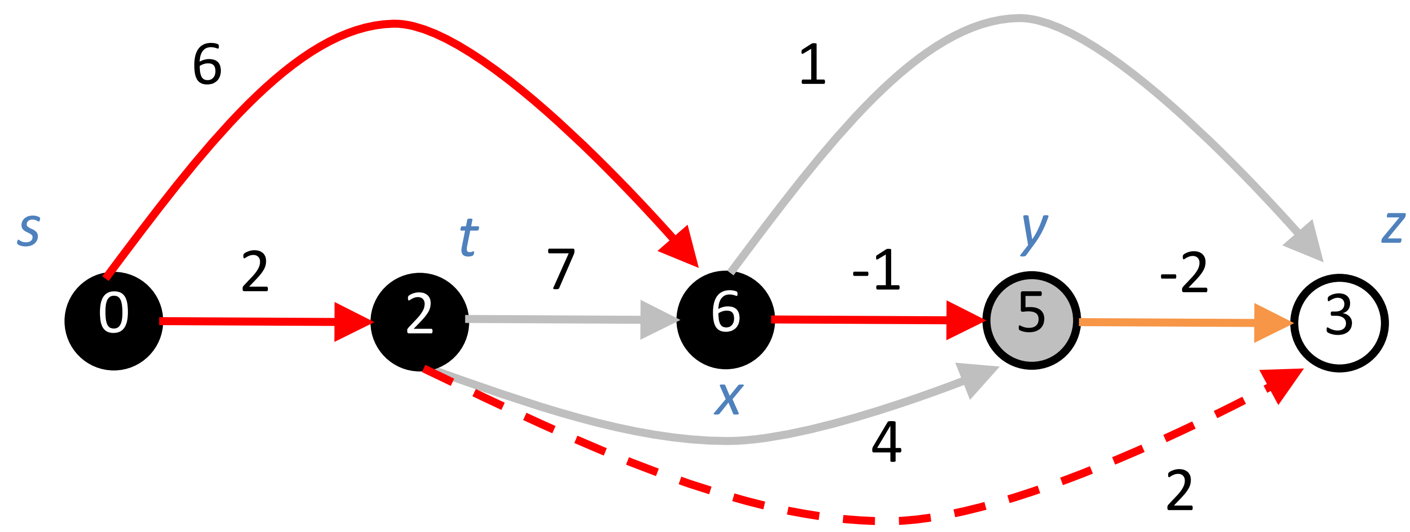

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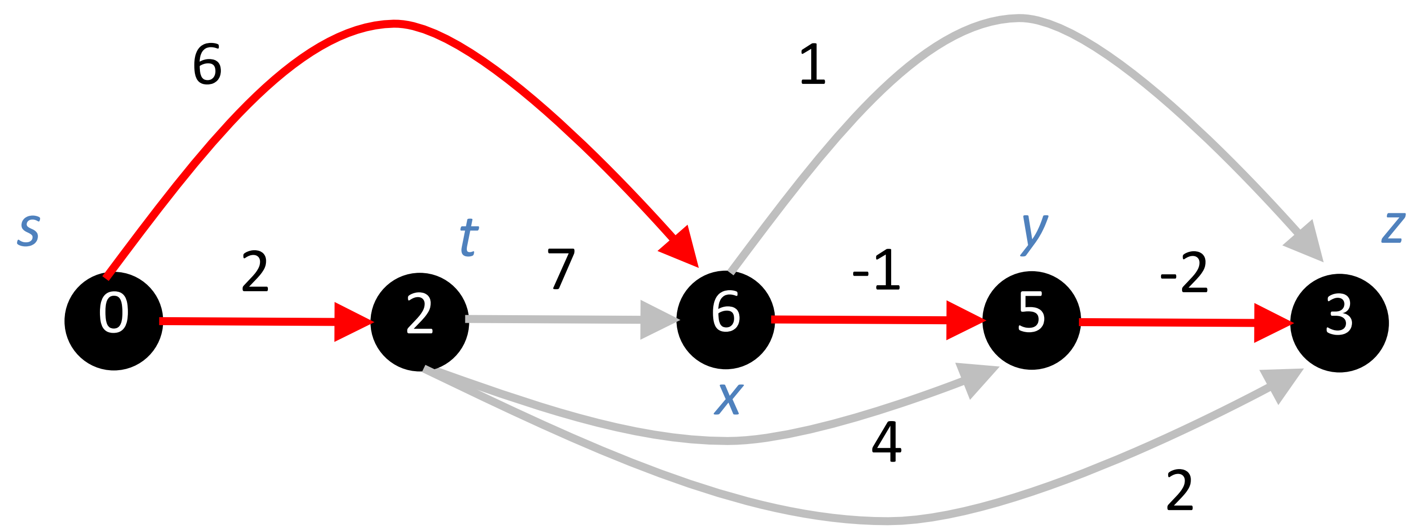


Single-source shortest paths in a DAG

```
DAG-SHORTEST-PATHS(V, E,w,s)
topologically sort the vertices
INIT-SINGLE-SOURCE(V,s)
for each vertex u in topological order do
   for each vertex v ∈ Adj[u] do
      RELAX(u,v,w)
```

Time: (V + E).

Correctness:

Because we process vertices in topologically sorted order, edges of **any** path must be relaxed in order of appearance in the path.

$\Rightarrow$ Edges on any shortest path are relaxed in order.

$\Rightarrow$ By path-relaxation property, correct.

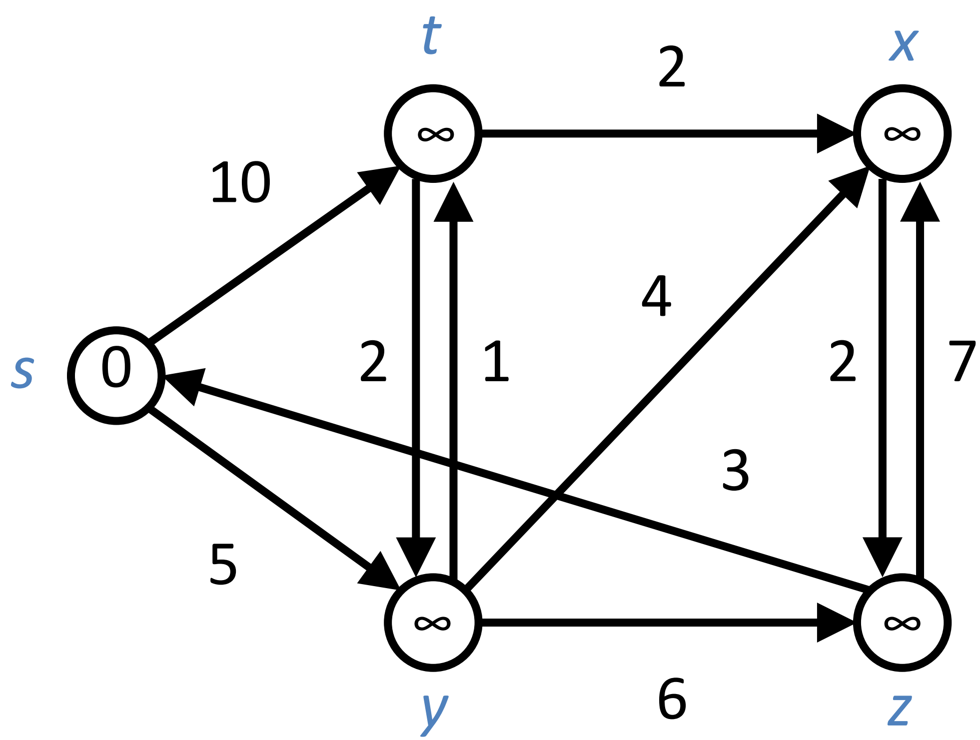
Dijkstra's algorithm

- No negative-weight edges.
- Weighted version of BFS:
  - Instead of a FIFO queue, uses a **priority queue**.
  - Keys are shortest-path weights (d[v]).
- Have two sets of vertices:
  - S = vertices whose final shortest-path weights are determined,
  - Q = priority queue = $V - S$.
- Similar Prim's algorithm, but computing d[v], and using shortest-path weights as keys.
- Greedy choice: At each step we choose the light edge.

Dijkstra's algorithm

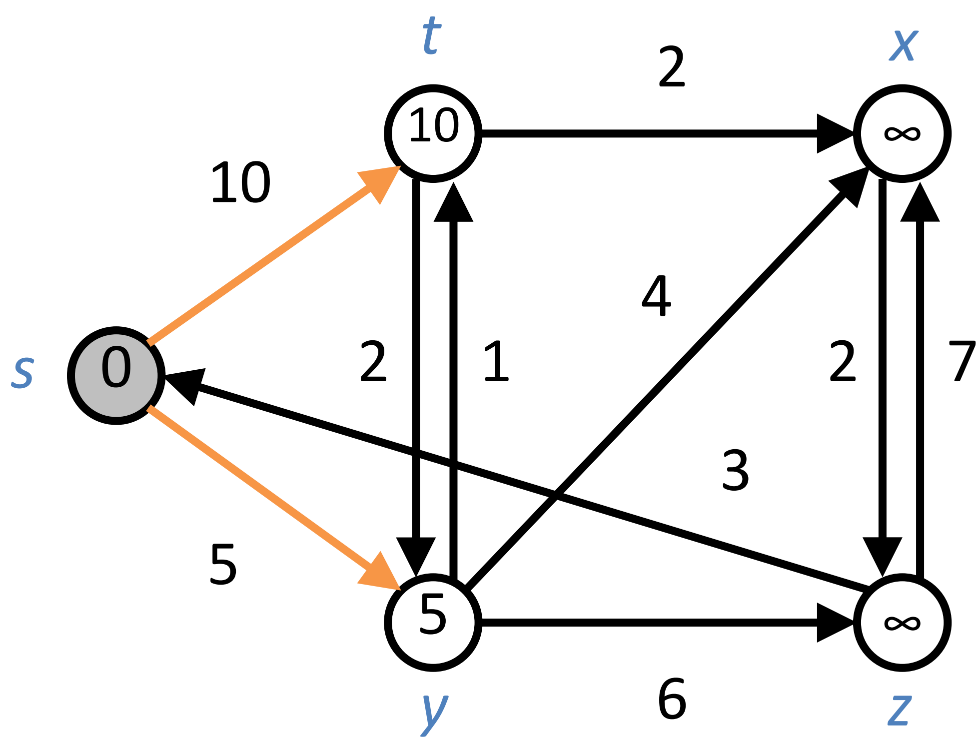
```
DIJKSTRA(V, E,w,s)
INIT-SINGLE-SOURCE(V,s)
S ← ∅
Q ← V
while Q ≠ ∅ do
   u ← EXTRACT-MIN(Q)
   S ← S ∪ {u}
   for each vertex v ∈ Adj[u] do
      RELAX(u,v,w)
```

Example



Q	s	t	y	x	z

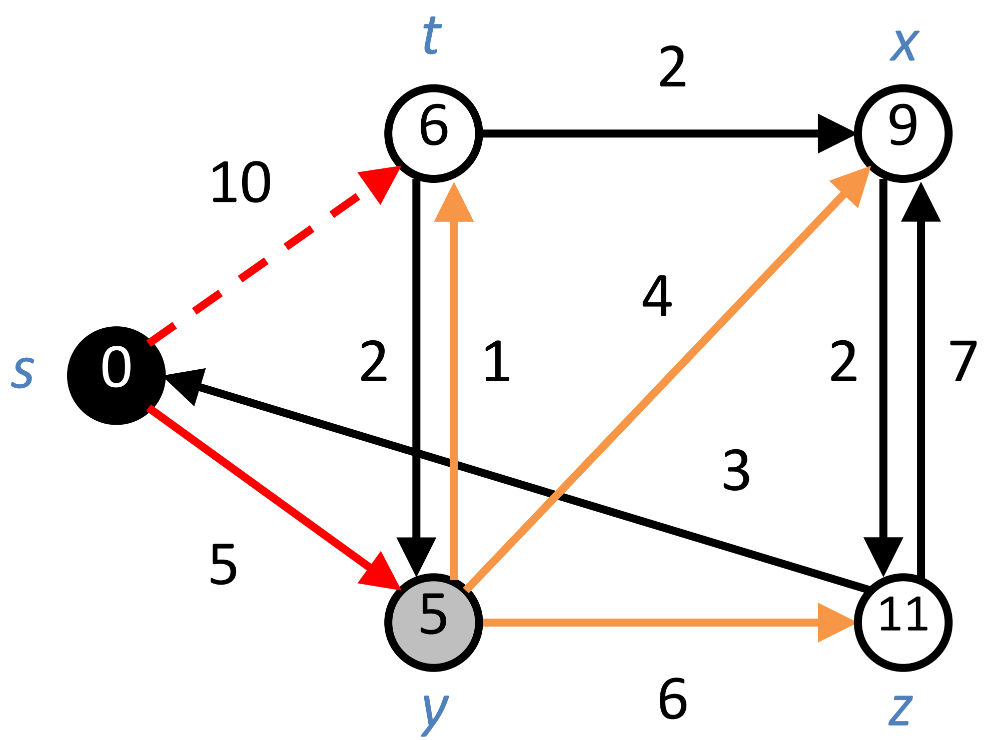
Example



Q

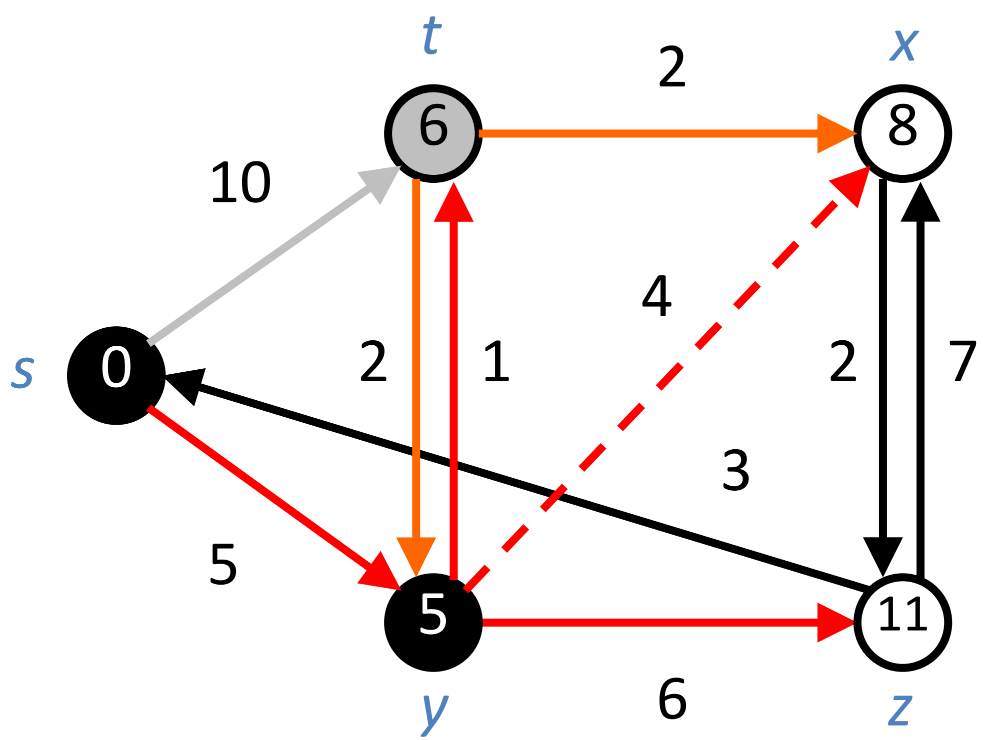
y	t	x	z	

Example



Q	t	x	z		

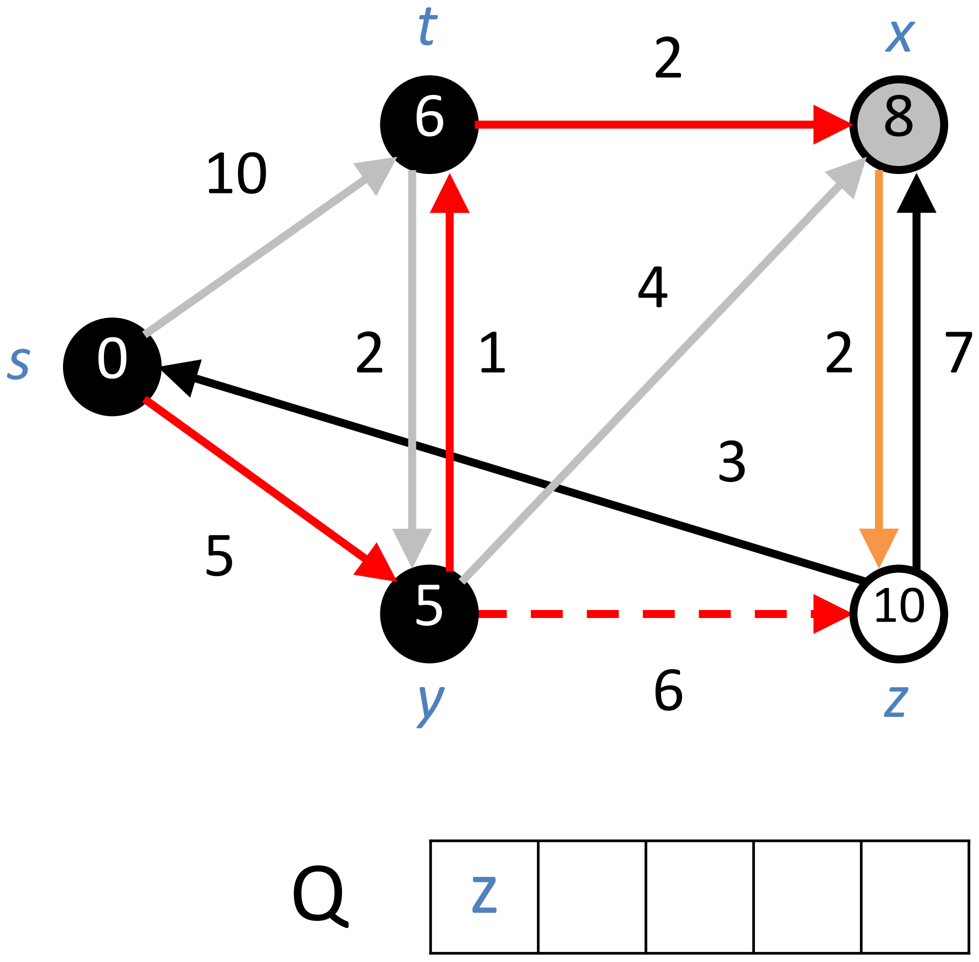
Example



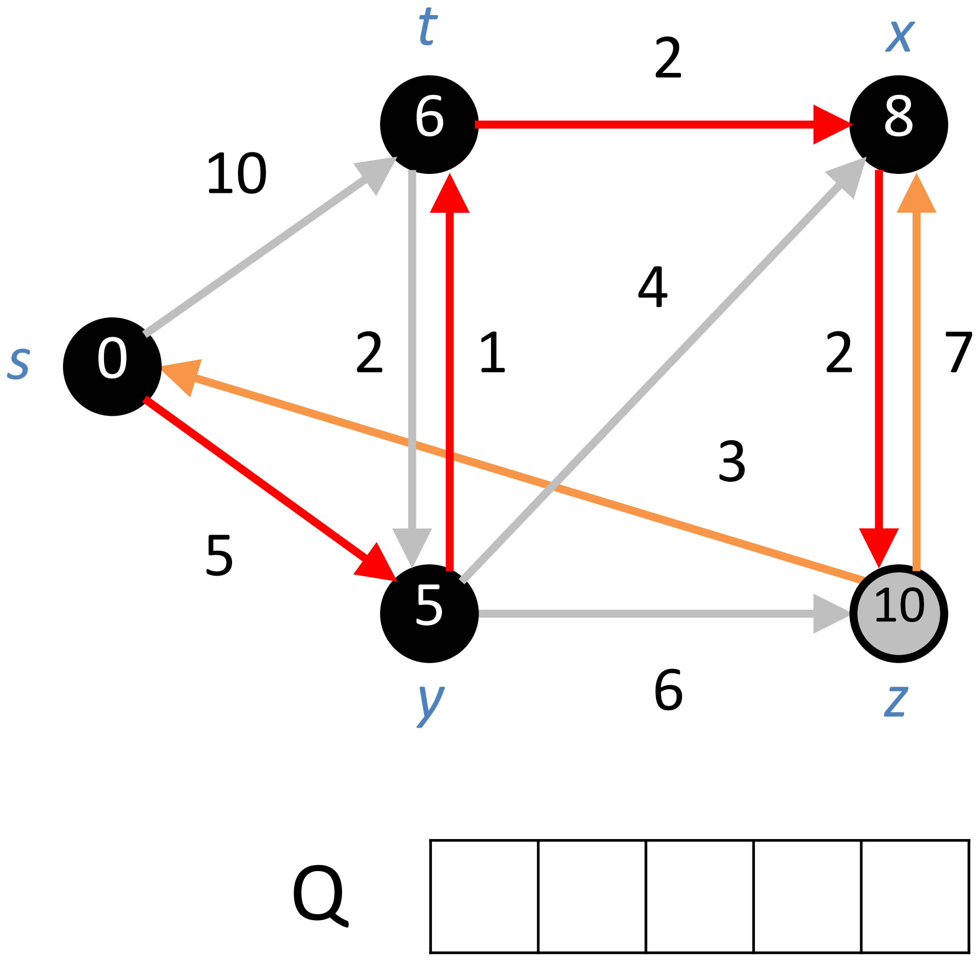
Q

x	z			

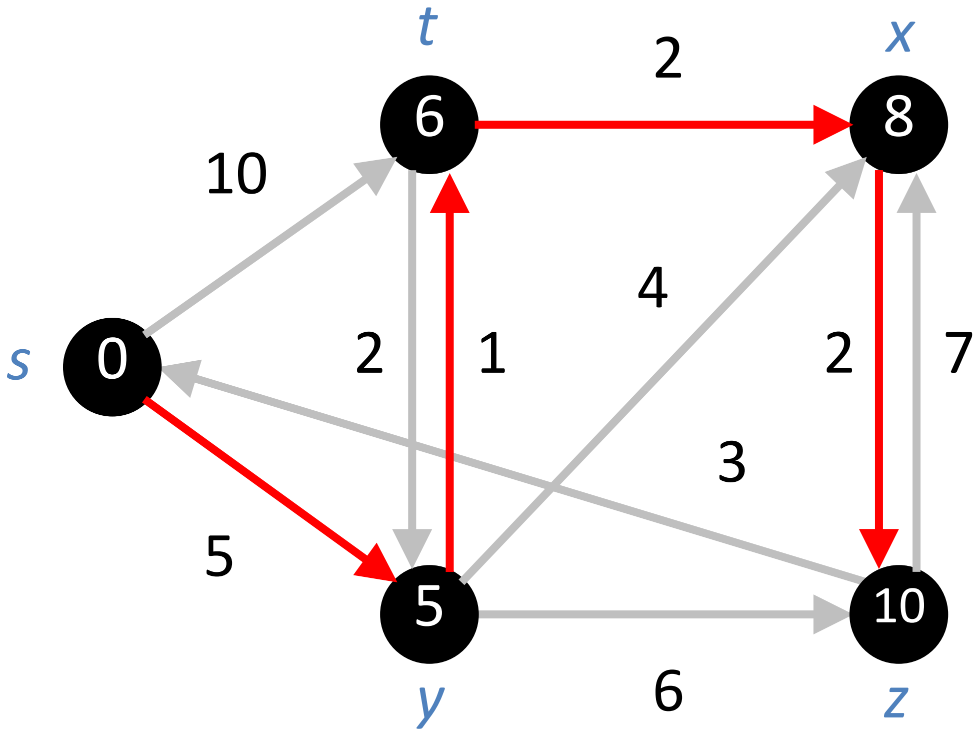
Example



Example



Example



Correctness

Loop invariant:
At the start of each iteration of the while loop, $d[v] = \delta(s,v)$ for all $v \in S$.

Initialization:
Initially, $S = \emptyset$, so trivially true.

Termination:
At end, $Q=\emptyset \Rightarrow S = V \Rightarrow d[v] = \delta(s,v)$ for all $v \in V$.

Maintenance:
Show that $d[u] = \delta(s,u)$ when u is added to S in each Iteration.

Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Suppose there exists u such that $d[u] \neq \delta(s,u)$.

Let u be the first vertex for which $d[u] \neq \delta(s, u)$ when u is added to S.

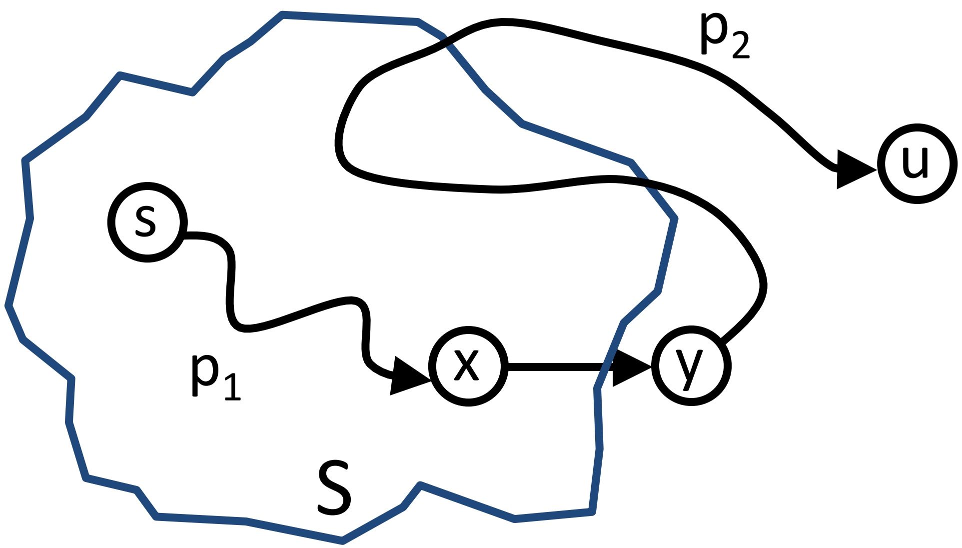
- $u \neq s$, since $d[s] = \delta(s,s) = 0$.
- Therefore, $s \in S$, so $S \neq \varnothing$.
- There must be some path $s \rightsquigarrow u$. Otherwise $d[u] = \delta(s,u) = \infty$ by no-path property.
- So, there is a path $s \rightsquigarrow u$. Thus, there is a shortest path $s \rightsquigarrow u$.

Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Just before u is added to S, path p connects a vertex in S (i.e., s) to a vertex in $V - S$ (i.e., u).

Let y be first vertex along p that is in $V - S$, and let $x \in S$ be y is predecessor.



Decompose p into $s \overset{p_1}{\leadsto} x \rightarrow y \overset{p_2}{\leadsto} u$.

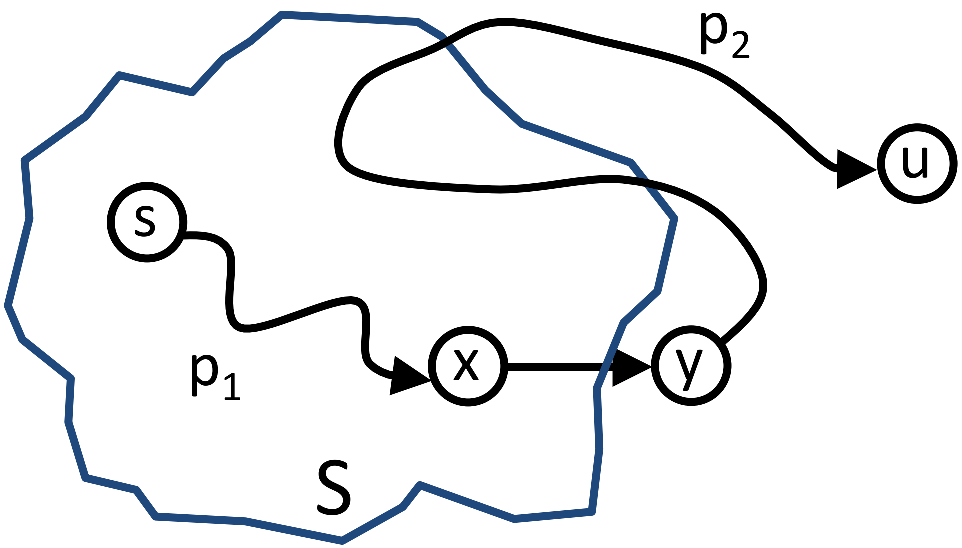
Correctness (cont'd)

Claim: $d[y] = \delta(s, y)$ when u is added to S.

Proof:

$x \in S$ and u is the first vertex such that $d[u] \neq \delta(s, u)$ when u is added to $S \Rightarrow d[x] = \delta(s, x)$ when x is added to S.

Relaxed (x, y) at that time, so by the convergence property, $d[y] = \delta(s, y)$.



Correctness (cont'd)

Show that $d[u] = \delta(s,u)$ when u is added to S in each iteration.

Now can get a contradiction to $d[u] \neq \delta(s, u)$:

y is on shortest path $s \leadsto u$, and all edge weights are nonnegative.

$\Rightarrow \delta(s, y) \leq \delta(s, u)$

$\Rightarrow d[y] = \delta(s,y)$

$\leq \delta(s,u)$

$\leq d[u]$ (upper-bound property)

In addition, since y and u were in Q when we chose u:

$d[u] \leq d[y] \Rightarrow d[u] = d[y]$.

Therefore, $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Contradicts assumption that $d[u] \neq \delta(s,u)$. ■

Analysis

Like Prim's algorithm, it depends on implementation of priority queue.

If binary heap, each operation takes $O(\lg V)$ time
$\Rightarrow O(E \lg V)$.

Note: We can achieve $O(V \lg V + E)$ with Fibonacci heaps.