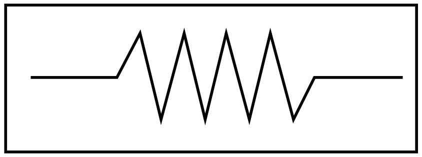
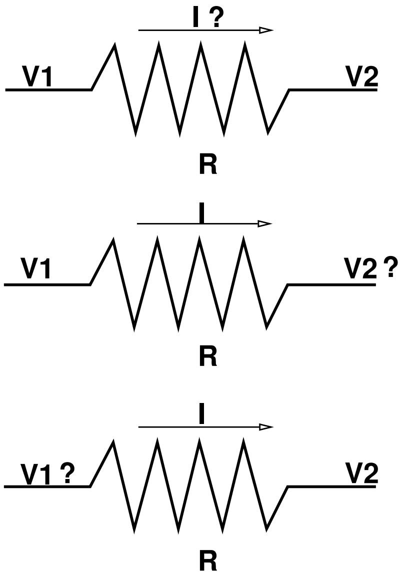
Objects, Re-use, and Causality



Object "resistor"

V1 − V2 = R*I



I = (V1−V2)/R

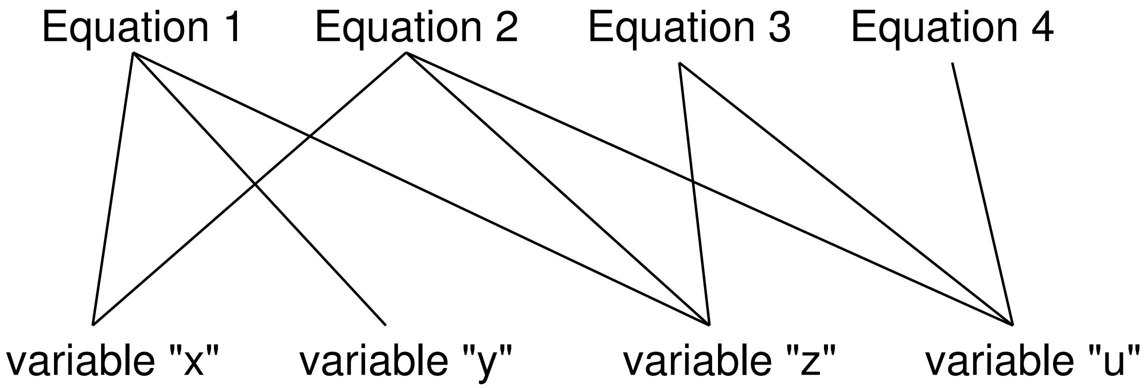
V2 = V1 − R*I

V1 = V2 + R*I

Causality Assignment

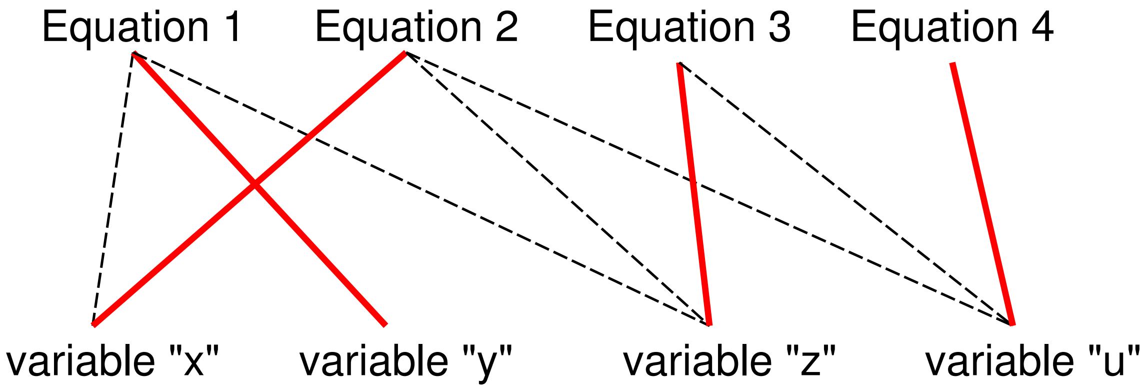
$$\begin{cases} x+y+z &= 0 \quad \text{Equation 1} \\ x+3z+u^2 &= 0 \quad \text{Equation 2} \\ z-u-16 &= 0 \quad \text{Equation 4} \\ u-5 &= 0 \quad \text{Equation 4} \end{cases}$$

Causality Assignment = bipartite (dependency) graph maximum cardinality matching



Equation 1 Equation 2 Equation 3 Equation 4

variable "x" variable "y" variable "z" variable "u"



Equation 1 Equation 2 Equation 3 Equation 4

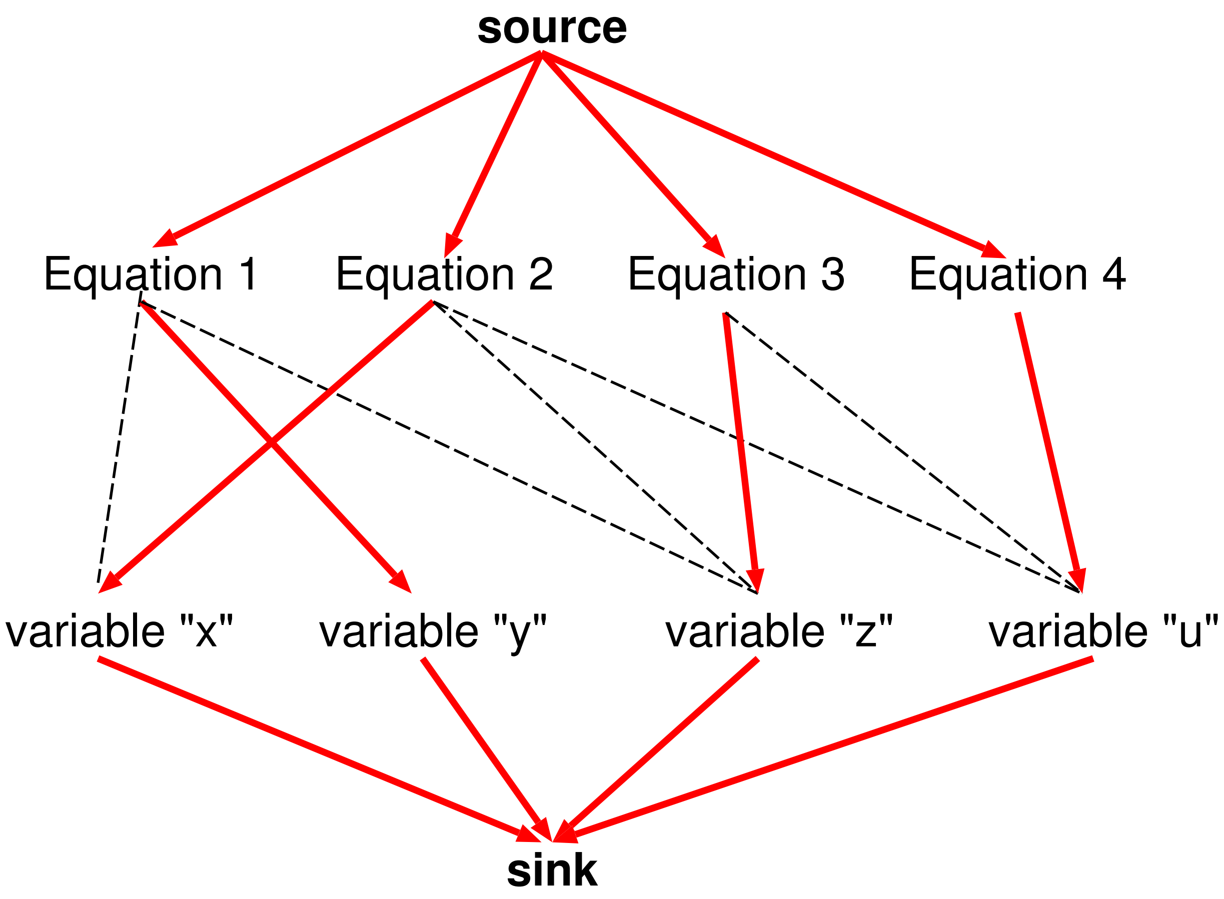
variable "x" variable "y" variable "z" variable "u"

Causality Assignment: causality assigned

$$\begin{cases} x+\underline{y}+z &= 0 \quad \text{Equation 1} \\ \underline{x}+3z+u^2 &= 0 \quad \text{Equation 2} \\ \underline{z}-u-16 &= 0 \quad \text{Equation 4} \\ \underline{u}-5 &= 0 \quad \text{Equation 4} \end{cases}$$

$$\begin{cases} \underline{y} &= -x-z \\ \underline{x} &= -3z-u^2 \\ \underline{z} &= u+16 \\ \underline{u} &= 5 \end{cases}$$

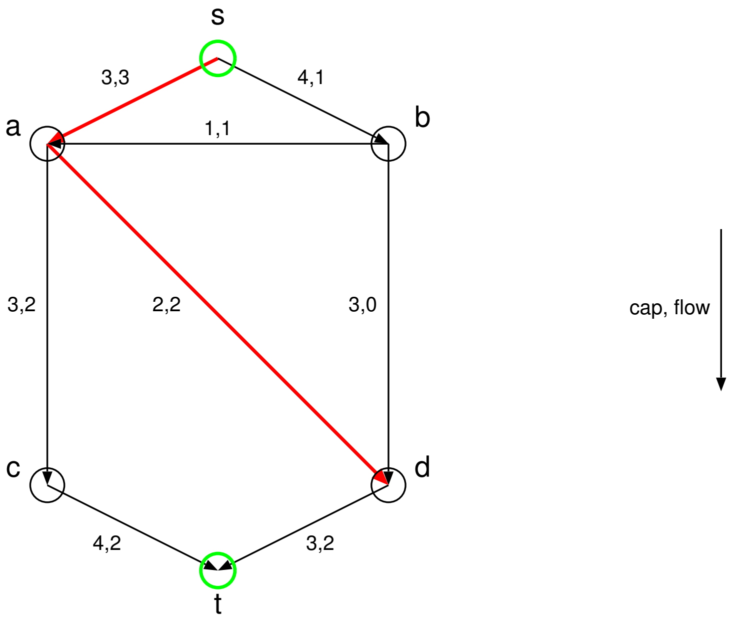
causality assignment: network flow



Network Flow Problems

$$G = [V, E]$$

Directed graph G with *source s* and *sink t*



Network Flow: definitions

Positive *capacity* $cap(v,w)$ on every edge $[v,w]$.

$cap(v,w) = 0$ if $[v,w]$ is not an edge.

A *flow* on G is any real-valued function f with properties:

1. *skew symmetry*. $f(v,w) = -f(w,v)$.
   $f(v,w) > 0$ is called a flow *from* v to w.
2. *capacity constraint*. $f(v,w) \leq cap(v,w)$. If $[v,w]$ is an edge such that $f(v,w) = cap(v,w)$, the flow is said to *saturate* $[v,w]$.
3. *flow conservation*. For every vertex v other than s and t

$$\sum_{w} f(v,w) = 0$$

Network Flow: *maximum flow*

The *value* $|f|$ of a flow f is the net flow out of the source

$$\sum_{v} f(s,v)$$

Maximum Flow Problem (Ford and Fulkerson).

Network Flow: *cut*

A *cut*: partition $X, \overline{X}$ of the vertex set V into two parts X and $\overline{X} = V - X$ such that X contains s and $\overline{X}$ contains t.

The *capacity* of a cut $X, \overline{X}$ is

$$cap(X, \overline{X}) = \sum_{v \in X, w \in \overline{X}} cap(v, w)$$

Flow across a cut is

$$f(X, \overline{X}) = \sum_{v \in X, w \in \overline{X}} f(v, w)$$

Network Flow: *max-flow min-cut theorem*

For any flow f, the flow across any cut $X, \overline{X}$ is equal to the flow value.
Capacity constraint $\rightarrow$ flow across cut cannot exceed capacity of the cut.
Maximum flow is not greater than the capacity of a *minimum cut*.

max-flow min-cut theorem: *maximum flow* = *minimum cut*

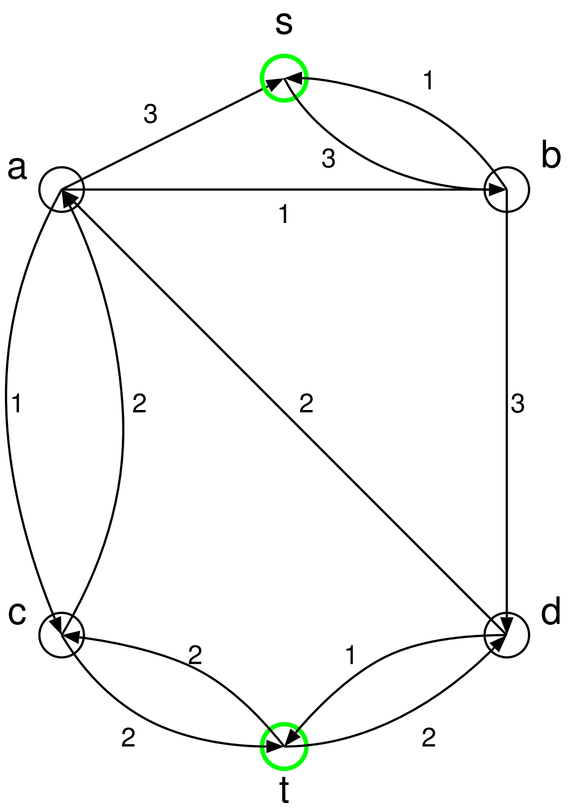
Network Flow: *residual graph*

Residual capacity for flow f

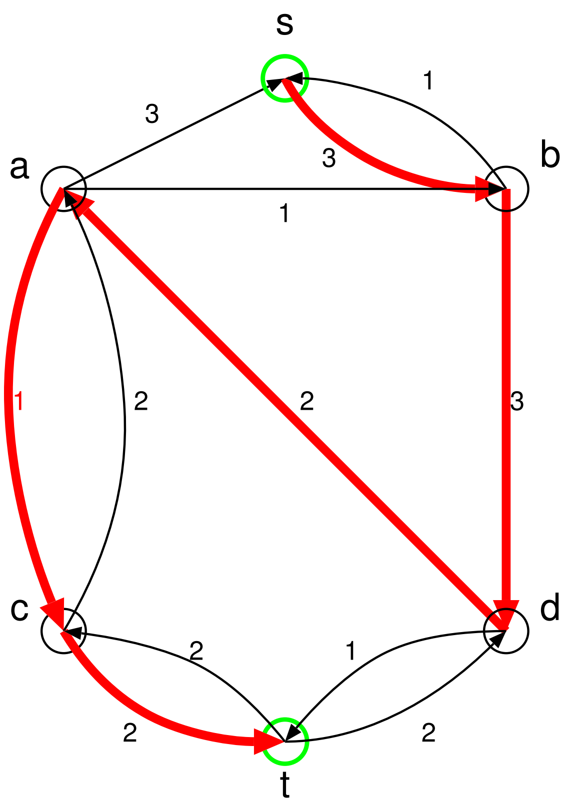
$$res(v,w) = cap(v,w) - f(v,w)$$

Up to $res(v,w)$ additional flow can be pushed along $[v,w]$.
Residual graph R is graph with edges $res(v,w)$.
Augmenting path from s to t.
Residual capacity is *minimum* $res(v,w)$.

Network Flow: *residual graph*



Network Flow: *augmenting path*



Ford Fulkerson

- Augmenting step
  1. Find an augmenting path p for the current flow.
  2. Increase the value of the flow
     by pushing $res(p)$ units of flow along p.
- Pathfinding step
  1. Find a path p_i from s to t in G^*.
  2. Let Δ_i be the minimum of $f^*(v,w)$ for $[v,w]$ an edge of p_i.
     For every edge $[v,w]$ on p_i, decrease $f^*(v,w)$ by Δ_i and delete $[v,w]$ from G^* if its flow is now zero.
  3. Increment i by one.

Path Finding: which path ?

- Edmonds and Karp:

  augmentation along path with maximum residual capacity.

- Dinic:

  augmentation along shortest augmenting path.

  Length: number of edges a path contains.

Dinic's algorithm: find *blocking flows* to saturate edges

1. Begin with zero flow.
2. Find a *blocking flow* f' on the *level graph* for the current flow f.

   Blocking flow: every path from the source s to the sink t contains a saturated edge.
3. Replace f by the flow $f+f'$ defined by:

   $$(f+f')(v,w) = f(v,w) + f'(v,w).$$
4. Repeat until the sink t is not in the level graph for the current flow.

Level Graph

- R: the residual graph for a flow f.
- *level* of v = the length of the shortest path from s to any vertex v in R.
- *Level graph* L for f = the subgraph of R containing
  - only the vertices reachable from s
  - only the edges $[v,w]$ such that

$$level(w) = level(v) + 1.$$

L contains every *shortest* augmenting path and can be constructed in $O(m)$ time by *breadth-first search*.

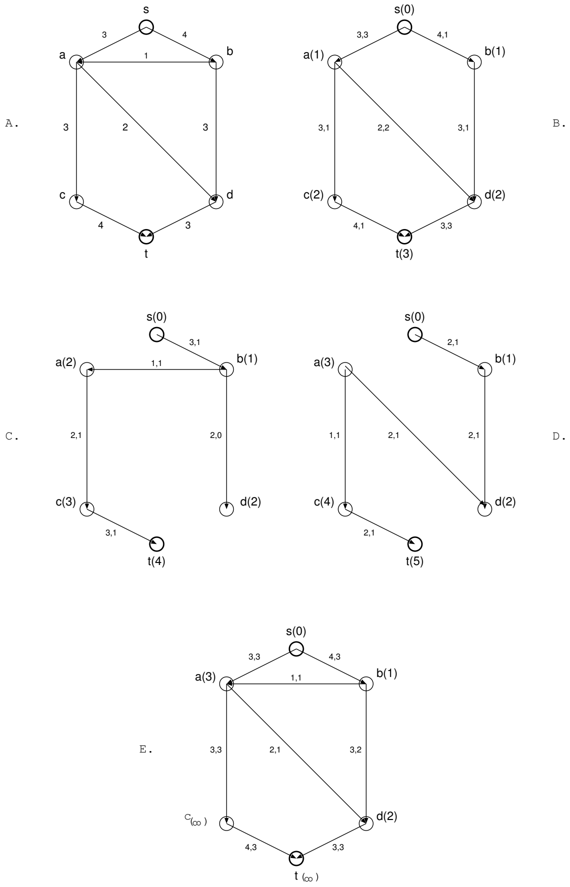
Finding a Blocking Flow (DFS)

- *Initialize*: Let $p = [s]$ and $v = s$. Go to *Advance*.
- *Advance*: If there is no edge out of v, go to *Retreat*. Otherwise, let $[v, w]$ be an edge out of v. Replace p by $p \& [w]$ and v by w. If $w \neq t$ repeat *Advance*; if $w = t$ go to *Augment*.
- *Augment*: Let Δ be the minimum of $(cap(v,w) - f(v,w))$ for $[v,w]$ an edge of p. Add Δ to the flow of every edge on p, delete from G all newly saturated edges, and go to *Initialize*.
- *Retreat*: If $v = s$ halt. Otherwise, let $[u, v]$ be the last edge on p. Delete v from p and $[u, v]$ from G, replace v by u, and go to *Advance*.

Dinic Performance
(m is number of nodes, n is number of edges)

- Finds a blocking flow in $O(nm)$ time, and a maximum flow in $O(n^2 m)$ time.
- On a unit network, Dinic's algorithm finds a blocking flow in $O(m)$ time, and a maximum flow in $O(n^{1/2} m)$ time. Unit network: edge capacities integer, each vertex v other than the source and the sink has either a single entering edge of capacity one, or a single outgoing edge of capacity one.
- On a network whose edge capacities are all one, Dinic's algorithm finds a maximum flow in $O(\text{min}\{n^{2/3} m, m^{3/2}\})$ time.

Example


A.
s
a
b
c
d
t
3
4
1
3
2
3
4
3
B.
s(0)
a(1)
b(1)
c(2)
d(2)
t(3)
3,3
4,1
3,1
2,2
3,1
4,1
3,3
C.
s(0)
a(2)
b(1)
c(3)
d(2)
t(4)
3,1
1,1
2,1
2,0
3,1
D.
s(0)
a(3)
b(1)
c(4)
d(2)
t(5)
2,1
1,1
2,1
2,1
2,1
E.
s(0)
a(3)
b(1)
c(∞)
d(2)
t (∞)
3,3
4,3
1,1
3,3
2,1
3,2
4,3
3,3

Symbolic Manipulation (Computer Algebra)

Simplification of expressions, re-writing of equations, symbolic solving, . . .

- Mathematica
- REDUCE
- AXIOM
- MACSYMA
- MuPAD (`http://www.mupad.de/`)

(muPAD) examples

```
>> 100!;

93326215443944152681699238856266700490715968264381621468592963895217599993\
22991560894146397615651828625369792082722375825118521091686400000000000000\
0000000000


>> (x+1)^4;

                                        4
                                 (x + 1)

>> expand(%);

                                 2      3    4
                        4 x + 6 x  + 4 x  + x  + 1

>>  x^2+2*x+1;

                                      2
                               2 x + x  + 1
```

(muPAD) examples

```
>> factor(%);

                              [1, x + 1, 2]

>> diff(x^2+2*x+1,x);

                                2 x + 2


>> int(x^2+2*x+1,x);

                                      3
                                 2   x
                              x + x  + --
                                     3

>> 2+3+4+x+4;

                                x + 13

>> 2+ 3+x+y+x*y+x^2+y^3+4;

                                    2    3
                       x + y + x y + x  + y  + 9
```

(muPAD) examples

```
>> solve({x+a*y-2,x-b*y+4},{x,y});

                    { {     2 b - 4 a        6   } }
                    { { x = ---------, y = ----- } }
                    { {       a + b        a + b } }

>> subs(%,a=3,b=4);

                          {{x = -4/7, y = 6/7}}

>> generate::C(x^2+4-sin(y));

                    "    t4 = -sin(y) + x*x + 4.0 ;"

>> generate::TeX(x^2+4-sin(y));

                   "- \\sin\\left(y\\right) + x^2 + 4"
```

Canonical Form

```
>> (y+2)+3 + x;

                                 x + y + 5

>> 5+y+x;

                                 x + y + 5

>> 2+3+2*y+x-y;

                                 x + y + 5

>> 2+x -y -x;

                                   2 - y
```

Canonical Form

(Davenport)

A representation of a mathematical object (e.g., polynomial) is *canonical* if two different representations always correspond to two different objects.

A correspondence f between a class O of objects and a class R of representations is a *representation* of O by R if each element of O corresponds to one or more elements of R (otherwise it is not represented) and each element of R corresponds to one and only one element of O (otherwise we do not know which element of O is represented).

The representation is canonical if f is *bijective*. With a canonical representation it is possible to check *equality* of objects by verifying that their representations are equal.

Normal Form

If O has the structure of a monoid, a weaker concept may be defined. A representation is called *normal* if zero has only one representation. Every canonical representation is normal, but the converse is false.

Having a unique representation for zero is important to be able to test for division by zero.

A normal representation over a group also gives us an algorithm to determine whether two elements a and b of O are equal. It is sufficient to check whether $a - b = 0$. In a canonical representation, it suffices to check whether a's and b's representations are identical.

Regular and Natural form

A representation should be *regular*

$A = x^2 + x$,
$A + 1 = (x^3 - 1)/(x - 1)$,
$A - x = x^2, \ldots$ is *not* regular.

Representations must be *natural*. Some form of simplification should occur. For polynomials in one variable, every power of x should appear at most once, and powers should be sorted in ascending or descending order.

Polynomials

Representations of polynomials: *dense* and *sparse*.

- Dense: vector of coefficients
- Sparse: list of (coeff, degree) tuples

$$(x^{1000}+1)(x^{1000}-1) = x^{2000}-1$$

Polynomial *in* a particular variable: $\sin(x) + 3 * \sin^2(x) - 2$
is polynomial in $\sin(x)$

Increasing or decreasing powers.

Polynomials in multiple variables

canonical, natural representation

Different types of ordering:

- *lexicographic*: alphabetically ordered. Within one variable name, ordered by powers. If the powers of that variable are the same, look at the next (lexicographic) variable. $x^2 + 2xy + x + y^2 + y + 1$
- *total degree, then lexicographic*: lexicographic distinction between same total degree, ordered by total degree. $x^2 + 2xy + y^2 + x + y + 1$
- *total degree, then inverse lexicographic*: $y^2 + 2xy + x^2 + y + x + 1$

Types, Domains, Algebraic Structures

Used to define *generic* operations

Definitions (Birkhoff & McLane)

1. Semigroup $S, +$
   - closure
   - associativity
2. Monoid
   - Semigroup with unit 0
3. Group
   - Identity 1
   - Inverse

4. Commutative (Abelian)
    - Semigroup
    - Monoid
    - Group
5. Ring $R, +, *$
    - $R, +$ Abelian Group
    - $R, *$ Monoid with unit 1
    - $*$ is distributive on both sides over $+$
6. Commutative Ring
    - Ring and $R, *$ is commutative
7. Field $=$ Commutative Ring
    - each non-zero element has multiplicative inverse

Computer Algebra $\sim$ Compilers

1. lexical analysis
2. syntactic analysis (grammar parsing)
3. intermediate representation: Abstract Syntax Tree (AST) and Symbol Table (ST)
4. operations (symbolic manipulation) on AST+ST
5. compiler compilers
    - Gentle `http://www.first.gmd.de/gentle`
    - TRAP `http://www.first.gmd.de/smile/trap`
    - ANTLR `http://www.antlr.org`
    - PCCTS `http://www.ocnus.com/pccts.html`

- Catalog of Compiler Construction Tools

  `http://www.first.gmd.de/cogent/catalog`

Internal model representation

- Abstract Syntax Tree + Symbol Table

$$b+2-(a+3)=x$$

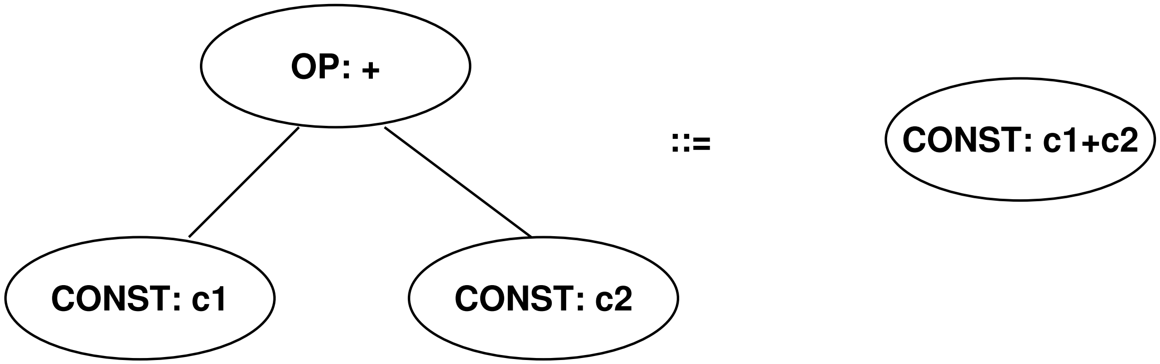
```
=[ +[b, +[2, -[a, 3]], x]
```

- From the AST + ST, a dependency graph can be built.
  1. for causality assignment,
  2. for equation re-write,
  3. for loop detection and sorting,
  4. for constant folding,
  5. for parameter expression lifting ($-K/g$),
  6. for output equation selection

Constant Folding Graph Grammar

$$c1 + c2$$

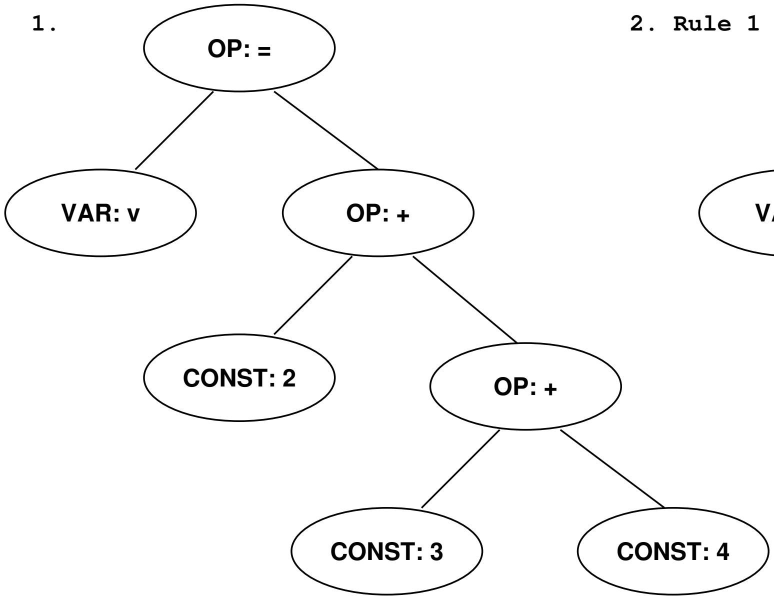
Rule 1.



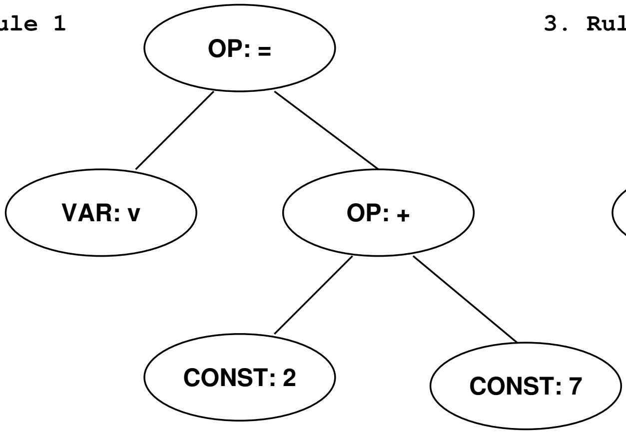
Constant Folding Transformation

$$v = 2 + (3 + 4)$$

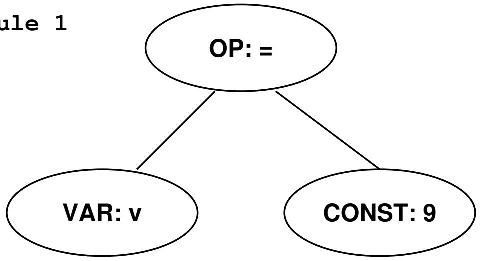
`1.`



`2. Rule 1`



`3. Rule 1`



Canonical Representation

To encode *associativity* and *commutativity* of operators.

A representation of a mathematical object is *canonical* if two different representations always correspond to two different objects.

1. n-ary operators
2. inv for each operator
3. lexicographic ordering

```
+[2, inv[3], a, inv[b]]
+[inv[1], a, inv[b]]
```

$$2-3+a-b \Rightarrow -1+a-b$$

$\longrightarrow$ **symbolic operations** (simplify, analyze, . . .)

$\longrightarrow$ **re-use AND performance !**

Object-Oriented Modelling of Physical Systems

1. Encapsulation, objects, classes, . . .
2. Types (Software vs. Dynamical systems)
    - Subtypes
    - Contravariance
    - Semantics of composition
3. Inheritance
4. Different levels of abstraction

Based on . . .

- WEST (bioactivated sludge waste water treatment)

  `www.hemmiswest.com`

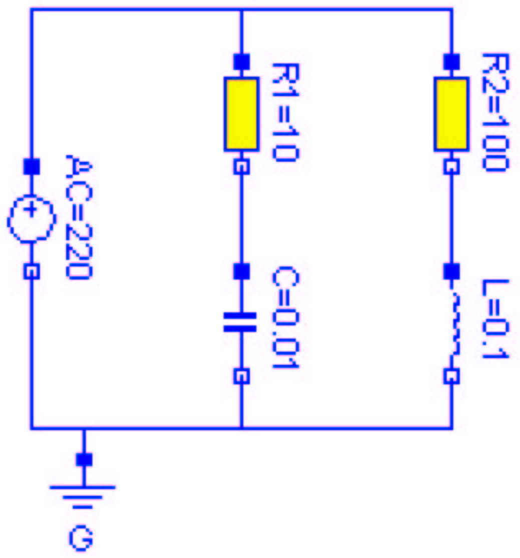
- Modelica (ESPRIT Basic Research, now Association)

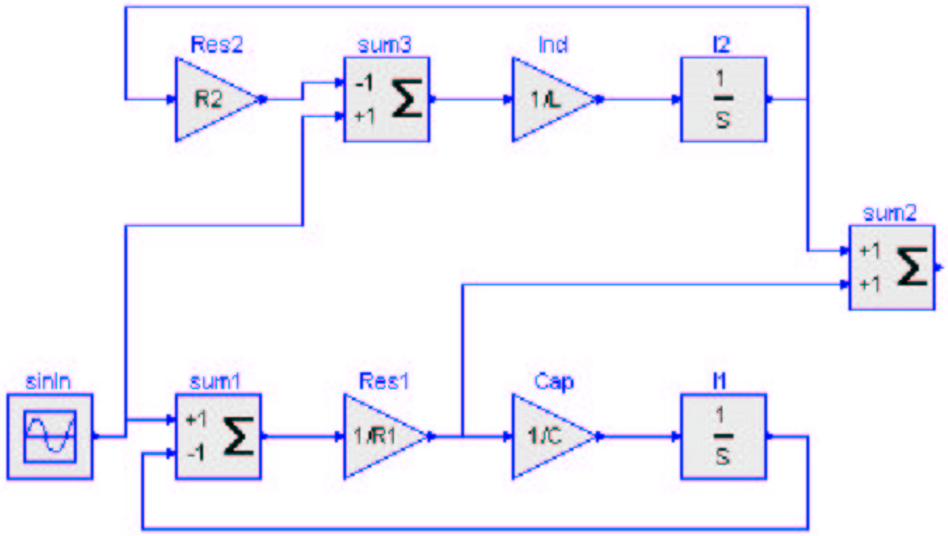
  `www.modelica.org`

  Aims:

  - Standard language for model exchange and re-use
  - Support non-causal, hybrid, hierarchical modelling
  - Semantics based on Hybrid DAEs
  - Separate model (goal: re-use, exchange)
    from its numerical solution (goal: accuracy, speed)
  - Library of basic models

Electrical example: Modelica vs. Matlab/Simulink





Electrical Types

```
type Time = Real (final quantity="Time", final unit="s");
type ElectricPotential = Real (final quantity="ElectricPotential",
                               final unit="V");
type Voltage = ElectricPotential;
type ElectricCurrent = Real (final quantity="ElectricCurrent",
                             final unit="A");
type Current = ElectricCurrent;
```

Electrical Pin Interface

```
connector PositivePin "Positive pin of an electric component"
       Voltage v "Potential at the pin";
  flow Current i "Current flowing into the pin";
end PositivePin;
```

Electrical Port

```
partial model OnePort
  "Component with two electrical pins p and n
   and current i from p to n"
  Voltage v "Voltage drop between the two pins (= p.v - n.v)";
  Current i "Current flowing from pin p to pin n";
  PositivePin p;
  NegativePin n;
equation
  v = p.v - n.v;
  0 = p.i + n.i;
  i = p.i;
end OnePort;
```

Electrical Resistor

```
model Resistor "Ideal linear electrical resistor"
  extends OnePort;
  parameter Resistance R=1 "Resistance";
  equation
    R*i = v;
end Resistor;
```

The circuit

```
model circuit
 Resistor  R1(R=10);
 Capacitor C(C=0.01);
 Resistor  R2(R=100);
 Inductor  L(L=0.1);
 VsourceAC AC;
 Ground    G;
equation
 connect(AC.p, R1.p);
 connect(R1.n, C.p);
 connect(C.n, AC.n);
 connect(R1.p, R2.p);
 connect(R2.n, L.p);
 connect(L.n, C.n);
 connect(AC.n, G.p);
end circuit;
```

Dynasim Modelica demo

Future

- multi-formalism
- multi-abstraction
- meta-modelling (AToM^3)