COMP 760 - Winter 2017 - Assignment 2

Due: March 7th, 2017

General rules: In solving these you may consult with each other.

1. (a) Prove that there exists a univeral constant K such that for every function $g : \{0,1\}^n \to \mathbb{R}$, we have

$$\sum_{S \neq \emptyset} \frac{1}{|S|} \widehat{g}(S)^2 \leq \frac{K\|g\|_2^2}{1 + \log(\|g\|_2/\|g\|_1)}.$$

   (b) Conclude that for $f : \{0,1\}^n \to \{0,1\}$, we have

$$\mathrm{Var[f]} \leq K \sum_i \frac{I_i(f)}{1 + \frac{1}{2}\log I_i(f)}.$$

   (c) Conclude the KKL inequality from the above inequality.

2. For a function $f : \{-1,1\}^n \to [-1,1]$, define $\partial_i f(x) = \frac{f(x) - f(x \oplus e_i)}{2}$, and let $I_i(f) = \|\partial_i f\|_1$. Let $\Delta f(x) = \sum_i |\partial_i f(x)|$. Prove that if f is of degree d, then

$$I_f := \sum_i I_i(f) \leq \|\Delta f\|_\infty \leq 2d^2.$$

   Hint: Use Markov brothers' inequality.

3. An $\oplus$-decision tree for a function $f : \{0,1\}^n \to \{0,1\}$ is a decision tree where every internal node is labeled with a subset $S \subseteq \{1,\dots,n\}$; On an input x, we traverses a path from the root to a leaf that is labeled with $f(x)$ where at a node labeled with S, we branch according to the value of $\oplus_{i \in S} x_i$. Prove that if f has a $\oplus$-decision tree with s leaves, then $\|\widehat{f}\|_1 := \sum_S |\widehat{f}(S)| \leq s$.

4. Suppose that $\|\widehat{f}\|_1 = M$ for a function $f : \mathbb{Z}_2^n \to \{-1,1\}$. Let $\alpha = |\widehat{f}(S)|$ and $\beta = |\widehat{f}(T)|$ denote the two largest Fourier coefficients (in absolute value).

   (a) Denoting $R = S\Delta T$, prove that either

$$\|\widehat{f|_{\chi_R=1}}\|_1 \leq M - \alpha \leq M - \frac{1}{M}, \qquad \|\widehat{f|_{\chi_R=-1}}\|_1 \leq M - \beta,$$

   or

$$\|\widehat{f|_{\chi_R=-1}}\|_1 \leq M - \alpha \leq M - \frac{1}{M}, \qquad \|\widehat{f|_{\chi_R=1}}\|_1 \leq M - \beta.$$

   (Here $f|_{\chi_R=1}$ means the restriction of f to the subspace $\{x \in \mathbb{Z}_2^n, \chi_R(x) = 1\} \cong \mathbb{Z}_2^{n-1}$).

   (b) Conclude that if $f : \{0,1\}^n \to \{-1,1\}$ is a boolean function with $\|\widehat{f}\|_1 = M$, then f can be computed by an $\oplus$-decision tree with at most n^{M^2} leaves.