COMP760, SUMMARY OF LECTURE 9.

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1. Fourier Analysis

For two functions $f, g : \mathbb{F}_2^n \to \mathbb{R}$ define

$$\langle f, g\rangle = \mathbb{E}[f(x)g(x)] = \frac{1}{2^n}\sum_x f(x)g(x).$$

This inner product turns the space of all functions $f : \mathbb{F}_2^n \to \mathbb{R}$ into a Hilbert space. Next we want to construct an orthonormal basis for this Hilbert space.

Definition 1 (Characters)**.** *For $S \subseteq \{1, \ldots, n\}$, define $\chi_S : \mathbb{F}_2^n \to \mathbb{R}$ as*

$$\chi_S : x \mapsto \prod_{i\in S}(-1)^{x_i} = (-1)^{\sum_{i\in S} x_i}.$$

Note $\chi_\emptyset \equiv 1$.

Observe that $\chi_S\chi_T = \chi_{S\triangle T}$, and that

$$\mathbb{E}[\chi_S(x)] = \begin{cases} 1 & S = \emptyset \\ 0 & S \neq \emptyset \end{cases}$$

It follows from these two observations (and the fact that there are 2^n of them) that characters form an orthonormal basis for the space of functions $f : \mathbb{F}_2^n \to \mathbb{R}$:

$$\langle \chi_S, \chi_T\rangle = \mathbb{E}\chi_S(x)\chi_T(x) = \begin{cases} 1 & S = T \\ 0 & S \neq T \end{cases}$$

Definition 2. *The Fourier expansion of a function $f : \mathbb{F}_2^n \to \mathbb{R}$ is the unique expansion of f in the Fourier basis:*

$$f = \sum \widehat{f}(S)\chi_S,$$

where $\widehat{f}(S) \in \mathbb{R}$ are called Fourier coefficients.

The following facts are very useful.

- $\widehat{f}(S) = \langle f, \chi_S\rangle \leq \mathbb{E}|f(x)|$.
- $\widehat{f}(\emptyset) = \mathbb{E}[f(x)]$.
- Plancharel: $\langle f, g\rangle = \sum \widehat{f}(S)\widehat{g}(S)$.
- Parseval: $\mathbb{E}[f(x)^2] = \langle f, f\rangle = \sum \widehat{f}(S)^2$.

Note that if we consider the space of functions $f : \{-1, 1\}^n \to \mathbb{R}$, then we have to replace the definition of the characters to Walsh functions $\chi_S : x \mapsto \prod_{i\in S} x_i$, which are just monomials. In this case the Fourier expansion is the polynomial representation:

$$f(x) = \sum_{S\subseteq[n]} \widehat{f}(S)\chi_S(x).$$

Definition 3. *Given a function* $f : \{0,1\}^n \to \{0,1\}$*, the (Fourier) degree of* f*, denote by* $\deg(f)$ *is the size of the largest* S *with* $\widehat{f}(S) \neq 0$.

Similar to approximate rank and sign-rank we can define the approximate degree, and the sign-degree of a function $f : \{0,1\}^n \to \mathbb{R}$:

- For $\epsilon \geq 0$, we the ϵ-approximate degree as $\deg_\epsilon(f) = \min_{\|f-g\|_\infty \leq \epsilon} \deg(g)$.
- The sign-degree is defined as $\deg_\pm(f) = \min \deg(g)$ over all g satisfying $f(x)g(x) > 0$ for all x with $f(x) \neq 0$.

Note that for a function $f : \{0,1\}^n \to \{-1,1\}$, and $0 \leq \epsilon < 1$ we have

$$\deg(f) \geq \deg_\epsilon(f) \geq \lim_{\epsilon \nearrow 1} \deg_\epsilon(f) = \deg_\pm(f).$$

As we saw earlier [Riv81, Corollary 1.4.1] we know that there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $d := \deg(p) = O(1/\alpha)$ and it satisfies

$$p([2/3, 4/3]) \subseteq [1-\alpha, 1+\alpha],$$

and

$$p([-4/3, -2/3]) \subseteq [-1-\alpha, -1+\alpha].$$

This shows that for every fixed $\epsilon \in (0,1)$,

$$\deg_{1/3}(f) = \Theta_\epsilon(\deg_\epsilon(f)).$$

In the light of this fact, without loss of generalization, we often refer to $\deg_{1/3}(f)$ as the approximate degree of f. The degree and approximate degree are closely related to two other complexity measures, decision tree complexity, and block sensitivity. We will not discuss the block sensitivity here, and refer the interested reader to the survey [BdW02]. The decision tree complexity of f, denoted by $\mathrm{dt}(f)$, is the smallest height of a decision tree that computes f. It is an easy exercise to see that $\deg_{1/3}(f) \leq \deg(f) \leq \mathrm{dt}(f)$. It follows from the results of [NS94, BBC$^+$01] that

$$\deg_{1/3}(f) \leq \deg(f) \leq \mathrm{dt}(f) \leq c \deg_{1/3}^6(f) \leq c \deg^6(f).$$

Hence $\deg_{1/3}(f), \deg(f)$ and $\mathrm{dt}(f)$ are polynomially related. The sensitivity of f is defined as $s(f) = \max_x |\{i \ : \ f(x) \neq f(x \oplus e_i)\}|$. Obviously $s(f) \leq \mathrm{dt}(f)$. It is a major open problem whether $s(f)$ is polynomially related to $\deg(f)$ (equivalently $\mathrm{dt}(f)$ or $\deg_{1/3}(f)$. We refer the reader to [HKP11] for more on the variations and the history of this problem.

There are various ways to construct communication problems from a function $f : \{0,1\}^n \to \{-1,1\}$. Probably the most natural one is to consider the function $g : (x,y) \mapsto f(x \oplus y)$. It is easy to see that the eigenvalues of M_g are $\{2^n \widehat{f}(T)\}_{T \subseteq [n]}$ corresponding to eigenvectors $\{\chi_T\}_{T \subseteq [n]}$.

Definition 4. *Given* $f : \{0,1\}^n \to \{-1,1\}$*, its* degree-$d$ threshold weight *is defined to be the minimum* $\sum_{|S| \leq d} |\lambda_S|$ *over all* integers λ_S *such that*

$$f(x) \equiv \mathrm{sign}(\sum_S \lambda_S \chi_S(x)).$$

If no such λ_S *can be found (equivalently* $d < \deg_\pm(f)$*) then we set* $W(f,d) = \infty$*. We also define*

$$W(f) = \min_{d=0,\ldots,n} W(f,d).$$

Note that since λ_S are integers, the function $\sum_S \lambda_S \chi_S(x)$ is integer-valued, and hence if it sign-represents $f : \{0,1\}^n \to \{-1,1\}$, then for all x,

$$\left|\sum_S \lambda_S \chi_S(x)\right| \geq 1.$$

Proposition 5. *For every $f : \{0,1\}^n \to \{-1,1\}$ we have*

$$R_{\frac{1}{2}-\frac{1}{2W(f)}}(f(x \oplus y)) = O(1).$$

Proof. Let

$$f(x) \equiv \mathrm{sign}(\sum_S \lambda_S \chi_S(x))$$

satisfy $\sum |\lambda_S| = W(f)$ and $\lambda_S \in \mathbb{Z}$. Alice and Bob choose a random S with their shared randomness such that

$$\Pr[S \text{ is chosen}] = \frac{|\lambda_S|}{W(f)}.$$

Then Alice sends $\chi_S(x)$ to Bob and Bob outputs $\mathrm{sign}(\lambda_S)\chi_S(x)\chi_S(y)$. Note

$$\begin{aligned}
\frac{1}{2} - \Pr_S[f(x \oplus y) \neq \mathrm{sign}(\lambda_S)\chi_S(x)\chi_S(y)] &= \frac{1}{2}\mathbb{E}_S\left[f(x \oplus y)\mathrm{sign}(\lambda_S)\chi_S(x)\chi_S(y)\right] \\
&= \frac{1}{2}\sum_S \frac{|\lambda_S|}{W(f)} f(x \oplus y)\mathrm{sign}(\lambda_S)\chi_S(x \oplus y) \\
&= \frac{f(x \oplus y)}{2W(f)} \sum \lambda_S \chi_S(x \oplus y) \\
&= \frac{1}{2W(f)}\left|\sum \lambda_S \chi_S(x \oplus y)\right| \geq \frac{1}{2W(f)}.
\end{aligned}$$

□

References

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